Entanglement between two interacting CFTs and generalized holographic entanglement entropy

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1.Introduction

Entanglement entropy (EE) is the quantity which measures the degree of quantum entanglement.

EE is a useful tool to study global properties of QFTs.

In the light of AdS/CFT correspondence, the geometries of gravitational spacetimes can be encoded in the quantum entanglement of dual CFTs.

The definition of (Renyi) entanglement entropy

We decompose the total Hilbert space into subsystems A and B.

$$H_{tot} = H_A \otimes H_B$$

We trace out the degrees of freedom of B and consider the reduced density matrix of A.

$$\rho_A = Tr_B \rho_{tot}$$

EE is defined as von Neumann entropy.

$$S_A := -tr_A \rho_A \log \rho_A$$

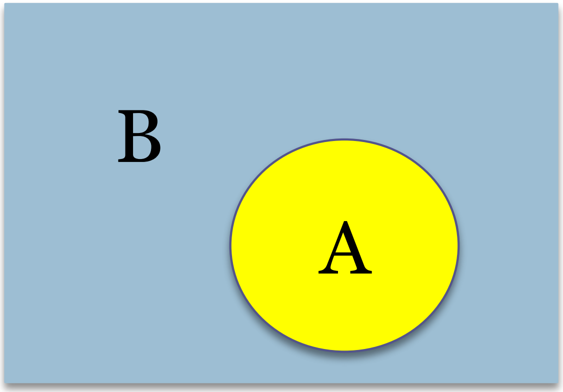
The Renyi entropy is the generalization of EE and defined as

$$S_A^{(n)} := \frac{1}{1-n} \log Tr({\rho_A}^n)$$

How to decompose the total Hilbert space?

Normally, the entanglement entropy is geometrically defined by separating the spatial manifold into the subsystem A and B.

$$H_{tot} = H_A \otimes H_B$$

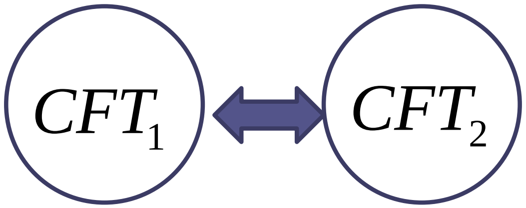


For this decomposition, extensive studies have been carried out and many properties of EE are known.

For example, area law, c-theorem, etc.

Instead, in this work, we analyze EE between two CFTs (called CFT_1 and CFT_2) which live in a common spacetime and interacting with each other, described by the action of the form:

$$S = \int dx^d [L_{CFT_1} + L_{CFT_2} + L_{\mathrm{int}}]$$



The total Hilbert space is decomposed as

$$H_{tot} = H_{CFT_1} \otimes H_{CFT_2}$$

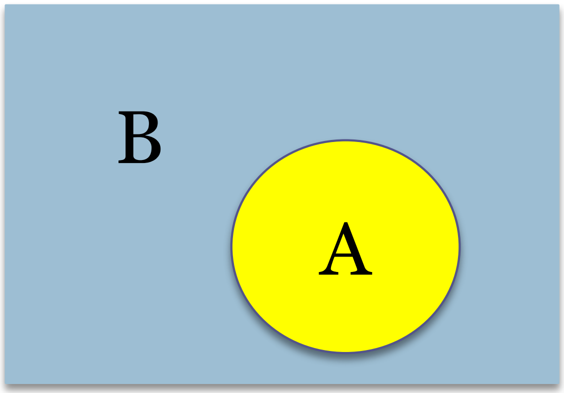
EE is defined as $S_1 := -tr\rho_1 \log \rho_1$ $\qquad \rho_1 = Tr_{CFT_2} \rho_{tot}$

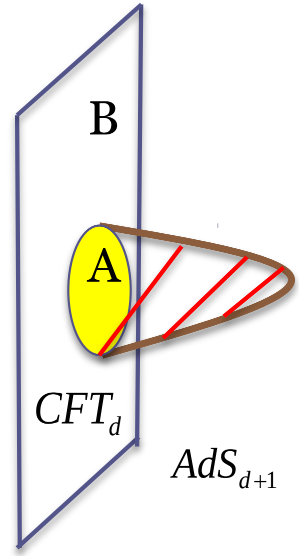
If $L_{\mathrm{int}} = 0$, $S_1 = 0$.

This EE may offer us a universal measure of strength of interactions.

