M5 branes on 3-manifolds

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Based on Review
(Yamazaki-Terashima, Dimofte-Gukov-Gaiotto,.. : 2011~)
+arXiv :1401.3595, 1409.6206, 1510.03884, 1510.05011
with N. Kim, S. Lee, M. Yamazaki, M. Romo

Introduction

String/M-theory : Solid Theoretical Frame-work

1) Very Rich,

Quantum Gravity

(Blackhole,)

Quantum field theories

(4d Class S,)

Mathematics

(Calabi-Yau,)

Introduction

String/M-theory : Solid Theoretical Frame-work

1) Very Rich, 2) Mysterious Dualities

Quantum Gravity

(Blackhole,)

Quantum field theories

(4d Class S,)

Mathematics

(Calabi-Yau,)

Introduction

String/M-theory : Solid Theoretical Frame-work

1) Very Rich, 2) Mysterious Dualities

Quantum Gravity

(Blackhole, AdS/CFT)

Quantum field theories

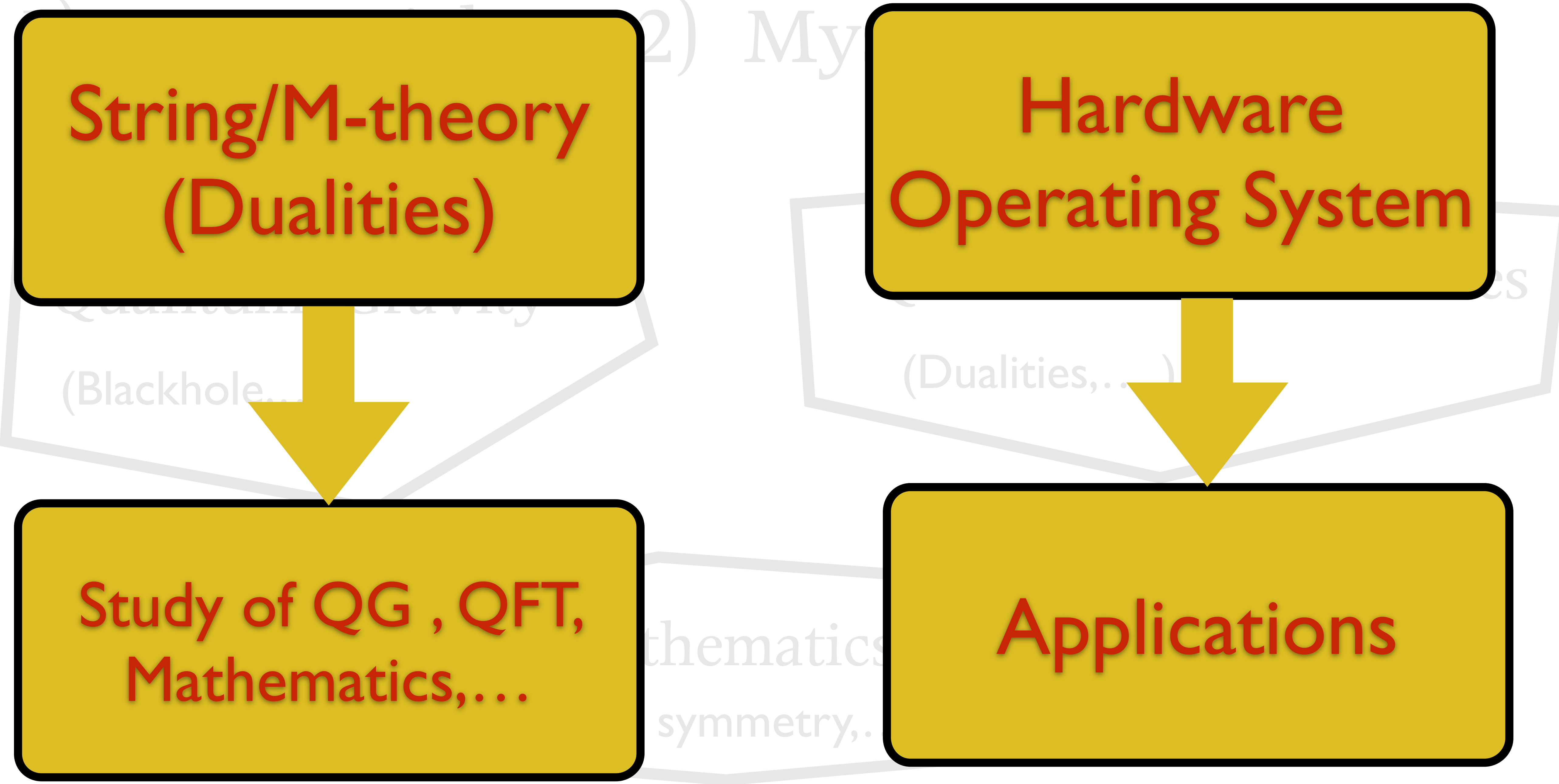
(4d Class S, S-duality)

Mathematics

(Calabi-Yau,Mirror-sym)

Introduction

Analogy with computer science



Introduction

M5s on 3-manifolds $\subset$ String/M-theory

1) Still Rich, 2) Still non-trivial Dualities

Quantum Gravity

(M-theory on AdS4)

Quantum field theories

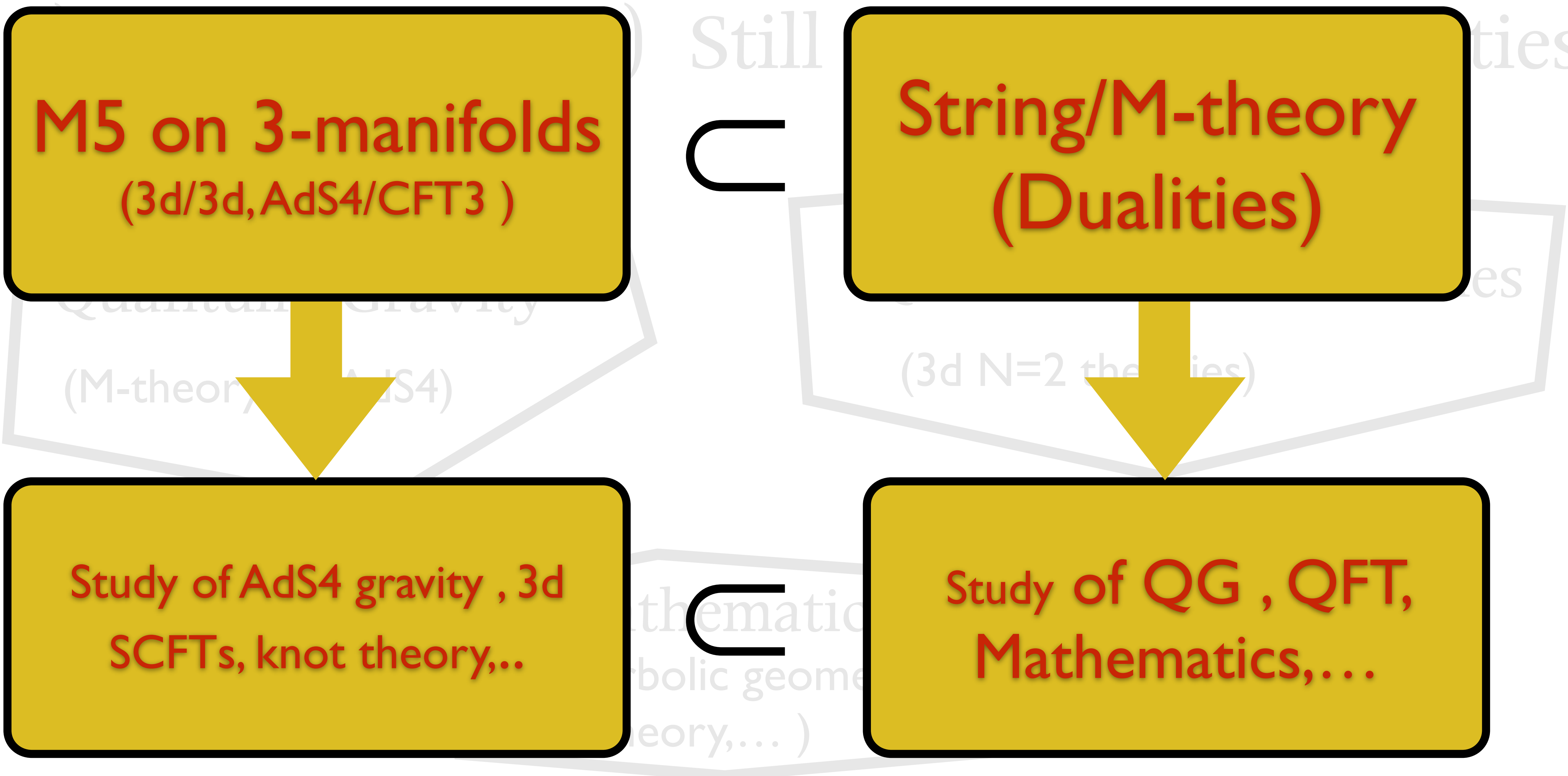
(3d N=2 theories)

