Partition Function of superconformal CS Theory by Fermi Gas Approach

Tomoki Nosaka and Sanefumi Moriyama / 1407.4268

YITP YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

Summary

- We studied the partition function of $\mathcal{N}=4$ superconformal quiver Chenr-Simons theories in "M theoretical region".
  (M2 branes in non-trivial background) (without $k\to\infty$)
- Applying Fermi Gas formalism, we analysed grand potential $J(\mu)=\log\sum_{N=0}^{\infty}e^{\mu N}Z(N)$ and explicitly obtained
  - **All-order** perturbative corrections in $1/\mu$ and k
  - Two kinds of non-perturbative corrections, **"worldsheet instantons"** and **"membrane instantons"**.
  - Non-trivial **cancellation of divergences** among instanton correction

$\mathcal{N}=4$ quiver Chern-Simons Theory

[Gaiotto-Witten, Hosomichi-Lee-Lee-Lee-Park, Imamura-Kimura]

- Characterized by M signs:

$$\{s_a\}_{a=1}^{M}=\{(+1)^{q_1},(-1)^{p_1},\cdots,(+1)^{q_m},(-1)^{p_m}\}$$

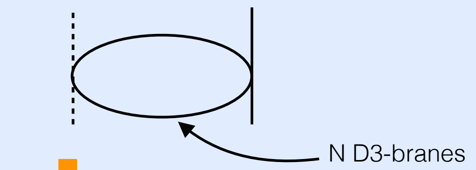
- Field Contents:

$U(N)_{k_a}$ vector multiplet: $(A_\mu^a,\sigma^a,\lambda^a,D^a)$ level: $k_a=\frac{k}{2}(s_a-s_{a-1})$

matter: (Y^a,ψ_Y^a,F_Y^a), (X^a,ψ_X^a,F_X^a) bifund. of $U(N)_{k_a}\times U(N)_{k_{a+1}}$

brane construction:

ABJM

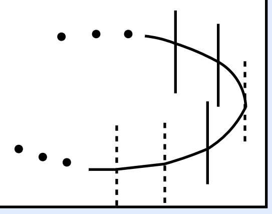


generalize

$\mathcal{N}=4$

$s_a=+1\to$ $(k,1)$ 5-brane

$s_a=-1\to$ NS5-brane



Partition function by localization [Kapustin-Willet-Yaakov]

localization locus: $\sigma^a=\mathrm{diag}(\lambda_{a,i})$, $D^a=\mathrm{diag}(-\lambda_{a,i})$, $others=0$

$$Z(N)=\frac{1}{(N!)^M}\int\prod_{a=1}^{M}\prod_{i=1}^{N}d\lambda_{a,i}e^{\frac{ik_a}{4\pi}\lambda_{a,i}^2}\prod_{a=1}^{M}\frac{\prod_{i>j}2\sinh\frac{\lambda_{a,i}-\lambda_{a,j}}{2}\prod_{i>j}2\sinh\frac{\lambda_{a+1,i}-\lambda_{a+1,j}}{2}}{\prod_{i,j}2\cosh\frac{\lambda_{a,i}-\lambda_{a+1,j}}{2}}$$

$Z_{\text{1-loop}}$ of vector multiplets

$Z_{\text{1-loop}}$ of hyper multiplets

$$\frac{\prod_{i>j}2\sinh(x_i-x_j)\prod_{i>j}2\sinh(y_i-y_j)}{\prod_{i,j}2\cosh(x_i-y_j)}=\det_{i,j}\frac{1}{2\cosh(x_i-y_j)},\qquad \int\prod_{i=1}^{N}dy_i\det_{i,j}A(x_i,y_j)\det_{i,j}B(y_i,z_j)=\det_{i,j}\left[\int dyA(x_i,y)B(y,z_j)\right]$$

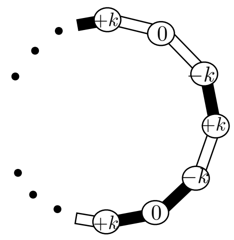
Fermi Gas formalism [Marino-Putrov]

$$Z(N)=\frac{1}{N!}\int\prod_{i=1}^{N}d\lambda_{1,i}e^{\frac{ik_1}{4\pi}}\sum_{\sigma\in S_N}(-1)^\sigma\prod_{i=1}^{N}\rho(\lambda_{1,i},\lambda_{1,\sigma(i)})$$

$$J(\mu)=\log\sum_{N=0}^{\infty}e^{\mu N}Z(N)=\mathrm{tr}\log(1+e^\mu\rho)$$

Inverse trsf: $Z(N)=\int\frac{d\mu}{2\pi i}e^{J-\mu N}$

1-particle Hamiltonian:



$: s_a=+1$

$: s_a=-1$

$$\rho=e^{-\widehat{H}}=\left[\frac{1}{2\cosh\frac{\widehat{q}}{2}}\right]^{q_1}\left[\frac{1}{2\cosh\frac{\widehat{q}}{2}}\right]^{p_1}\cdots\left[\frac{1}{2\cosh\frac{\widehat{q}}{2}}\right]^{q_m}\left[\frac{1}{2\cosh\frac{\widehat{q}}{2}}\right]^{p_m}$$

with $[\widehat{Q},\widehat{P}]=i\hbar\quad(\hbar=2\pi k)$

Merits:

- $k\to 0$ (comprementary against 't Hooft limit) is easy : "classical limit"!
- Very quick derivation of $F\approx N^{\frac{3}{2}}$

(In inverse trsf. integration is dominated by μ^* s.t. $\frac{dJ}{d\mu}=N$

$\approx$ " $\#$ of states with $\widehat{H}<\mu$ " $\approx\mu^2$ ➡ $\mu^*\approx\sqrt{N}, J\approx\mu^3$ ➡ $Z(N)\approx e^{N^{\frac{3}{2}}}$)

Three methods of analysis:

	k	$1/\mu$
1. Volume inside Fermi surface:	perturbative	perturbative
2. WKB expansion:	perturbative	non-perturbative
3. Direct calculation of $Z_k(N)$:	non-perturbative	non-perturbative

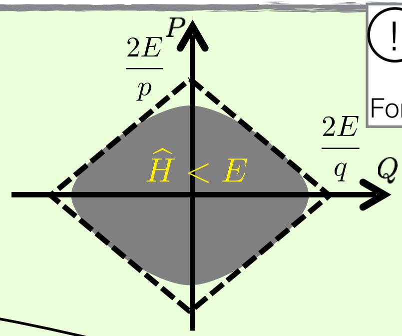
1. ➡ $Z_{\text{pert}}(N)=e^AC^{-\frac{1}{3}}\mathrm{Ai}(C^{-\frac{1}{3}}(N-B))$ with explicit form of B,C
2. ➡ A , $\mathcal{O}(e^{-\mu})$ in $J(\mu)$: we call **"membrane instanton"**
3. ➡ $\mathcal{O}(e^{-\frac{\mu}{k}})$ in $J(\mu)$: we call **"worldsheet instanton"**

Method 1. Volume Inside the Fermi Surface

- Semi-classical $\widehat{H}$ is calculated by Baker-Champbell-Hausdorff formula:

$$\widehat{H} = -q\widehat{U} - p\widehat{T} + \cdots \quad \left(U = \log 2\cosh\frac{Q}{2} \quad T = \log 2\cosh\frac{P}{2} \right)$$

E : large

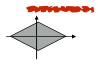


(!) Here we restrict on the cases of $\{s_a\}_{a=1}^{M} = \{(+1)^q, (-1)^p\}$
For general N=4 quiver, please ask me.

$$\cdot J(\mu) = \int_0^\infty dE \frac{dn}{dE} \log(1 + e^{\mu - E})$$

$n(E)$: phase space volume of $\widehat{H} < E$

$$n(E) = CE^2 + B - \frac{\pi^2 C}{3} + \mathcal{O}(e^{-E})$$



: deviation from the polygon

with $C = \dfrac{2}{k\pi^2 qp}$

$$B = \frac{\pi^2 C}{3} - \frac{1}{6k}\left[\frac{q}{p} + \frac{p}{q}\right] + \frac{kqp}{24}$$

$$Z(N)_{\text{pert}} = \int \frac{d\mu}{2\pi i} e^{\frac{C}{3}\mu^3 + B\mu + A - \mu N} = e^A C^{-\frac{1}{3}} \text{Ai}[C^{-\frac{1}{3}}(N-B)]$$

:contains **all-order** perturbative corrections.

Method 2. WKB expansion of J

For ABJM(q=p=1)

$$J(\mu) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{n\mu} \text{tr} e^{-n\widehat{H}}$$

- $\text{tr} \to \int \frac{dQdP}{2\pi\hbar}$
- Each coefficients in $\hbar$-expansion can be integrated by the formula $\displaystyle\int dx \frac{1}{(2\cosh\frac{x}{2})^n} = \frac{\sqrt{4\pi}}{2^n} \frac{\Gamma[\frac{n}{2}]}{\Gamma[\frac{n+1}{2}]}$