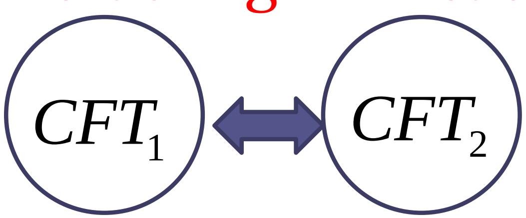
Holographic Interpretation

Geometrically defined EE in CFT → Area of minimal surface in AdS





EE between two CFTs interacting with each other



This EE is closely related to an internal compact space such as S^5 in the $AdS_5 \times S^5$ correspondence

This should be distinguished from a system of two entangling CFTs without interactions, where its gravity dual is given by an AdS black hole geometry.

Contents

2.Our models

We consider two solvable relativistic models.

(1) The massless interaction model

The action is given by

$$S = \frac{1}{2}\int d^d x[(\partial_\mu \phi)^2 + (\partial_\mu \psi)^2 + \lambda \partial_\mu \phi \partial^\mu \psi].$$

We diagonalize the action by the orthogonal transformation

$$S = \frac{1}{2}\int d^d x[A_+(\partial_\mu \phi')^2 + A_-(\partial_\mu \psi')^2],$$

where $A_\pm = 1 \pm \frac{\lambda}{2}$, $\begin{pmatrix}\phi' \\ \psi'\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 \\ -1 & 1\end{pmatrix}\begin{pmatrix}\phi \\ \psi\end{pmatrix}$

$$H = \frac{1}{2}\int d^{d-1}x\,\left[A_+^{-1}\left(\pi_{\phi'}^2 + A_+^2(\nabla\phi')^2\right) + A_-^{-1}\left(\pi_{\psi'}^2 + A_-^2(\nabla\psi')^2\right)\right],$$

$$= \frac{1}{2}\int dx^{d-1}[\frac{1}{A_+A_-}(\pi_\phi^2 + \pi_\psi^2 - \lambda\pi_\phi\pi_\psi) + (\nabla\phi)^2 + (\nabla\psi)^2 + \lambda\nabla\phi\nabla\psi]$$

$$-2 \leq \lambda \leq 2.$$

(2) The massive interaction model

The action is given by

$$S = \frac{1}{2}\int d^d x \left[(\partial_\mu \phi)^2 + (\partial_\mu \psi)^2 - (\phi, \psi)\begin{pmatrix} A & C \\ C & B \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix}\right],$$

We diagonalize the action by the orthogonal transformation

$$S = \frac{1}{2}\int d^d x \, [(\partial_\mu \phi')^2 + (\partial_\mu \psi')^2 - m_1^2 \phi'^2 - m_2^2 \psi'^2],$$

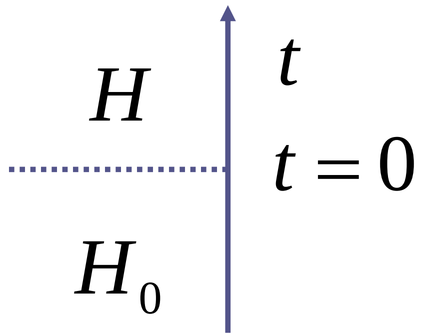
where $\begin{pmatrix} \phi' \\ \psi' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix}.$

The three parameters (A,B,C) are equivalently expressed in terms of (θ, m_1, m_2)

We will trace out ϕ and consider the EE of ψ in the following two cases:

First, we consider the EE of the ground state for the total Hamiltonian.

Next, we consider the time evolution of the EE generated by the total Hamiltonian.
We choose the initial state to be the ground state for the free Hamiltonian which is the Hamiltonian of free massless fields, i.e. we prepare the ground state for the free Hamiltonian and switch on the interaction at $t = 0$.



3.How to compute EE

We present two different but equivalent methods of calculations:

(1) A real time formalism based on wave functionals.

It is useful to obtain exact results.

Today's talk

(2) An Euclidean replica formalism using boundary states.

It is useful to obtain perturbative expansions systematically.
It will be applicable to interacting models which is not solvable.

A real time formalism based on wave functionals

We calculate EE directly from the wave functional.

Because the Hamiltonians of our models are quadratic, the wave functions of the ground states and of time-evolving states whose initial states are the ground states for the free Hamiltonian are Gaussian wave functions.

We consider generally the EE for the Gaussian wave function.