Mathematics

(Knot theory)

Introduction

M5s on 3-manifolds $\subset$ String/M-theory



Introduction

$M5s$ on 3-manifolds $\subset$ String/M-theory

M5 on 3-manifolds

(3d/3d, AdS4/CFT3)

$\subset$

String/M-theory

(Dualities)

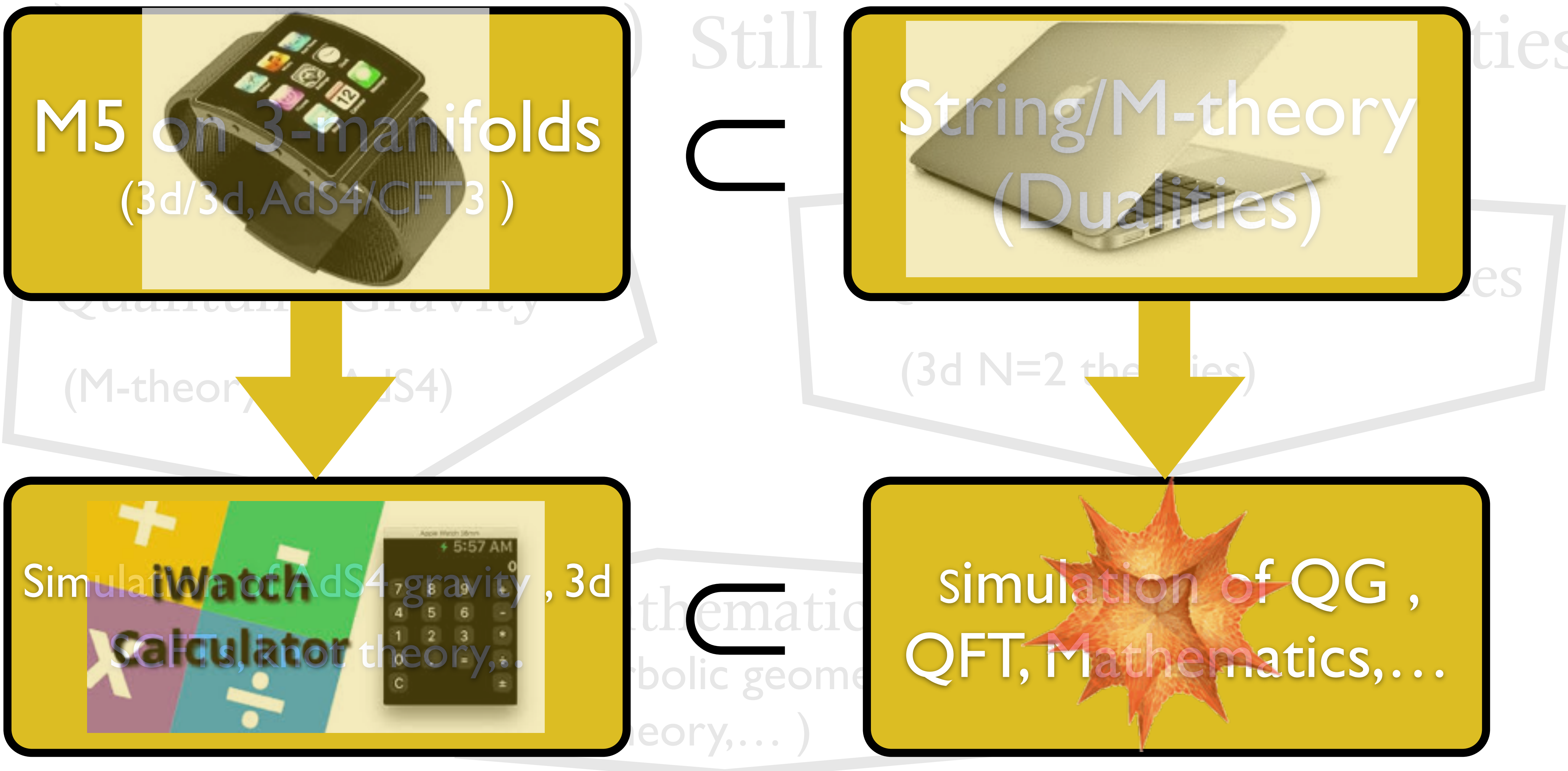
Study of AdS4 gravity , 3d SCFTs, knot theory,..

$\subset$

Study of QG , QFT, Mathematics,. . .

Introduction

M5s on 3-manifolds ⊂ String/M-theory



Introduction

M5s on 3-manifolds $\subset$ String/M-theory

Goal :
Understand the hardware/OS
of iWatch

Outline

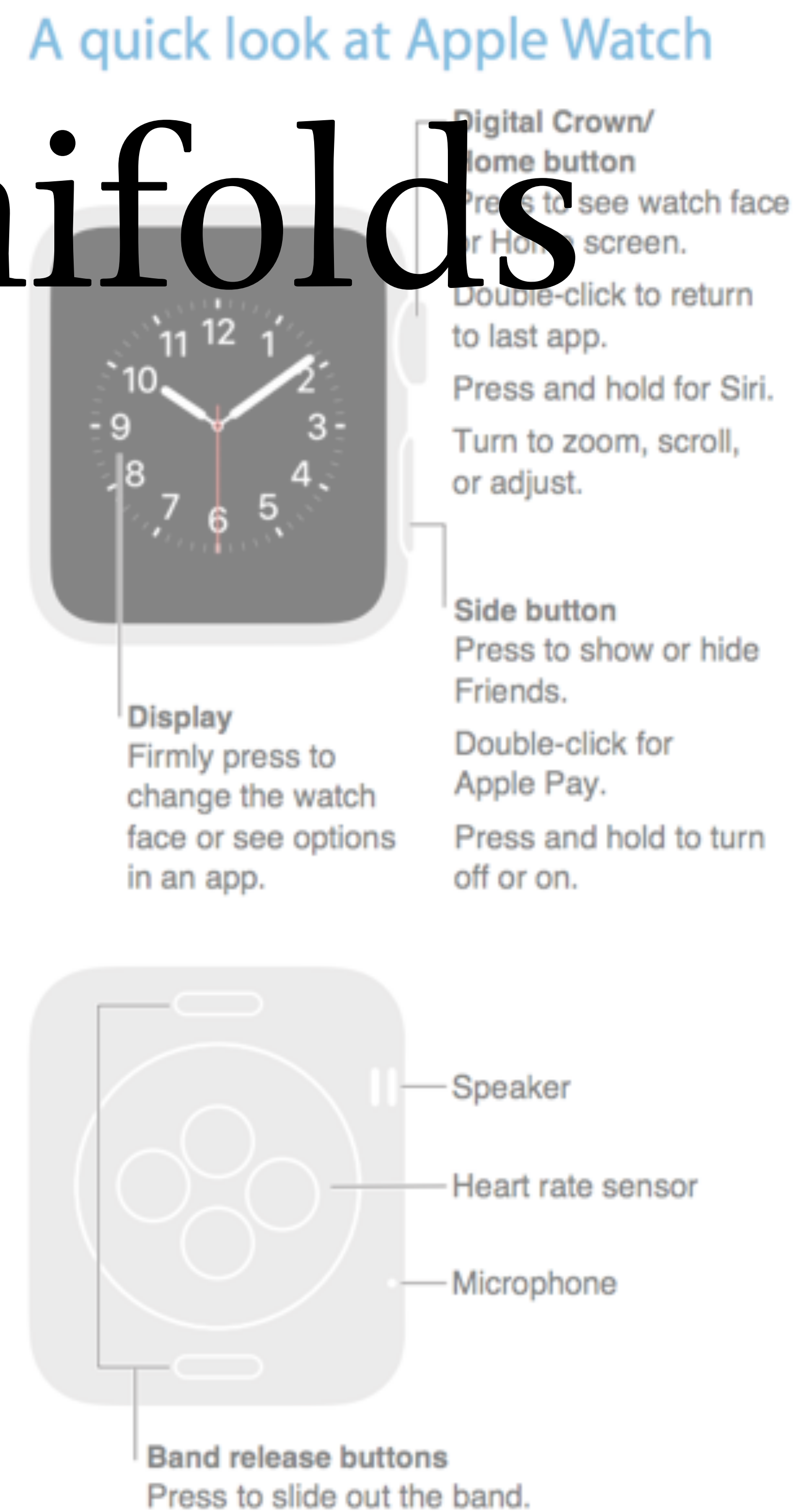
*. M5s on 3-manifolds
   (Manual of iWatch)

