Instantons in Lifshitz Field Theories

Toshiaki Fujimori

arXiv:1507.06456 (JHEP 1510 (2015) 021)

Toshiaki Fujimori, Muneto Nitta (Keio Univ.)

Developments in String Theory and Quantum Field Theory

@ YITP, Nov. 12th, 2015

1. Introduction

Instanton : non-perturbative effects

4d gauge theory

- Yang-Mills instanton

2d non-linear σ model

- sigma model instanton

common properties

instanton
asymptotic freedom
generation of mass gap
etc

- 2d NL σ M is a toy model of 4d gauge theory

any other models?

Lifshitz-type field theories

- Lifshitz scaling (anisotropic scaling)

$$t \to \lambda^z t, \qquad x \to \lambda x$$

- kinetic term in Lifshitz type theories **: no Lorentz symmetry**

$$S = \int dt d^d x \left[-\phi \left(\partial_t^2 - \Delta^z \right) \phi + \cdots \right]$$

- improved UV behavior $z > 1$ propagator $\sim \dfrac{1}{E^2 - p^{2z}}$

weighted power counting →

- renormalizability and unitarity
- Derrick's argument··· new instanton

Examples of Lifshitz field theories

- Lifshitz sigma model $\mathbf{CP}^N$ [Das-Murthy, 2009]

  $z = d$ $O(N)$ [Anagnostopoulos et al, 2010]

  - asymptotic freedom
  - generation of mass gap (large N)

- Lifshitz-Yang-Mills theory [Horava, 2008]

  [Kanazawa-Yamamoto, 2014] lattice

  - asymptotic freedom

- Horava-Lifshitz gravity [Horava, 2009]

  - renormalizable gravity theory

Examples of Lifshitz field theories

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  - asymptotic freedom

- this talk

instantons in Lifshitz-type sigma model and gauge theory

2. Instantons in Lifshitz-type sigma models

supersymmetry in Lifshitz-type models

- SUSY algebra ・・・ (SUSY)^2 = time translation

in 2+1-dimensions

$$\{Q, \bar{Q}\} = 2i\partial_t$$

- supercharges : **Weyl spinors under SO(2) spatial rotation**

$$Q \to e^{i\frac{\theta}{2}} Q, \qquad \bar{Q} \to e^{-i\frac{\theta}{2}} \bar{Q},$$

2. Instantons in Lifshitz-type sigma models

supersymmetry in Lifshitz-type models

- SUSY algebra ・・・ (SUSY)^2 = time translation

in 2+1-dimensions

$$\{Q, \bar{Q}\} = 2i\partial_t \quad (\text{ + other bosonic symmetry})$$

- supercharges : **Weyl spinors under SO(2) spatial rotation**

$$Q \to e^{i\frac{\theta}{2}} Q, \qquad \bar{Q} \to e^{-i\frac{\theta}{2}} \bar{Q},$$

superfield formalism

- supercharges

$$Q = \frac{\partial}{\partial\theta} + i\bar{\theta}\partial_t, \qquad \bar{Q} = \frac{\partial}{\partial\bar{\theta}} + i\theta\partial_t$$

$$D = \frac{\partial}{\partial\theta} - i\bar{\theta}\partial_t, \qquad \bar{D} = \frac{\partial}{\partial\bar{\theta}} - i\theta\partial_t$$

only time derivative

- algebra

$$\{Q,\bar{Q}\} = -\{D,\bar{D}\} = 2i\partial_t \qquad 0 = \{Q,D\},\cdots$$

supermultiplets

- real superfield $\bar{\Phi} = \Phi$

$$\Phi = \phi + \theta\psi + \bar{\psi}\bar{\theta} + \theta\bar{\theta}F$$

real scalar + Dirac fermion + real auxiliary field

- chiral multiplet $\bar{D}\Phi = 0$

$$\Phi = \phi + \theta\psi - i\theta\bar{\theta}\partial_t\phi$$

- Fermi multiplet, vector multiplet ····

action for real multiplets

$$\mathcal{L} = \int d\theta d\bar{\theta} \left[\frac{1}{2} G_{ab} \underline{D\Phi^a \bar{D}\Phi^b} + W(\Phi, \underline{\partial_i \Phi, \cdots}) \right]$$