We consider the following Gaussian wave function,

$$\langle\{\phi,\psi\}|\Psi\rangle = N \exp\left\{ -\frac{1}{2}\int d^{d-1}x d^{d-1}y \Big[\phi(x)G_1(x,y)\phi(y) + \psi(x)G_2(x,y)\psi(y) + 2\phi(x)G_3(x,y)\psi(y)\Big]\right\}.$$

$G_i(x,y)$ are the complex valued functions.

$$\langle\{\phi,\psi\}|\Psi\rangle = N\exp\left\{-\frac{1}{2}\int d^{d-1}x d^{d-1}y\Big[\phi(x)G_1(x,y)\phi(y)+\psi(x)G_2(x,y)\psi(y) + 2\phi(x)G_3(x,y)\psi(y)\Big]\right\}.$$

We trace out ϕ and obtain

$$\mathrm{Tr}\rho_\psi^n = N'^n\int D\psi_1\cdots D\psi_n \exp\left[-\int d^{d-1}x d^{d-1}y(\psi_1(x),\cdots,\psi_n(x))M_n(x,y)\begin{pmatrix}\psi_1(y)\\ \vdots\\ \psi_n(y)\end{pmatrix}\right]$$

where

$$M_n = \begin{pmatrix} \mathrm{Re}X & Y & 0 & \cdots & 0 & Y\\ Y & \mathrm{Re}X & Y & \cdots & 0 & 0\\ 0 & Y & \mathrm{Re}X & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \mathrm{Re}X & Y\\ Y & 0 & 0 & \cdots & Y & \mathrm{Re}X\end{pmatrix}.$$

$$X = G_2 - G_3(G_1+G_1^*)^{-1}G_3, \qquad Y = \frac{-1}{4}\left[G_3(G_1+G_1^*)^{-1}G_3^* + G_3^*(G_1+G_1^*)^{-1}G_3\right]$$

We can diagonalize M_n

$$\mathrm{Tr}\rho_\psi^n = \prod_i \frac{(1-\xi_i)^n}{(1-\xi_i^n)}. \qquad \text{where} \qquad \xi_i = \frac{1}{z_i}\left[1-\sqrt{1-z_i^2}\right]$$

z_i is the eigenvalues of Z.

$$Z = -2\,(\mathrm{Re}X)^{-1}Y = -2\left[Y^{-1}\mathrm{Re}G_2 + 2 + Y^{-1}\mathrm{Im}G_3(\mathrm{Re}G_1)^{-1}\mathrm{Im}G_3\right]^{-1}$$

$$S_\psi^{(n)} \;=\; (1\,-\,n)^{-1}\ln\mathrm{Tr}\rho_\psi^n \qquad S_\psi = -\mathrm{Tr}\rho_\psi\ln\rho_\psi = -\frac{\partial}{\partial n}\ln\mathrm{Tr}\rho_\psi^n|_{n=1}$$

$$S_\psi^{(n)} \equiv \sum_i s^{(n)}(\xi_i) = \sum_i (1-n)^{-1}\left[n\ln(1-\xi_i) - \ln(1-\xi_i^n)\right],$$

$$S_\psi \equiv \sum_i s(\xi_i) = \sum_i \left[-\ln(1-\xi_i) - \frac{\xi_i}{1-\xi_i}\ln\xi_i\right]$$

In our model, the mode sum becomes the momentum sum.

4.EE of ground states and time evolution

The massless interaction case $S = \frac{1}{2}\int d^d x[(\partial_\mu\phi)^2 + (\partial_\mu\psi)^2 + \lambda\partial_\mu\phi\partial^\mu\psi].$

EE for the ground state.

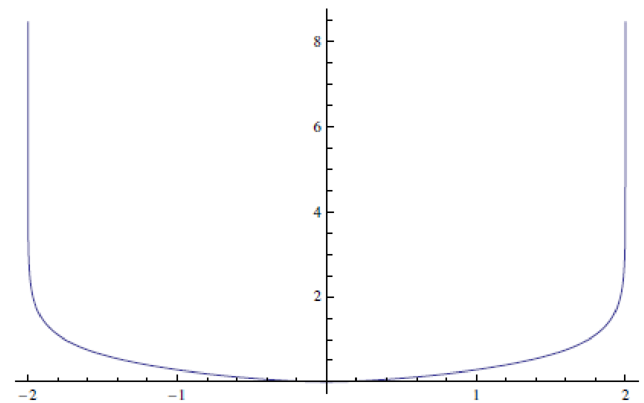
$$S_1^{(n)} = s_1^{(n)}(\lambda)\cdot\sum_{k\neq 0} 1, \qquad S_1 = s_1(\lambda)\cdot\sum_{k\neq 0} 1,$$