*. Computation tools
   (Programing Languages)

*. Summary and discussion

M5s on 3-manifolds

- Basic Set-up
- Dualities



M5s on 3-manifolds

11d $\mathbb{R}^{1,2} \times T^* M \times \mathbb{R}^2$

N $M5s$ $\mathbb{R}^{1,2} \times M$ (M : Closed 3-manifold)

IR world-volume theory : $T_N[M]$

M5s on 3-manifolds

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N $M5s$ $\mathbb{R}^{1,2} \times M$ (M : Closed 3-manifold)

IR world-volume theory : $T_N[M]$

Or, equivalently

6d A_{N-1} (2,0) theory on $\mathbb{R}^{1,2} \times M$

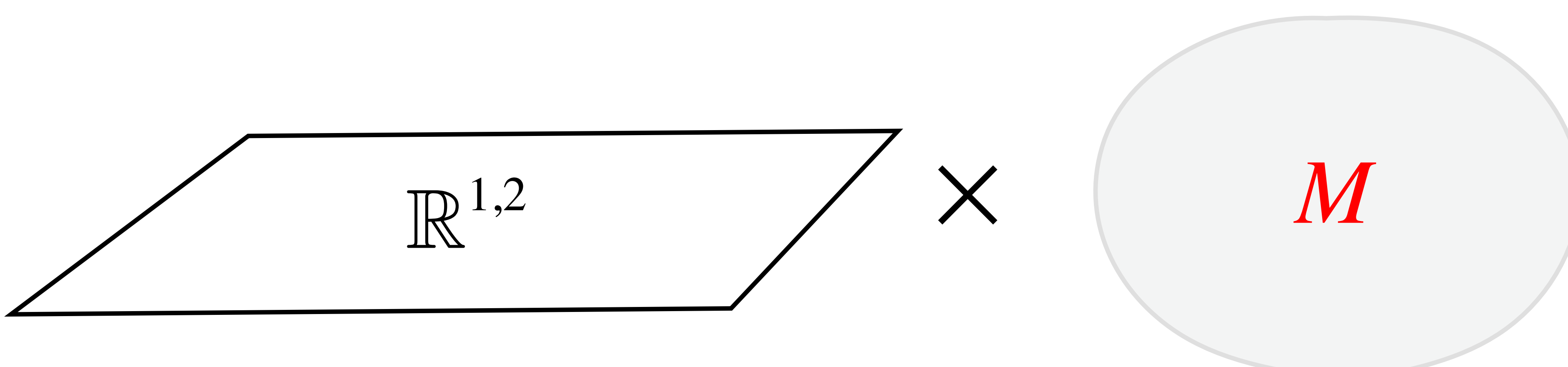
$\longrightarrow$ $T_N[M]$

Top'l twisting : $A^{SO(3)_R} = \omega_M$

$SO(3)_M \times SO(3)_R \supset SO(3)_{\text{diag}}$

Q : 2 2 $\underline{1} \oplus 3$

$\Rightarrow 1/4\ BPS \Rightarrow 3d\ \mathcal{N} = 2$



Codimension 4 defects in the 6d theory

	N M5	:	0	1	2	3	4	5				7	8	9	#
Defect branes	M5	:	0			3						7	8	9	#
	M2	:	0			3			6						

6d A_{N-1} (2,0) theory

+ codimension 4 defect

Codimension 4 defects in the 6d theory

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6d A_{N-1} (2,0) theory

+ codimension 4 defect

↓ S^1 -reduction

			0	1	2	3	4	5	6	7	8	9	#
	N D4	:	0	1	2		4	5					
Defect branes	D4	:	0							7	8	9	#
	F1	:	0						6				

5d $\mathcal{N}=2$ SU(N) SYM

+ Wilson line

$$W_R = \mathrm{Tr}_R(P e^{\int A+i\phi})$$

Codimension 4 defects in the 6d theory

	Brane	0	1	2	3	4	5	6	7	8	9	#
N	M5	0	1	2	3	4	5					
Defect branes	M5	0			3				7	8	9	#
	M2	0			3			6				

6d A_{N-1} (2,0) theory

+ codimension 4 defect

Labelled by R

R : unitary rep of SU(N)

↓ S^1 -reduction

	Brane	0	1	2	3	4	5	6	7	8	9	#
N	D4	0	1	2		4	5					
Defect branes	D4	0							7	8	9	#
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Codimension 2 defects in the 6d theory

N	M5	:	0 1 2 3 4 5
Defect branes	M5	:	0 1 2 3 7 8 9 #

6d A_{N-1} (2,0) theory
+ codimension 2 defect

Codimension 2 defects in the 6d theory

N	M5	:	0 1 2 3 4 5
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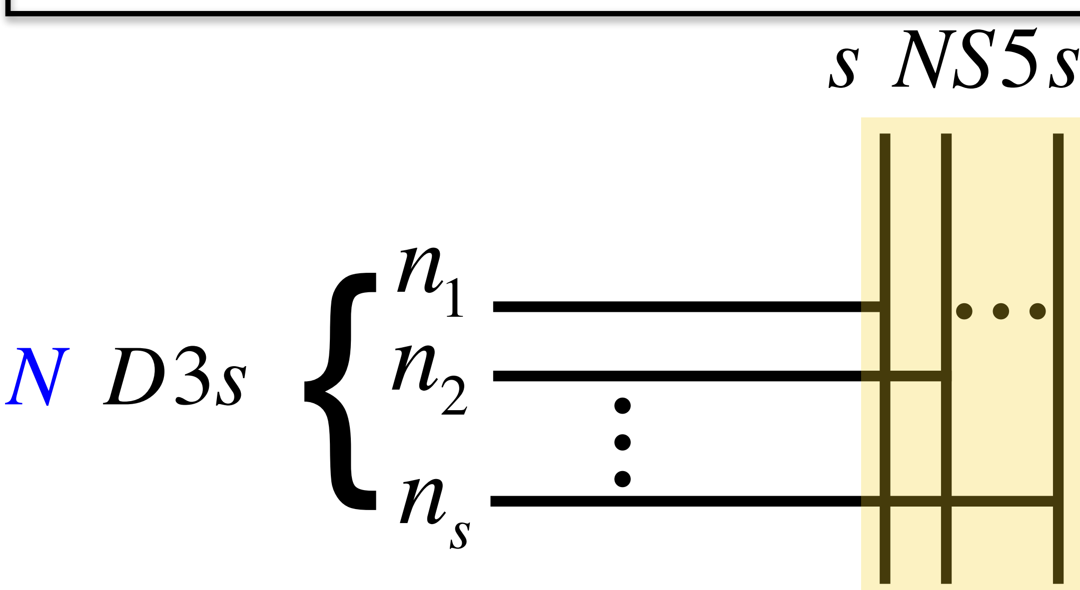
6d A_{N-1} (2,0) theory
+ codimension 2 defect

$\downarrow$ S^1 -reduction

N	D4	:	0 1 2 4 5
Defect branes	D4	:	0 1 2 7 8 9 #

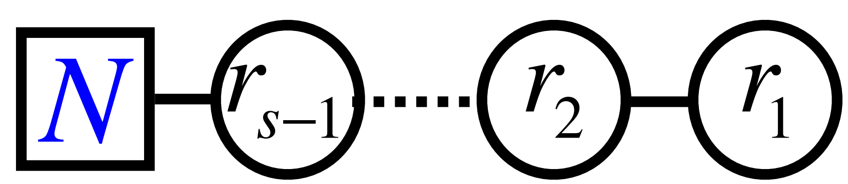
5d $\mathcal{N}$=2 SU(N) SYM
+ 3d T_ρ[SU(N)] theory