time derivative

spatial derivative

action for real multiplets

$$\mathcal{L} = \int d\theta d\bar{\theta} \left[\frac{1}{2} G_{ab} \underline{D\Phi^a \bar{D}\Phi^b} + W(\Phi, \underline{\partial_i \Phi, \cdots}) \right]$$

time derivative — spatial derivative

⬇ integrating out F

$$\mathcal{L} = g_{ab} \left[\frac{1}{2} \partial_t \phi^a \partial_t \phi^b + i\psi^a \mathcal{D}_t \bar{\psi}^b \right] - R_{abcd} \psi^a \psi^b \bar{\psi}^c \bar{\psi}^d$$

$$+ \underline{\left(\frac{\partial W}{\partial \phi^a} - \frac{\partial W}{\partial \partial_i \phi^a} + \cdots \right)}^2 \quad + \text{fermionic terms}$$

spatial derivative

bosonic part of action

$$S = \int dt\, d^2x \left(G_{ab}\partial_t\phi^a \partial_t\phi^b + G^{ab}\frac{\delta W}{\delta\phi^a}\frac{\delta W}{\delta\phi^b} \right)$$

- "superpotential" ・・・ functional on each time slice

$$W(t) = \int d^2x\, \mathcal{W}(\phi, \partial_i\phi, \cdots)$$

[P. Horava, "Membranes at Quantum Criticality," (2009)]

"detailed balance condition"

bosonic part of action

$$S = \int dt\, d^2x \left(G_{ab}\partial_t\phi^a\partial_t\phi^b + G^{ab}\frac{\delta W}{\delta\phi^a}\frac{\delta W}{\delta\phi^b} \right)$$

- "superpotential" ・・・ functional on each time slice

$$W(t) = \int d^2x\, \mathcal{W}(\phi, \partial_i\phi, \cdots)$$

[P. Horava, "Membranes at Quantum Criticality," (2009)]

- supersymmetric ground state $Q|0\rangle = \bar{Q}|0\rangle = 0$

$$\langle\phi|0\rangle \propto e^{-W[\phi]} \rightarrow \frac{\langle 0|\hat{\mathcal{O}}|0\rangle}{\langle 0|0\rangle} = \frac{\int \mathcal{D}\phi\, \mathcal{O} e^{-2W}}{\int \mathcal{D}\phi\, e^{-2W}}$$

2d theory

BPS bound and BPS equation

- Bogomol'nyi completion

$$S = \int dt d^2x \left[\frac{1}{2} G_{ab} \left(\partial_t \phi^a + G^{ac} \frac{\delta W}{\delta \phi^c} \right) \left(\partial_t \phi^b + G^{bd} \frac{\delta W}{\delta \phi^d} \right) - \partial_t W \right]$$

- BPS bound $\quad S \geq W(t = -\infty) - W(t = \infty)$

$$\boxed{\partial_t \phi^a = -G^{ab} \frac{\delta W}{\delta \phi^b}} \quad \Longleftarrow \quad Q\psi^a = 0$$

BPS eq. = gradient flow for W

an explicit example of W

- target space = simple Lie group G

$$\pi_3(G) = \mathbb{Z}$$

instanton number

- "superpotential" W : action of 2d chiral model

$$W(t) = \alpha \int d^2x \, \mathrm{Tr}\Big[(iU^\dagger \partial_i U)(iU^\dagger \partial_i U)\Big]$$

W for 3d Lifshitz sigma model

- target space = simple Lie group G $\pi_3(G) = \mathbb{Z}$

instanton number

- "superpotential" W : action of 2d chiral model

$$W(t) = \alpha \int d^2x \, \mathrm{Tr}\left[(iU^\dagger \partial_i U)(iU^\dagger \partial_i U)\right]$$

$$+\xi \int d^2x \int_{-\infty}^{t} dt \, \epsilon_{IJK} \mathrm{Tr}\left[(U^\dagger \partial_I U)(U^\dagger \partial_J U)(U^\dagger \partial_K U)\right]$$