where

$$s_1^{(n)} = (1-n)^{-1}[n\ln(1-\xi) - \ln(1-\xi^n)], \qquad s_1 = -\ln(1-\xi) - \frac{\xi}{1-\xi}\ln\xi.$$

$$\xi = \frac{1}{z}\left[1-\sqrt{1-z^2}\right] \qquad z = \frac{\lambda^2}{8-\lambda^2}.$$

$s_1(\lambda)$



λ

In this case, each mode contributes identically the entropies.
In this way, EE between two scalar field theories has the volume law divergence.

The volume law divergence can be seen clearly when the volume of space is finite.

We regularize the UV divergent mode sum by the smooth momentum cutoff.

When the space is a (d-1) torus of size L,

$$S_1^{(n)} = s_1^{(n)}(\lambda)\left(c_{d,d-1}\left(\frac{L}{\epsilon}\right)^{d-1} + c_{d,d-2}\left(\frac{L}{\epsilon}\right)^{d-2} + \cdots + c_{d,0}\right),$$

$$S_1 = s_1(\lambda)\left(c_{d,d-1}\left(\frac{L}{\epsilon}\right)^{d-1} + c_{d,d-2}\left(\frac{L}{\epsilon}\right)^{d-2} + \cdots + c_{d,0}\right),$$

where ε is a UV cutoff length.
c are constants.

The massless interaction case

Time evolution

For $|\lambda| << 1$

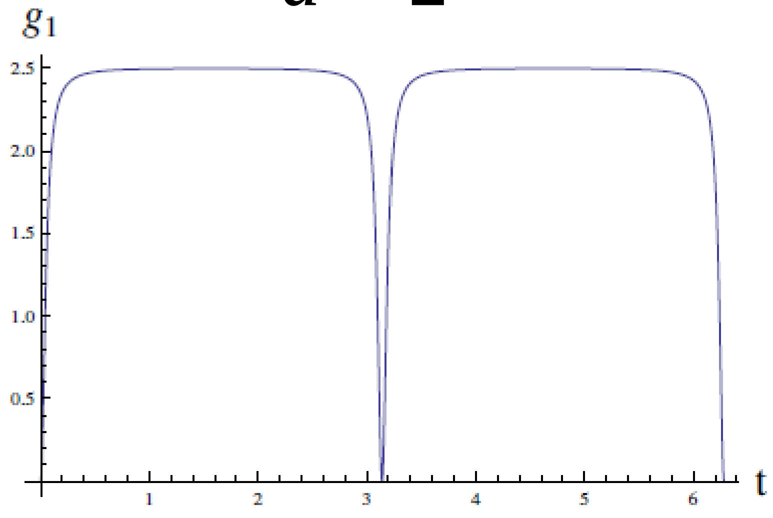
$$S_{\psi}^{(n)} \simeq -\frac{n}{1-n}\frac{\lambda^2}{4}\sum_{k\neq 0}\sin^2(wt) \quad (n>1), \qquad S_{\psi} \simeq -\frac{\lambda^2}{4}\ln\lambda^2\sum_{k\neq 0}\sin^2(wt).$$

$d=2 \qquad x \sim x+2\pi$

EE has the periodicity $t \sim t+\pi$

When ε is infinitesimally small, EE almost instantaneously reaches the maximum within the time of order ε

$d=2$



A plot of $g_1 = \frac{n-1}{n\lambda^2}S_{\text{ent}}^{(n)}$ as a function of t

We chose $\epsilon = 0.1$

The massive interaction case

EE for the ground state.

$$\frac{S_3}{(2\pi)^{1-d}V} \equiv \int d^{d-1}\mathbf{k}\, s_3(k) = \int d^{d-1}\mathbf{k} \left[-\ln(1-\xi(k)) - \frac{\xi(k)}{1-\xi(k)} \ln \xi(k) \right],$$