[Gaiotto,Witten '08]



$\rho = [n_1, n_2, \ldots, n_s]$ $(n_i \geq n_{i+1})$
$(N = n_1 + \ldots + n_s)$

Partitions of N



$r_1 = n_s, r_2 = n_s + n_{s-1}, \ldots$

Codimension 2 defects in the 6d theory

	N M5	:	0 1 2 3 4 5
Defect branes	M5	:	0 1 2 3 7 8 9 #

$\downarrow$ S^1 -reduction

	N D4	:	0 1 2 4 5
Defect branes	D4	:	0 1 2 7 8 9 #

6d A_{N-1} (2,0) theory

+ codimension 2 defect

★ Labelled by $(\rho, \mathfrak{M}_\alpha)$

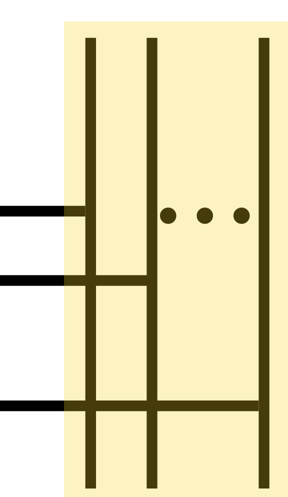
★ $\alpha = 1 \ldots \mathrm{rank}(H_\rho)$

5d $\mathcal{N}=2$ SU(N) SYM

+ 3d T_ρ[SU(N)] theory

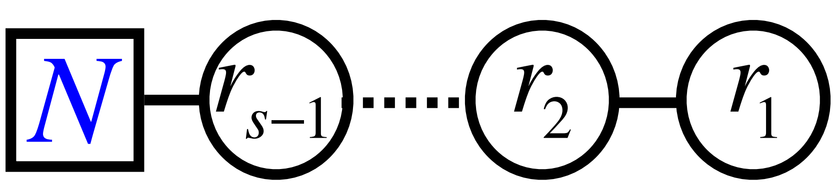
[Gaiotto,Witten '08]

s $NS5s$



$\rho = [n_1, n_2, \ldots, n_s]$ $(n_i \geq n_{i+1})$

$(N = n_1 + \ldots + n_s)$



$r_1 = n_s, r_2 = n_s + n_{s-1}, \ldots$

T_ρ[SU(N)] : 3d $\mathcal{N}=4$

Flavor symmetry $SU(N) \times H_\rho$

$H_\rho := S(\prod_{\alpha=1}^{N} U(l_\alpha))$

l_α : the number of times that α appears in ρ

Codimension 2 defects in the 6d theory

Simplest codimension two defect :

$\rho = [N-1, 1], \; H_\rho = S(U(1) \times U(1)) = U(1)$

called 'simple' or 'minimal'

Maximal codimension two defect :

$\rho = [1^N], \; H_\rho = SU(N)$

called 'full' or 'maximal'

M5s on 3-manifolds + defects

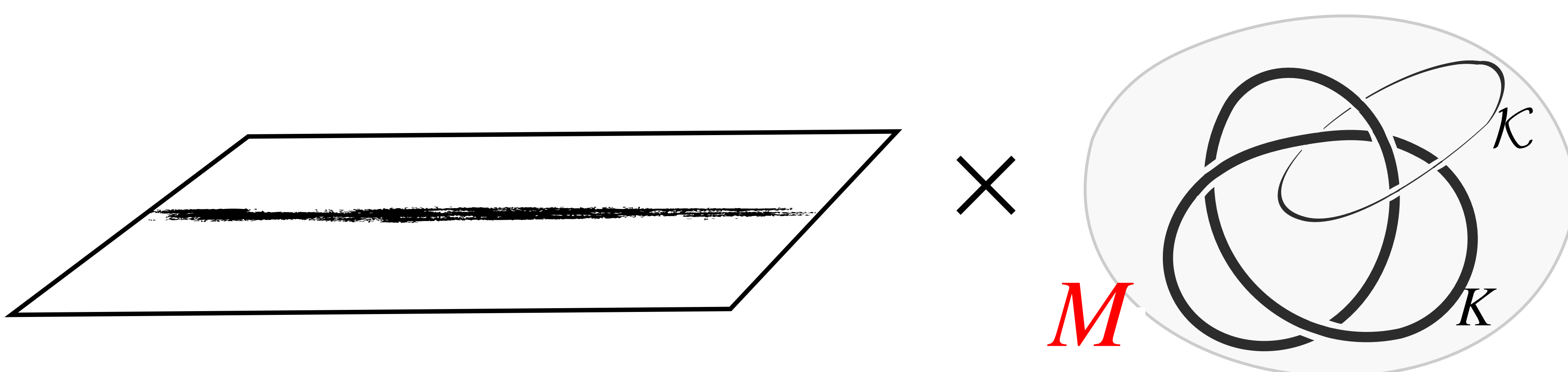
Adding Defects to the system

6d A_{N-1} (2,0) theory on $\mathbb{R}^{1,2} \times M$

+ co-dimension 2 defect $\mathbb{R}^{1,2} \times K$ of type ρ

+ co-dimension 4 defect $\mathbb{R}^{1} \times \mathcal{K}$ of type R

$\longrightarrow$ $T_N[M, K, \rho]$ + line defect $L(R, \mathcal{K})$



$T_N[M, K, \rho]$: $3d$ $\mathcal{N} = 2$ theory w/ flavor symmetry H_ρ

Supersymmetric ptns

$Z[T_N[M, K, \rho] + L(R, \mathcal{K})\ on\ B]$

Rigid SUSY background B $(e.g,\ \mathrm{S}^2 \times_q S^1,\ \mathrm{S}_b^3/\mathbb{Z}_k)$

$$Z = \int [d\Phi]_B \exp\left(iS[\Phi]\right)$$

Supersymmetric ptns

$Z[T_N[M,K,\rho] + L(R,\mathcal{K}) \text{ on } B]$

Rigid SUSY background B (e.g, $S^2 \times_q S^1$, $S^3_b / \mathbb{Z}_k$)

$$Z = \int [d\Phi]_B \exp\left(iS[\Phi]\right)$$

$S^2 \times_q S^1$: (generalized) Superconformal index

$$\mathcal{I}(m_\alpha, u_\alpha; q) = Z_{S^2 \times S^1} = \mathrm{Tr}_{\mathcal{H}(S^2, m_\alpha)} (-1)^{2j_3} q^{j_3 + \frac{R}{2}} \prod_{\alpha=1}^{\mathrm{rank}(H_\rho)} u_\alpha^{F_\alpha}$$

S^3_b : Squashed 3-sphere ptn $Z_{S^3_b}(m_\alpha)$

$$\{ b^2 |z|^2 + b^{-2} |w|^2 = 1 \} \subset \mathbb{C}^2$$

Entanglement(Reyni) Entropy

3d-3d Correspondence

[Yamazaki, Terashima: '11]
[Dimofte, Gukov, Gaiotto: '11]

$$Z[T_N[M]\text{ on }B] = Z[SL(N,\mathbb{C})_{(k,\sigma)}\ CS\text{ on }M]$$

3d-3d Correspondence

$Z[A_{N-1}$ (2,0) theory $on\ B \times M]$

[Yamazaki,Terashima: '11]
[Dimofte,Gukov, Gaiotto: '11]

$Z[T_N[M]$ on $B]$ $=$ $Z[SL(N,\mathbb{C})_{(k,\sigma)}\ CS$ on $M]$

[Jafferis,Cordova: '13]
[Yamazaki,Lee: '13]
[Yagi '13]

1. Independent on relative size between B and M

2. SL(N) CS theory after reduction on B

$$\mathcal{L}_{CS} = \frac{1}{2\hbar} CS[\mathcal{A}] + \frac{1}{2\tilde{\hbar}} CS[\overline{\mathcal{A}}], \quad \frac{4\pi}{\hbar} = k + \sigma, \quad \frac{4\pi}{\tilde{\hbar}} = k - \sigma$$

$$k = k,\ \sigma = k\frac{1-b^2}{1+b^2}\ (S^3/\mathbb{Z}_k) \quad \text{and} \quad k = 0,\ \frac{4\pi i}{\sigma} = \log q\ (S^2 \times_q S^1)$$

$$\mathcal{A} = A_\mu + i\phi_\mu$$

	$SO(3)_M \times SO(3)_R$	$\xrightarrow{\text{top twisting}}$	$SO(3)_{\text{diag}}$
$A:$	$\mathbf{3}\ ,\ \mathbf{1}$	$\longrightarrow$	$\mathbf{3}\,(A_\mu)$
$\phi:$	$\mathbf{1}\ ,\ \mathbf{3}\oplus\mathbf{1}\oplus\mathbf{1}$	$\longrightarrow$	$\mathbf{3}(\phi_\mu)\oplus\mathbf{1}\oplus\mathbf{1}$