Wess-Zumino-Witten term

bosonic part ca supersymmetric Lifshitz model

$$L = \mathrm{Tr}\left\{ \underbrace{(iU^\dagger \partial_t U)^2}_{\mathcal{O}(\partial_t^2)} + \underbrace{\left[\alpha \partial_i (iU^\dagger \partial_i U) + i\xi (iU^\dagger \partial_{[i} U)(iU^\dagger \partial_{j]} U) \right]^2}_{\mathcal{O}(\partial_x^4)} \right\}$$

z=2 Lifshitz scaling invariance $t \to \lambda^2 t, \quad x_i \to \lambda x_i$

- BPS bound

$$S \geq \xi \int dt d^2x \, \epsilon^{ijk} \mathrm{Tr} \left[(U^\dagger \partial_i U)(U^\dagger \partial_j U)(U^\dagger \partial_k U) \right] \;\; \propto n\xi$$

$\pi_3(G) = \mathbb{Z}$ **topological charge (Skyrmion)**

BPS equation = gradient flow

$$iU^\dagger \partial_t U = \alpha \partial_i (iU^\dagger \partial_i U) + \xi [U^\dagger \partial_1 U, U^\dagger \partial_2 U]$$

simplified case

- $\alpha = 0$ → $$iU^\dagger \partial_t U = \xi [U^\dagger \partial_1 U, U^\dagger \partial_2 U]$$
- area-preserving diff. $$x_i \to x_i', \quad \det \left(\frac{\partial x_i'}{\partial x_j} \right) = 1$$

BPS equation = gradient flow

$$iU^\dagger \partial_t U = \alpha \partial_i (iU^\dagger \partial_i U) + \xi [U^\dagger \partial_1 U, U^\dagger \partial_2 U]$$

simplified case

- $\alpha = 0$ ➡ $$iU^\dagger \partial_t U = \xi [U^\dagger \partial_1 U, U^\dagger \partial_2 U]$$

- area-preserving diff. $x_i \to x'_i, \quad \det\left(\frac{\partial x'_i}{\partial x_j}\right) = 1$

(• another simplification : $\alpha = \xi$ ➡ $iU^\dagger \partial_t U = \xi \partial_z (iU^\dagger \partial_{\bar{z}} U)^\dagger$)

symmetric ansatz for instanton

- fixed point of $U(1) \subset G_V \times$ spatial rotation

for $G = SU(2)$

$$U = V^\dagger \begin{pmatrix} \phi & -\sqrt{1-|\phi|^2} \\ \sqrt{1-|\phi|^2} & \bar{\phi} \end{pmatrix} V, \qquad V = \exp\left(ik\theta\,\frac{\sigma_3}{2}\right)$$

reduced BPS equation

$$\left[\partial_t + i\partial_\rho + \frac{i}{2}(\phi\,\partial_\rho\bar{\phi} - \bar{\phi}\,\partial_\rho\phi)\right]\phi = 0$$