EE has UV divergence for $d \geq 5$ and the leading UV divergent term is

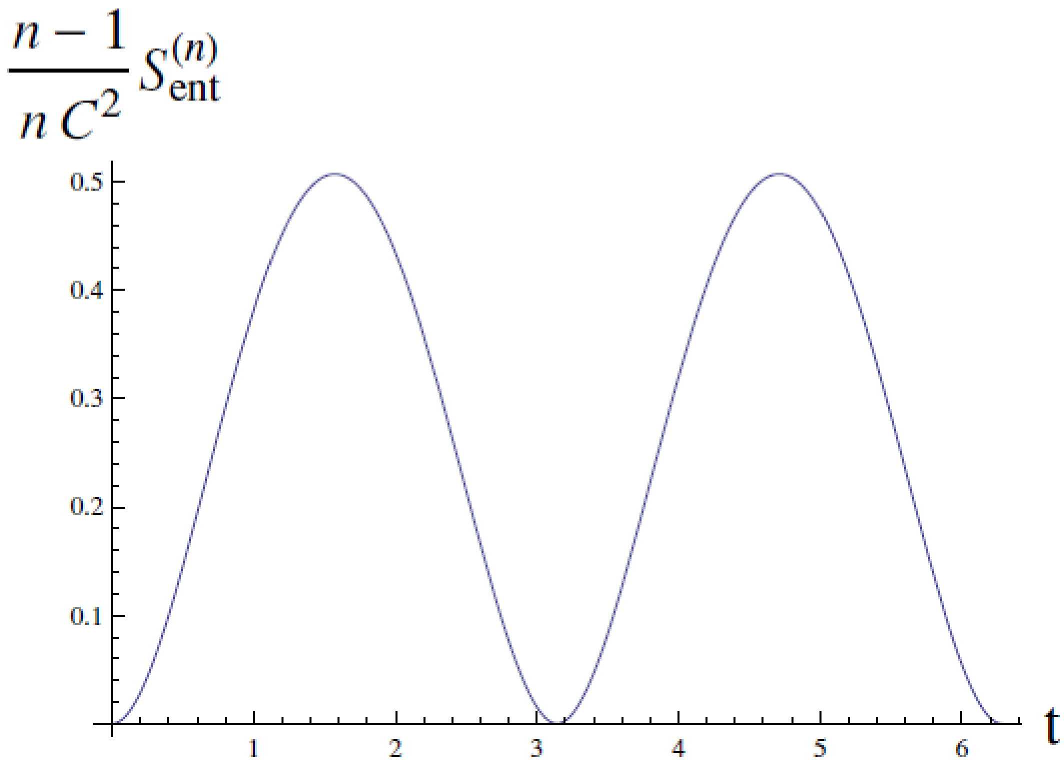
$$\begin{aligned} \frac{S_3}{(2\pi)^{1-d}V} &= \frac{1}{4}\sin^2\theta\cos^2\theta(m_1^2-m_2^2)^2 \int d\Omega_{d-1} \int^{\Lambda} dk k^{d-6} \ln k \\ &= \frac{1}{4}\sin^2\theta\cos^2\theta(m_1^2-m_2^2)^2 \int d\Omega_{d-1} \times \begin{cases} \frac{1}{2}(\ln\Lambda)^2 & \text{for} \quad d=5 \\ \frac{1}{d-5}\Lambda^{d-5}\ln\Lambda & \text{for} \quad d \geq 6. \end{cases} \end{aligned}$$

For $d \leq 4$ EE is UV finite. This is because the massive interaction is negligible in the high energy region.

The massive interaction case

Time evolution

For $d = 2$ $\quad x \sim x + 2\pi$



EE grows slowly in the massive interaction case.
ε

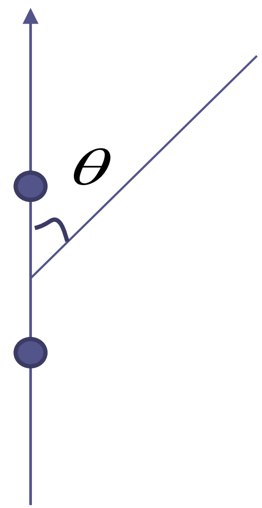
5.Generalized holographic entanglement entropy

First, we consider the case where N/2 D3 branes are placed at $\vec{y} = (-l, 0, 0, 0, 0, 0)$ and the other N/2 D3-branes at $\vec{y} = (l, 0, 0, 0, 0, 0)$

$$ds^2 = H^{-1/2}(\vec{y})dx^{\mu}dx_{\mu} + H^{1/2}(\vec{y})dy^i dy^i,$$

where $\mu = 0, 1, 2, 3$ and $i = 1, 2, \cdots, 6.$

$$H = \frac{R^4}{2(r^2 + l^2 - 2rl\cos\theta)^2} + \frac{R^4}{2(r^2 + l^2 + 2rl\cos\theta)^2}.$$



The dual gauge theory has the unbroken gauge group $SU(N/2)\times SU(N/2)$ and fields which are bi-fundamental with respect to each of the two SU(N/2) groups are massive with the mass

$$M = \frac{l}{\pi\alpha'}$$

If we focus on the physics below this mass scale M, we can regard that the system is described by two SU(N/2) gauge theories (CFTs) interacting each other. We call them CFT1 and CFT2.