3d-3d Correspondence + Defects

$Z[A_{N-1}$ (2,0) theory *on* $B\times M$ + codimension 2 *on* $B\times K$ + codimension 4 *on* $(S^1)_\pm \times \mathcal{K}]$

$Z[T_N[M,K]$ on $B + L(R,K)$ on $(S^1)_\pm]$

$Z[SL(N,\mathbb{C})\, CS$ on $M + V_\rho(K) + (\mathrm{W}_R)_\pm(\mathcal{K})]$

[Kim, Yamazaki, Romo, DG: '15]

3d-3d Correspondence + Defects

$Z[A_{N-1}\ (2,0)\ \text{theory}\ on\ B\times M + \text{codimension 2}\ on\ B\times K + \text{codimension 4}\ on\ (S^1)_{\pm}\times\mathcal{K}]$

$Z[T_N[M,K]\ \text{on}\ B + L(R,K)\ \text{on}\ (S^1)_{\pm}]$

$Z[SL(N,\mathbb{C})\ CS\ \text{on}\ M + V_\rho(K) + (\mathrm{W}_R)_{\pm}(\mathcal{K})]$

[Kim, Yamazaki, Romo, DG: '15]

Codimension 4 in 6d (2,0) theory $\to \mathrm{Tr}_R P\exp(\int_{\{\pm\}\times\mathcal{K}}(A\pm i\phi))$ in 5d

$\to (W_R)_+(\mathcal{K}) \coloneqq \mathrm{Tr}_R P\exp(\int_{\mathcal{K}}\mathcal{A})\ ,\ (W_R)_-(\mathcal{K}) \coloneqq \mathrm{Tr}_R P\exp(\int_{\mathcal{K}}\overline{\mathcal{A}})$ in SL(N) CS theory

3d-3d Correspondence + Defects

$Z[A_{N-1}$ (2,0) theory *on* $B \times M$ + codimension 2 *on* $B \times K$ + codimension 4 *on* $(S^1)_\pm \times \mathcal{K}]$

$Z[T_N[M, K]$ on $B + L(R,K)$ on $(S^1)_\pm]$

$Z[SL(N,\mathbb{C})\ CS$ on $M + V_\rho(K) + (\mathrm{W}_R)_\pm(\mathcal{K})]$

[Kim, Yamazaki, Romo, DG: '15]

Codimension 4 in 6d (2,0) theory $\to \mathrm{Tr}_R P\exp(\int_{\{\pm\}\times\mathcal{K}} (A \pm i\phi))$ in 5d

$\to (W_R)_+(\mathcal{K}) := \mathrm{Tr}_R P\exp(\int_{\mathcal{K}} \mathcal{A})$, $(W_R)_-(\mathcal{K}) := \mathrm{Tr}_R P\exp(\int_{\mathcal{K}} \overline{\mathcal{A}})$ in SL(N) CS theory

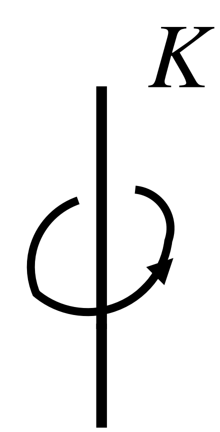
Codimension 2 in 6d (2,0) theory $\to$ coupling $\mathrm{T}_\rho[\mathrm{SU(N)}]$ in 5d

$\to V_{\rho,\mathfrak{M}}(K)$ = Modromay defect around K in SL(N) CS theory

$$\log\left(P\exp\int_{\text{around }K}\mathcal{A}\right) \sim \mathrm{diag}(\overbrace{\mathfrak{M}_1,\ldots,\mathfrak{M}_1}^{n_1},\overbrace{\mathfrak{M}_2,\ldots,\mathfrak{M}_2}^{n_2},\ldots,\overbrace{\mathfrak{M}_s,\ldots,\mathfrak{M}_s}^{n_s})$$

$\mathfrak{M}_i = 2\pi b m_i$ for $Z_{S_b^3}(m_i)$

$\mathfrak{M}_i = (\log q/2)m_i + \log u_i$ for $\mathcal{I}(m_i, u_i; q)$



$M \to M \setminus K := M - N_K$

Holography

- $6d\ A_{N-1}$ (2,0) theory on $\mathbb{R}^{1,2} \times M$

$6d\ A_{N-1}$ (2,0) on $\mathbb{R}^{1,5}$ → $3d\ \mathcal{N}=2\ SCFT\ T_N[M]$ on $\mathbb{R}^{1,2}$

UV → IR

Holography

- $6d\ A_{N-1}$ (2,0) theory on $\mathbb{R}^{1,2} \times M$

$6d\ A_{N-1}$ (2,0) on $\mathbb{R}^{1,5}$ → $3d\ \mathcal{N}=2\ SCFT\ T_N[M]$ on $\mathbb{R}^{1,2}$

UV IR

- Corresponding Holographic RG [Gauntlett, Kim, Waldram : `00] [Pernici,Sezgin : `85]

$AdS_7 \times S^4$ → Pernici-Sezgin AdS_4 solution

$$ds_{11}^2 = \frac{(1+\sin^2\theta)^{1/3}}{g^2}\left[ds^2(AdS_4) + ds^2(M) + \frac{1}{2}(d\theta^2 + \frac{\sin^2\theta}{1+\sin^2\theta}d\phi^2) + \frac{\cos^2\theta}{1+\sin^2\theta}d\tilde{\Omega}^2\right]$$

for closed hyperbolic M ($R_{\mu\nu} = -2g_{\mu\nu}$)

- Holography : $3d\ T_N[M]$ theory = M – theory on Pernici-Sezgin AdS_4 solution

for closed hyperbolic M ($R_{\mu\nu} = -2g_{\mu\nu}$)

Holography + Defects

- $6d\ A_{N-1}$ (2,0) theory on $\mathbb{R}^{1,2} \times M$

$6d\ A_{N-1}$ (2,0) on $\mathbb{R}^{1,5}$ → $3d\ \mathcal{N}=2$ SCFT $T_N[M, K, \rho]$ on $\mathbb{R}^{1,2}$

+Co-dimension 2 defect ρ

+Co-dimension 4 defect R

+Line operator $L(R, \mathcal{K})$

Holography + Defects

- $6d\ A_{N-1}$ (2,0) theory on $\mathbb{R}^{1,2} \times M$

$\boxed{6d\ A_{N-1}\ (2,0)\ \text{on}\ \mathbb{R}^{1,5}}$ → $\boxed{3d\ \mathcal{N}=2\ SCFT\ T_N[M,K,\rho]\ \text{on}\ \mathbb{R}^{1,2}}$

+Co-dimension 2 defect ρ

+Co-dimension 4 defect R

+Line operator $L(R,\mathcal{K})$

- Corresponding Holographic RG

$\boxed{AdS_7 \times S^4}$ → $\boxed{\text{Pernici-Sezgin AdS}_4\text{solution}}$ $AdS_4 \times M \times \tilde{S}^4$

??

$+M5\ on\ AdS_4 \times K \times S^1\ (\rho = \text{simple})$

$+M2\ on\ AdS_3\ \ (R = \square)$

$M5s\ on\ AdS_3 \times S^3\ \ (R = A_k\ , k \sim o(N))$

$+M2\ on\ AdS_2 \times \mathcal{K}\ \ (R = \square)$

$M5s\ on\ AdS_2 \times \mathcal{K} \times S^3\ (R = A_k\ , k \sim o(N))$

Holography + Defects

- $6d\ A_{N-1}$ (2,0) theory on $\mathbb{R}^{1,2} \times M$

$6d\ A_{N-1}$ (2,0) on $\mathbb{R}^{1,5}$ → $3d\ \mathcal{N}=2\ SCFT\ T_N[M,K,\rho]$ on $\mathbb{R}^{1,2}$

+Co-dimension 2 defect ρ

+Co-dimension 4 defect R

+Line operator $L(R,\mathcal{K})$

- Corresponding Holographic RG

$AdS_7 \times S^4$ → Pernici-Sezgin AdS_4 solution $AdS_4 \times M \times \tilde{S}^4$

??