ϕ : half plane $\to$ disc

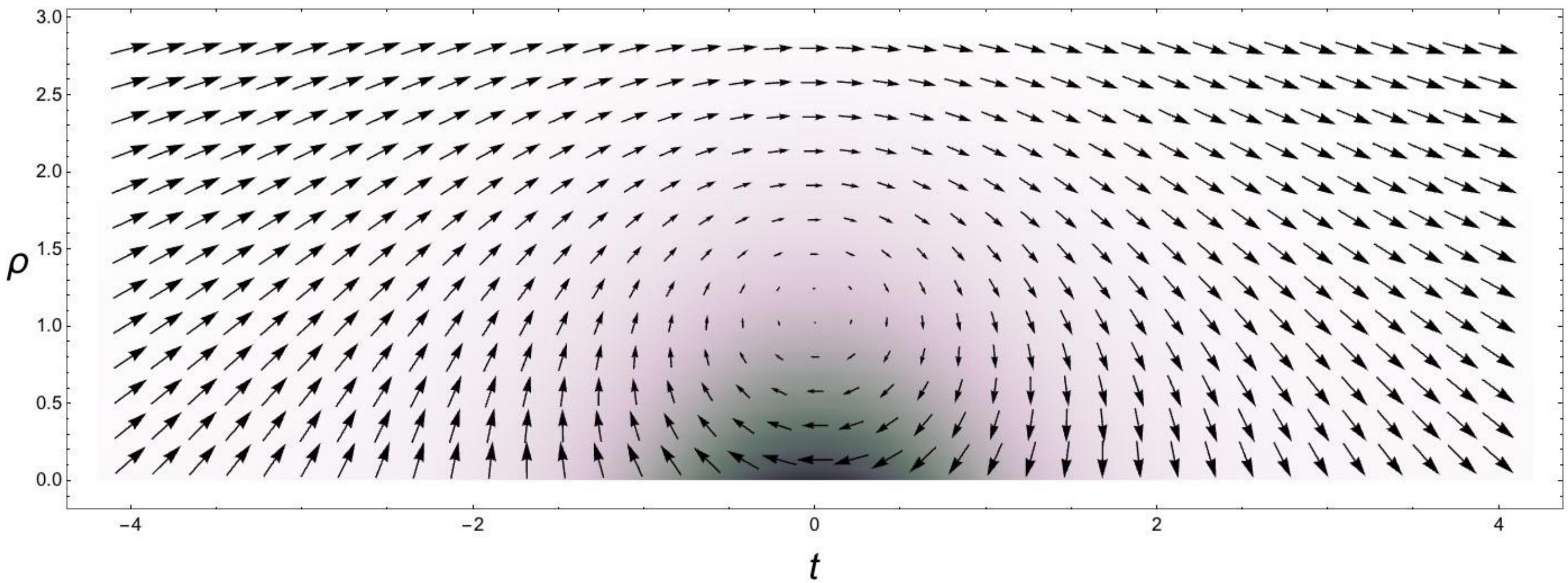
$(t, \rho \equiv x_1^2 + x_2^2)$ $\qquad |\phi| \leq 1$

numerical solution

$$\left[\partial_t + i\partial_\rho + \frac{i}{2}(\phi\,\partial_\rho\bar{\phi} - \bar{\phi}\,\partial_\rho\phi)\right]\phi = 0$$

ϕ : half plane $\to$ disc

$(t, \rho \equiv x_1^2 + x_2^2)$ $\quad |\phi| \leq 1$



- arrows = (Re ϕ , Im ϕ)
- shading = action density

exact solution

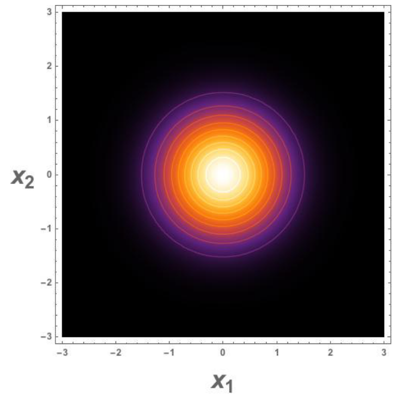
- compactification $\underset{t}{S^1} \times \mathbb{R}^2$

- fixed point of $U(1) \subset G_V \times$ spatial rotation

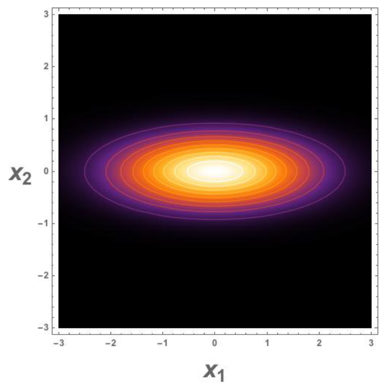
  $U(1) \subset G_A \times$ time translation