We would like to argue that the entanglement entropy between these two CFTs are given by the holographic entanglement formula

$$S_{\mathrm{ent}} = \frac{\mathrm{Area}(\gamma)}{4G_N},$$

by choosing γ to be the area of minimal surface which separates the two groups of N/2 D3-branes.

From the symmetrical reason, it is clear that γ is given by

$$\gamma : t = 0, \quad \theta = \frac{\pi}{2}.$$

$$S_{\mathrm{ent}} \simeq \frac{4}{15\pi^2} \cdot \frac{N^2 V_3 r_{\mathrm{UV}}^5}{R^6 l^2} = \frac{16 N^2 V_3}{15\pi^3} \lambda g \Lambda^3,$$

where V3 is the volume in the (x1,x2,x3) direction and r_{UV} is the cut off in the radial direction.

$$r_{\mathrm{UV}} = \Lambda R^2. \qquad\qquad r_{\mathrm{UV}} \ll l.$$

g is the dimensionless coupling which parametrize the strength of interactions between CFT1 and CFT2.

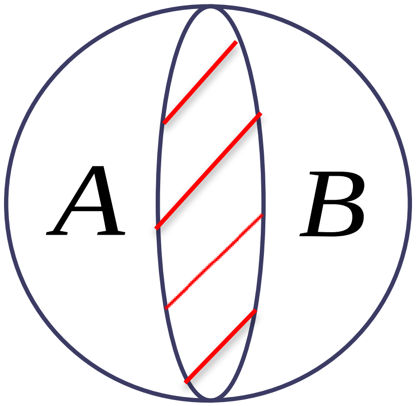
$$g = \frac{\Lambda^2}{M^2},$$

Our holographic analysis reproduced the volume law which we found in the scalar field calculations for the massless interaction.

A proposal for generalized holographic entanglement entropy

We divide the internal space S^5 into A and B.

Our main assumption is that this separation corresponds to a factorization of Hilbert space in the dual CFT.



$$\mathcal{H}_{\rm CFT} = \mathcal{H}_A \otimes \mathcal{H}_B.$$

EE between two CFTs is given by the holographic formula

$$S_{\rm ent} = \frac{{\rm Area}(\gamma)}{4G_N},$$

γ chosen to be the minimal surface whose boundary coincides with that of A i.e. $\partial A = \partial\gamma$

We studied coincident D3-branes i.e. $AdS_5 \times S^5$ and reproduced the volume law.

6.Conclusion

- We analytically computed the entanglement entropy between two free scalar field theories in d spacetime dimensions, which are interacting each other in several ways. We considered massless and massive interactions and compute EE for the ground states and time evolution.

In the massless interaction case

$$S_{ent} \propto V_{d-1}\Lambda^{d-1}$$

In the massive interaction case

For $d \le 4$, EE is UV finite For $d \ge 5$, $S_{ent} \propto V_{d-1}\Lambda^{d-5} \log \Lambda$

- We proposed a holographic calculation of entanglement entropy between the two interacting CFTs. We divide the internal space into subregions. We reproduce the volume law of EE which is same as the massless interaction.

$$\mathcal{H}_{\mathrm{CFT}} = \mathcal{H}_A \otimes \mathcal{H}_B .$$

How about the off diagonal elements?

$$\begin{array}{r c} \begin{array}{r} \frac{N}{2}\Big\{ \\ \frac{N}{2}\Big\{ \end{array} & \begin{bmatrix} \Phi_{AA} & \Phi_{AB} \\ \Phi_{BA} & \Phi_{BB} \end{bmatrix} \\ & \begin{array}{cc} \underbrace{\qquad}_{\frac{N}{2}} & \underbrace{\qquad}_{\frac{N}{2}} \end{array} \end{array}$$

This problem is similar to the EE in the gauge theory.