$+M5\ on\ AdS_4 \times K \times S^1$ (ρ = simple)

$+M2\ on\ AdS_3$ ($R = \square$)

$M5s\ on\ AdS_3 \times S^3$ ($R = A_k$, $k \sim o(N)$)

$+M2\ on\ AdS_2 \times \mathcal{K}$ ($R = \square$)

$M5s\ on\ AdS_2 \times \mathcal{K} \times S^3$ ($R = A_k$, $k \sim o(N)$)

- Holography : $3d\ T_N[M]$ theory = M – theory on Pernici-Sezgin AdS_4 solution

Line operator $L(R,\mathcal{K})$ — $M2\ on\ AdS_2 \times \mathcal{K}$ ($R = \square$); $M5s\ on\ AdS_2 \times \mathcal{K} \times S^3$ ($R = A_k$, $k \sim o(N)$)

$T_N[M, K, \rho = simple] - T_N[M]$ — $M5\ on\ AdS_4 \times K$

Computational tools

- 3d SCFT : Localization
- SL(N) CS Theory
  State-integral models
- Supergravity at large N

Localization in 3d theories

$3d \;\; \mathcal{N}=2$ theory

Gauge group G, Chiral matters Φ in R,

CS interactions $\vec{\mathrm{k}}$, superpotential $\mathrm{W}(\Phi)$

Localization on $B = \mathrm{S}^2 \times S^1, \; \mathrm{S}_b^3 / \mathbb{Z}_k$

$$Z = \int [d\Phi]_B \exp\left(iS\left[\Phi; (G, R, \vec{k}, W(\Phi)) \right] \right)$$

$$\xrightarrow{\text{Localized}} \int d\phi_0 e^{iS[\phi_0]} Z^{1-loop}[\phi_0] \quad \text{(finite dimensional integration)}$$

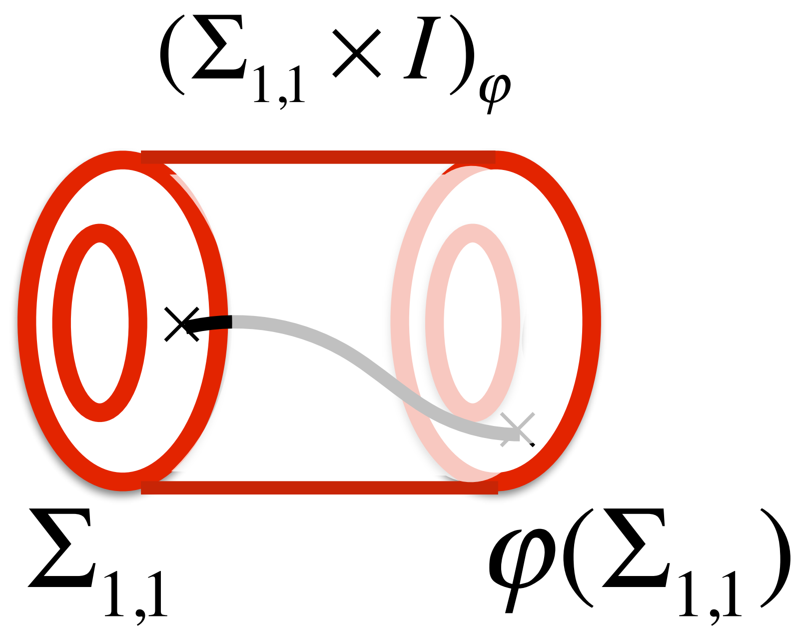
If we know the Lagrangian description of $T_N[M, K, \rho]$
But most of $T_N[M, K, \rho]$, we do not know.

Lagrangian description of $T_N[M, K, \rho]$

Duality wall theory

$T[SU(N),\varphi]$

$4d\ \mathcal{N}=2^*$	$4d\ \mathcal{N}=2^*$
τ	$\varphi(\tau)$



$3d\ \mathcal{N}=2$ theory with $SU(N)\times SU(N)\times U(1)_{\text{axial}}$

$(\ U(1)_R\times U(1)_{\text{axial}}\subset SO(4)_R\ )$

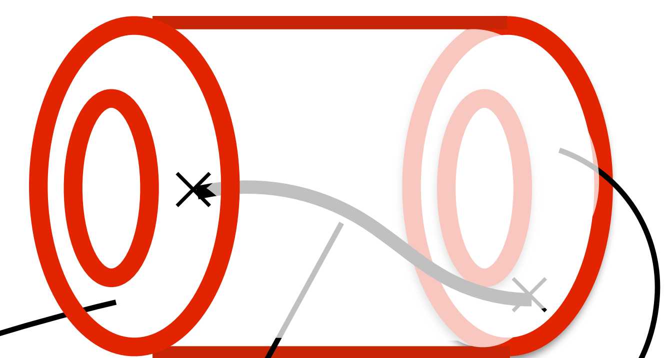
Lagrangian description of $T_N[M, K, \rho]$

Duality wall theory

$T[SU(N), \varphi]$

$4d\ \mathcal{N}=2^*$	$4d\ \mathcal{N}=2^*$
τ	$\varphi(\tau)$

$(\Sigma_{1,1} \times I)_\varphi$

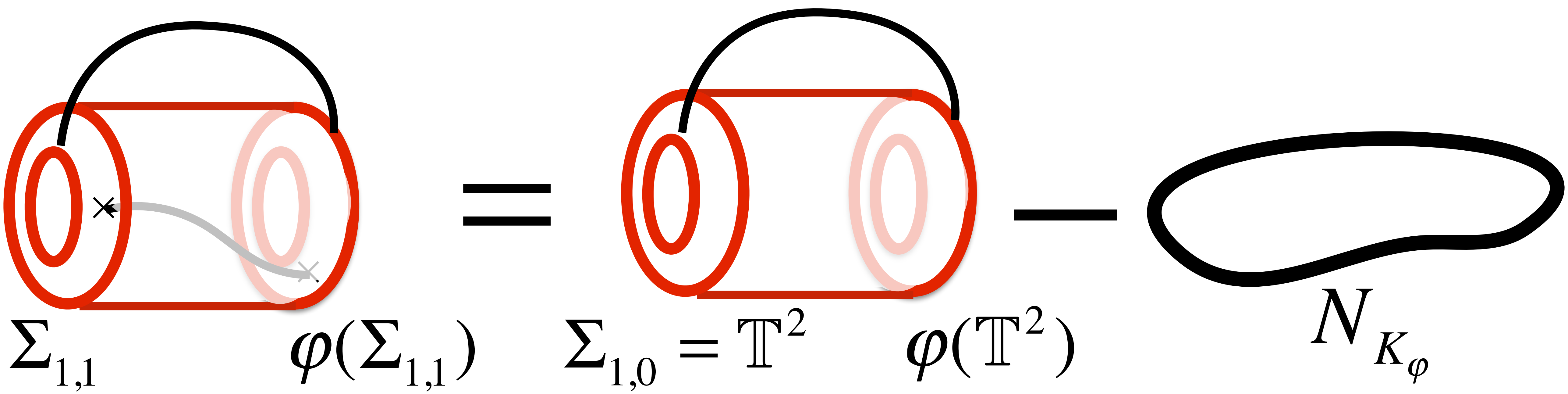


$\Sigma_{1,1}$ $\varphi(\Sigma_{1,1})$

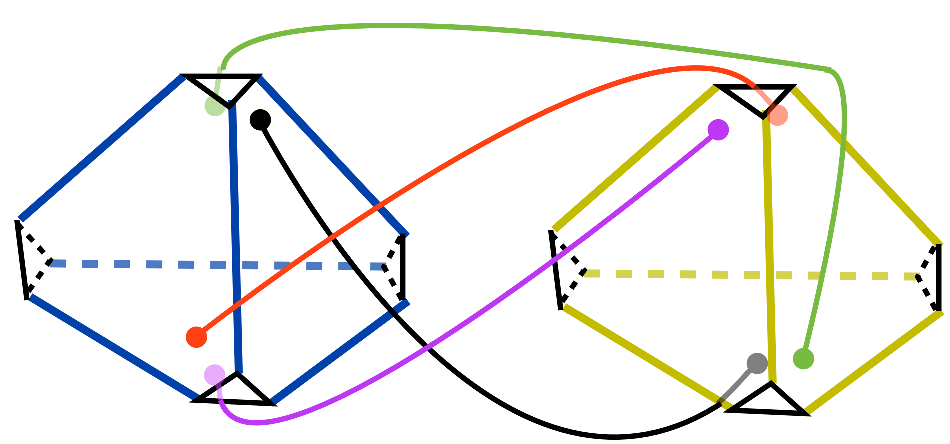
$3d\ \mathcal{N}=2$ theory with $SU(N) \times SU(N) \times U(1)_{\text{axial}}$