Sum over n

generalized hypergeometric series

$${}_4F_3\left[\bullet, \bullet, \frac{e^{2\mu}}{16}\right]$$

generalize

generalize

For $\mathcal{N} = 4$

$$\int \frac{dx}{(2\cosh\frac{x}{2})^{qn}} = \frac{\sqrt{4\pi}}{2^{qn}} \frac{\Gamma[\frac{nq}{2}]}{\Gamma[\frac{nq+1}{2}]}$$

$$\Gamma[qx] = \frac{q^{qx}}{\sqrt{q(2\pi)^{q-1}}} \prod_{i=0}^{q-1} \Gamma\left[x + \frac{i}{q}\right]$$

$${}_{q+p+2}F_{q+p+1}\left[\bullet, \bullet, \frac{e^{2\mu}}{4^{q+p}}\right]$$

Expand around $\mu = \infty$

Example of results:

$$\Sigma(q) = 2, \Sigma(p) = 1 : J(\mu) = \frac{\mu^3}{3\pi^2 k} + \left[-\frac{1}{12k} + \frac{k}{12}\right]\mu + \frac{9\zeta(3)}{2\pi^2 k} - \frac{k}{4} - \frac{\pi^2 k^3}{720} + \cdots + \left[-\frac{4}{\pi k} + \frac{\pi k}{3} + \frac{\pi^3 k^3}{180} + \cdots\right]e^{-\mu} + \cdots$$

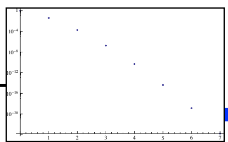
$:C$ $\quad :B$ $\quad :A$ $\quad$: **"membrane instantons"**

Method 3. Direct calculation of $Z_k(N)$

- $\Sigma(q) = 2, \Sigma(p) = 1$, k is fixed to an integer → possible to calculate $\text{tr}\rho^n$ recursively with n
- Read off $(Z_k(1), Z_k(2), \cdots)$ from $J = \text{tr}\log(1 + e^\mu \rho) = \sum_{N=0}^{\infty} e^{\mu N} Z_k(N)$

- Fit with

$$Z\left[J = \frac{C}{3}\mu^3 + B\mu + A + \gamma e^{-\frac{2\mu}{k}} + \cdots\right] = e^A C^{-\frac{1}{3}}\left[\text{Ai}[C^{-\frac{1}{3}}(N-B)] + \gamma \text{Ai}\left[C^{-\frac{1}{3}}\left(N - B + \frac{2}{k}\right)\right] + \cdots\right]$$



to determine unknown coefficients $(\gamma, \cdots)$

Results:

$$J^{k=2}_{\text{non-pert}} = \frac{2\mu+2}{\pi}e^{-\mu} + \mathcal{O}(e^{-2\mu}), \quad J^{k=3}_{\text{non-pert}} = \frac{8}{3}e^{-\frac{2\mu}{3}} + \mathcal{O}(e^{-\frac{4\mu}{3}}),$$

$$J^{k=4}_{\text{non-pert}} = 2\sqrt{2}e^{-\frac{\mu}{2}} + \mathcal{O}(e^{-\mu}), \quad J^{k=5}_{\text{non-pert}} = \frac{8}{\sqrt{5}}e^{-\frac{2\mu}{5}} + \mathcal{O}(e^{-\frac{4\mu}{5}}),$$

$$J^{k=6}_{\text{non-pert}} = \frac{8}{\sqrt{3}}e^{-\frac{\mu}{3}} + \mathcal{O}(e^{-\frac{2\mu}{3}})$$

: **"worldsheet instantons"**

Cancellation of divergences among non-perturbative effects

Philosophy:

- The coefficient of "worldsheet/membrane instanton" alone may diverge at some k
- Matrix model itself is well defined, so must be finite. But how?

→ Ans: At such k, however, the exponents coincide and divergences cancel.

[Hatsuda-Moriyama-Okuyama]

Demonstration:

- Extraporate the results in Method 2 and 3:

$$J^{\text{membrane}}_{\text{non-pert}} = -\frac{2}{\tan\frac{\pi k}{2}}e^{-\mu} + \mathcal{O}(e^{-2\mu}) \quad \approx^{k\sim 2} \left[-\frac{4}{\pi(k-2)} + \frac{\pi(k-2)}{3} + \cdots\right]e^{-\mu} + \cdots$$

$$J^{\text{worldsheet}}_{\text{non-pert}} = \frac{4\cos\frac{\pi}{k}}{\sin^2\frac{2\pi}{k}}e^{-\frac{2\mu}{k}} + \mathcal{O}(e^{-\frac{4\mu}{k}}) \quad \approx^{k\sim 2} \left[\frac{4}{\pi(k-2)} + \frac{2(1+\mu)}{\pi} + \cdots\right]e^{-\mu} + \cdots$$

Both has pole at $k = 2$,
but sum up into **finite** value.
→ finite $J^{k=2}_{\text{non-pert}}$ in Way 3. is reproduced!

Future Works

- Extend to more general theory (Method 2 and 3 for general $\mathcal{N} = 4$; $\mathcal{N} = 3$)
- SUGRA interpretation for { quiver-dependence of A, B ; "instantons" }
- Directry interpretate non-perturbative effects from the original Chern-Simons theories