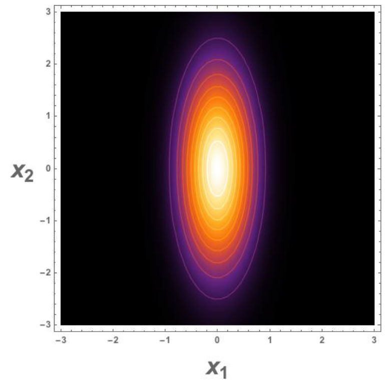
$$\phi = \exp\left(-2\pi n \frac{\rho + it}{\beta}\right)$$

$$S_{sol} = 8\pi^2 n \xi$$



area preserving diff.





3. Instantons in Lifshitz gauge theories

W for gauge theories

- spatial dimension = 2n-1

Chern-Simons term

$$W = \int_{\mathbb{R}^{2n-1}} CS_{2n-1}$$

- topological charge

$$S \geq \int_{\mathbb{R}^{2n}} \mathrm{Tr}\left[F \wedge \cdots \wedge F\right]$$

- n=2 : standard Yang-Mills in 4d spacetime

(5+1)d Lifshitz-Yang-Mills theory

- (5+1)d action **dimensionless : renormalizable**

$$S = \frac{1}{g^2}\int dt d^5x \, \mathrm{Tr}\left[F_{0i}^2 + (\epsilon_{ijklm}F_{jk}F_{lm})^2\right]$$

z=3 Lifshitz scaling

$$t \to \lambda^3 t, \qquad x_i \to \lambda x_i,$$

anisotropic Weyl transf.

$$dt^2 + (dx^i)^2 \to \lambda(t,x)^6 dt^2 + \lambda(t,x)^2 (dx^i)^2$$

(5+1)d Lifshitz-Yang-Mills theory

- BPS equation $$F_{0i} = \frac{1}{4}\epsilon_{ijklm}F_{jk}F_{lm}$$
- topological charge $$S = \frac{1}{g^2}\int_{\mathbb{R}^6} \underbrace{\mathrm{Tr}(F \wedge F \wedge F)}_{dCS_5}$$

symmetric ansatz

- fixed point of $SO(5) \subset$ gauge transf. $\times$ spatial rotation
- anisotropic Weyl transf.