$(\ U(1)_R \times U(1)_{\text{axial}} \subset SO(4)_R\ )$

Topology on mapping torus



$= S^3 -$  $(\varphi = ST^{-1}S^{-1}T)$

$=$ 

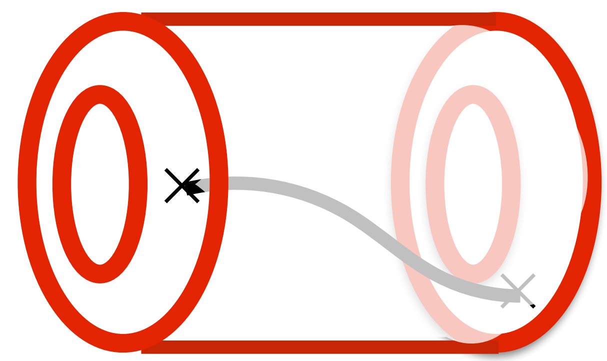
Lagrangian description of $T_N[M, K, \rho]$

$T[SU(N), \varphi]$ theory

$4d\ \mathcal{N}=2^*$ | $4d\ \mathcal{N}=2^*$

τ | $\varphi(\tau)$

$(\Sigma_{1,1} \times I)_\varphi$



$\Sigma_{1,1}$ $\varphi(\Sigma_{1,1})$

$\mathrm{Tr}(T[SU(N), \varphi])$

theory obtained by

gauging diagonal SU(N)

of $T[SU(N), \varphi])$

$(\Sigma_{1,1} \times S^1)_\varphi$

$:= (\Sigma_{1,1} \times [0,1]) / \sim ,$

$(x,0) \sim (\varphi(x), 0)\ .$

$$\mathrm{Tr}(T[SU(N), \varphi]) = T_N[M = (\mathbb{T}^2 \times S^1)_\varphi, K_\varphi, \rho = \text{simple}]$$

State-integrals in SL(N) CS theory

$$Z = \int [d\mathcal{A}]_{(M\backslash K,\rho)} \exp\left(iS_{CS}[\mathcal{A},\overline{\mathcal{A}};k,\sigma]\right)$$

$$\longrightarrow \int dX \exp\left(\frac{1}{2\hbar} X \cdot B^{-1}A\, X + ..\right) \prod \psi_{\hbar}(X\ ) \text{ (finite dimensional integration)}$$

- Firstly developed for ‘maximal’ co-dimension 2 defects

[Dimofte : `11]
[Dimofte, Gabella, Goncharov: `13]

- Extend to include for co-dimension 4 defects and non-maximal ρ for some examples

[Kim, Yamazaki, Romo, DG: `15]

- Can be used to find field theory description for $T_N[M,K,\rho]$: Abelian description

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Supergravity at large N

- Free energy at large N

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- Simple Codimension 2 defect

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$$\mathcal{F}_b(T_N[M]) \coloneqq \log|Z_{S_b^3}(T_N[M])| = \frac{\mathrm{vol}(M)}{12\pi}(b+b^{-1})^2 N^3 + (\text{subleading in } 1/N)$$

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for $k \sim o(N)$

- Simple Codimension 2 defect

$$\mathcal{F}_{b=1}(T_N[M, K, \rho = \text{simple}]) - \mathcal{F}_{b=1}(T_N[M]) := \frac{\ell(K)}{3} N^2 + (\text{subleading in } 1/N)$$

Consistency Checks

- Localization/State-integral model

$$\mathcal{I}(T_{N=3}[M=(\mathbb{T}^2\times S^1)_\varphi, K_\varphi, \rho=\text{simple}])_{\text{localization}}$$
$$=Z[SL(3)_{k=0}\,\text{on}\ M=(\mathbb{T}^2\times S^1)_\varphi, K_\varphi, \rho=\text{simple}])_{\text{state-intgral}}$$

checked in q expansion

$$\mathcal{I}(T_{N=2}[M=(\mathbb{T}^2\times S^1)_\varphi, K_\varphi, \rho=\text{simple}]+\text{loop operators})_{\text{localization}}$$
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- State-integral/Holography computation

Holographic computations show that the b-dependence of

$(S_b^3)-$free-energy is very simple at large N (only b^{-2}, b^0, b^2) .

In CS theory, $\hbar = 2\pi i b^2$. $Z(SL(N)\,\text{CS on } M) \xrightarrow{\hbar\to 0} \frac{1}{\hbar}S^0 + S^1 + S^2\hbar + \dots$

$$\lim_{N\to\infty}\frac{1}{N^3}S_0 = -\frac{i}{6}\text{vol}(M), \quad \lim_{N\to\infty}\frac{1}{N^3}S_1 = -\frac{1}{6\pi}\text{vol}(M),$$

$$\lim_{N\to\infty}\frac{1}{N^3}S_2 = \frac{i}{24\pi^2}\text{vol}(M),$$

$$\lim_{N\to\infty}\frac{1}{N^3}S_j = 0 \ (j\geq 3),$$

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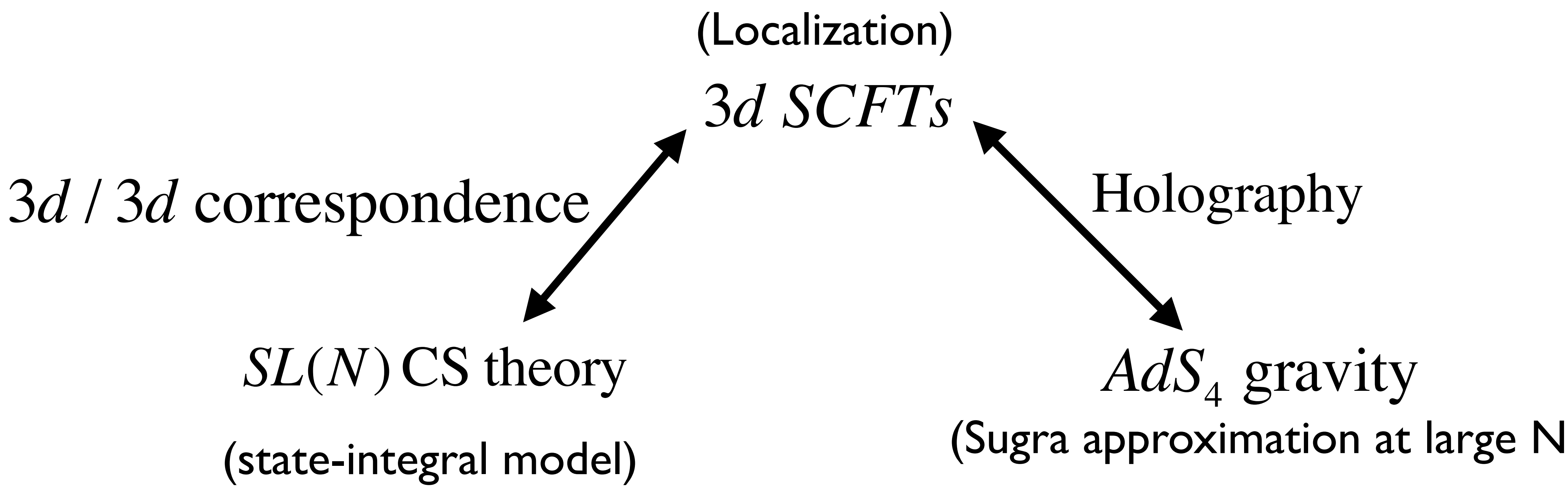
$\lim_{N\to\infty}\frac{1}{N^3}S_0 = -\frac{i}{6}\text{vol}(M)$, $\lim_{N\to\infty}\frac{1}{N^3}S_1 = -\frac{1}{6\pi}\text{vol}(M)$, analytically confirmed

$\lim_{N\to\infty}\frac{1}{N^3}S_2 = \frac{i}{24\pi^2}\text{vol}(M)$, numerically confirmed

$\lim_{N\to\infty}\frac{1}{N^3}S_j = 0 \ (j\geq 3)$,

Conclusion

- Various aspects of M5s on 3-manifold + defects



- Applications?