BPS equation ➡ anisotropic vortex eq. on half-plane

symmetric ansatz

- SO(5) symmetric SU(4) instanton in 6d

$(t, \rho \equiv |x_i|^3)$

→ anisotropic U(1) vortex on half-plane

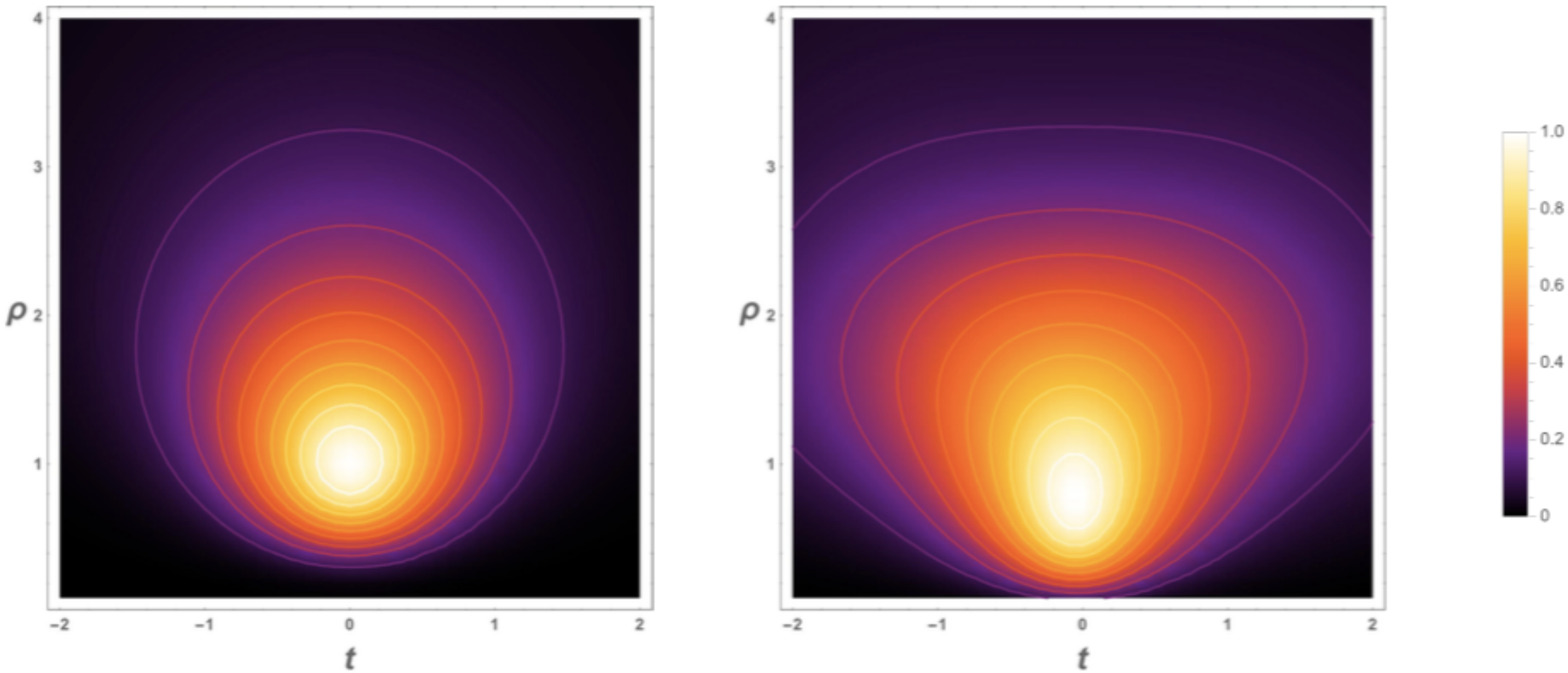
$$F_{t\rho} = -\frac{1}{9\rho^2}(|\phi|^2 - 1)^2 \qquad \mathcal{D}_t\phi = i(|\phi|^2 - 1)\mathcal{D}_\rho\phi$$

- SO(3) symmetric SU(2) instanton in 4d **[Witten, 1977]**

→ integrable U(1) vortex on hyperbolic plane

$$F_{t\rho} = -\frac{1}{\rho^2}(|\phi|^2 - 1) \qquad \mathcal{D}_t\phi = i\mathcal{D}_\rho\phi$$

4d and 6d instantons (action density)



(a) 4d instanton (b) 6d instanton numerical

isotropic **anisotropic**

$SO(2) \in SL(2,\mathbb{R})$: isometry of hyperbolic plane

4. Summary

- Instantons in Lifshitz-type field theories
- (2+1)d, z=2 non-linear sigma model
- (5+1)d, z=3 gauge theory

future works

- beta function, etc
- localization

$$Q|0\rangle = \bar{Q}|0\rangle = 0 \quad \Rightarrow \quad \frac{\langle 0|\hat{\mathcal{O}}|0\rangle}{\langle 0|0\rangle} = \frac{\int \mathcal{D}\phi\, \mathcal{O} e^{-2W}}{\int \mathcal{D}\phi\, e^{-2W}}$$

(d+1)-dim   d-dim

known example: 2d NLSM (non-Lifshtiz)

$$S = \int d^2x \, g_{ab} \partial_\mu \phi^a \partial^\mu \phi^b + \cdots$$

- 2d Kahler NLSM $\quad W = \int_{\mathbb{R}} \frac{i}{2} \left(d\phi^a \partial_a - d\bar{\phi}^a \bar{\partial}_a \right) K$
- gradient flow $\quad (\partial_t + i\partial_x)\phi^i = 0$ **instantons in NLSM**
- action at saddle point $\quad S = \int_{\mathbb{R}^2} \phi^* \omega$ **Kahler form**