Thank you
for
your attention

CS ptn on mapping torus

- For the case, $M-\bigcup_{i=1}^{h} N_{K_i} = (\Sigma_{g,h}\times S^1)_\varphi$
- Regarding the circle as 'time'

$$Z[SL(N)\,CS\ \ (\Sigma_{g,h}\times S^1)_\varphi\ ,(\rho_a,\overrightarrow{\mathfrak{M}_\alpha})] = \mathrm{Tr}_{\mathcal{H}_N(\Sigma_{g,h},(\rho_a,\overrightarrow{\mathfrak{M}_\alpha}))}\hat{\varphi}$$

$$\mathcal{H}_N\left(\Sigma_{g,h},(\rho_\alpha,\mathfrak{M}_\alpha)\right):\ \text{Quantization of}\left(P_N\left(\Sigma_{g,h},(\rho_\alpha,\mathfrak{M}_\alpha)\right),\omega_{k,\sigma}\right)$$

$$P_N\left(\Sigma_{g,h},(\rho_\alpha,\mathfrak{M}_\alpha)\right) := \{\text{Flat } SL(N) \text{ connections on } \Sigma_{g,h} \text{ with } b.c\ (\rho_\alpha,\mathfrak{M}_\alpha)\}$$

$$\omega_{k,\sigma} := \frac{1}{\hbar}\int_{\Sigma_{g,h}} \mathrm{Tr}(\delta A\wedge\delta A) + \frac{1}{\tilde{\hbar}}\int_{\Sigma_{g,h}} \mathrm{Tr}(\delta\bar{A}\wedge\delta\bar{A})\,,\ (\frac{1}{\hbar}=\frac{k+\sigma}{8\pi}\,,\ \frac{1}{\tilde{\hbar}}=\frac{k-\sigma}{8\pi})\,.$$

- Find a good coordinate system of the phase-space

Loop coordinates : $\mathrm{Tr}_R(\mathrm{Hol}(\gamma_i))$, $\{\gamma_i\}$: generators of $\pi_1(\Sigma_{g,h})$

Difficult to quantize (complicated relations and $\omega_{k,\sigma}$)

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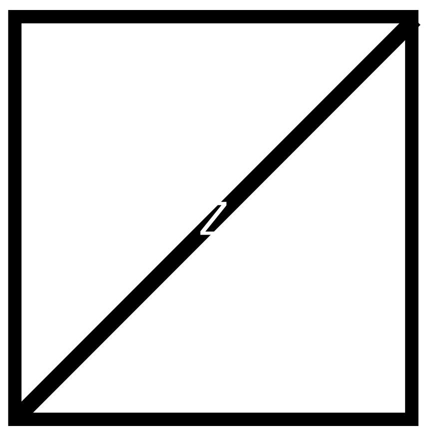
- Find a good coordinate system of the phase-space

Fock-Goncharov Coordinates for $\rho_\alpha =$ `maximal'

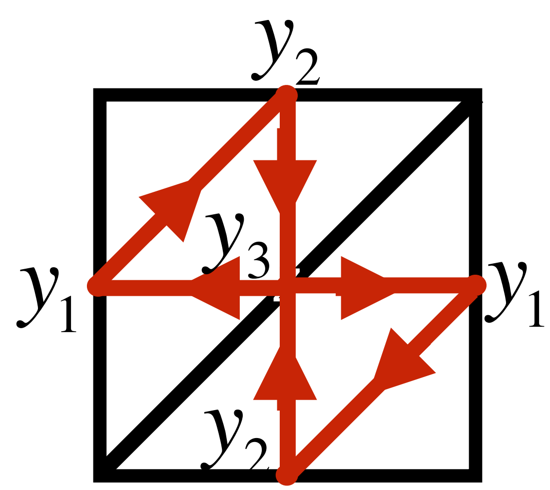
(linear relations and simple $\omega_{k,\sigma}$)

FG coordinates

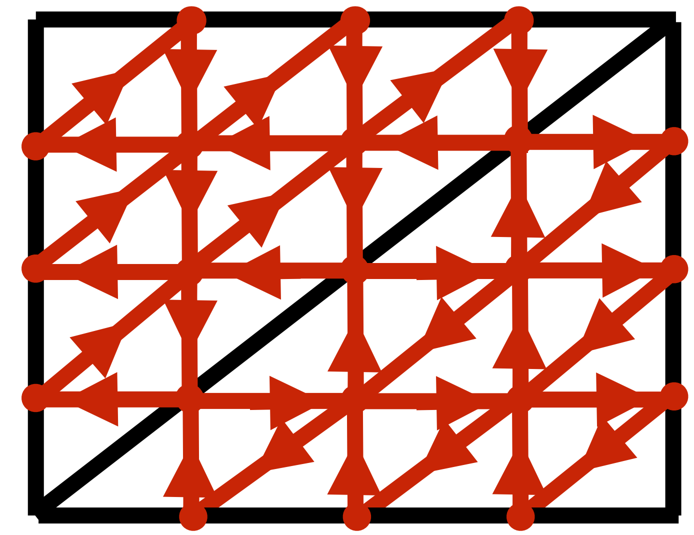
- Ideal triangulation -〉 Tessellation -〉 FG quiver



ideal triangulation *of* $\Sigma_{1,1}$



$N=2$



$N=4$

- N=2, once-punctured torus

$$P_{N=2}\left(\Sigma_{1,1},(\rho,\mathfrak{M})\right)=\{y_1,y_2,y_3 : y_1y_2y_3=e^{\mathfrak{M}}\}$$

$$\omega_{\mathrm{k},\sigma}:\{Y_i,Y_j\}_{P.B}=\hbar Q_{ij}\ ,\{\bar{Y}_i,\bar{Y}_j\}_{P.B}=\tilde{\hbar}Q_{ij}\ ,Q_{12}=Q_{23}=Q_{31}=2\ .$$

- Holonomy computation

$$\mathrm{Hol}(\gamma_x)=\begin{pmatrix}\frac{(y_1+1)y_3}{\sqrt{y_1y_3}} & \sqrt{\frac{y_3}{y_1}} \\ \frac{1}{\sqrt{y_1y_3}} & \frac{1}{\sqrt{y_1y_3}}\end{pmatrix},\quad \mathrm{Hol}(\gamma_y)=\begin{pmatrix}\frac{y_2+1}{\sqrt{y_2y_3}} & -\sqrt{y_2y_3} \\ -\sqrt{\frac{y_2}{y_3}} & \sqrt{y_2y_3}\end{pmatrix}$$

Cluster algebra

- Cluster Algebra generated by FG quiver $(q := e^{\hbar}, \tilde{q} := e^{\tilde{\hbar}})$

$$\mathcal{A}_Q := \{ \mathsf{y}_i, \bar{\mathsf{y}}_{i\,(i\in I)} \,|\, \mathsf{y}_j \mathsf{y}_i = q^{Q_{ij}} \mathsf{y}_i \mathsf{y}_j \,,\ \bar{\mathsf{y}}_j \bar{\mathsf{y}}_i = \tilde{q}^{Q_{ij}} \mathsf{y}_i \mathsf{y}_j \,,\ \bar{\mathsf{y}}_j \mathsf{y}_i = \mathsf{y}_i \bar{\mathsf{y}}_j \ \}$$

- Mutation

$$\hat{\mu}_k \, \mathsf{y}_i \, \hat{\mu}_k^{-1} = q^{\frac{1}{2} Q_{ik} [Q_{ik}]_+} \mathsf{y}_i \mathsf{y}_k^{[Q_{ik}]_+} \prod_{m=1}^{|Q_{ki}|} \left(1 + q^{\mathrm{sgn}(Q_{ki})(m-\frac{1}{2})} \mathsf{y}_k^{-1} \right)^{-\mathrm{sgn}(Q_{ki})}$$

$$(\mu_k Q)_{ij} := \begin{cases} -Q_{ij} & (i = k \text{ or } j = k) \,, \\ Q_{ij} + [Q_{ik}]_+ [Q_{kj}]_+ - [Q_{jk}]_+ [Q_{ki}]_+ & (i, j \neq k) \,, \end{cases}$$

- Representation of MCG (mapping class group)

$$\hat{\varphi} := \hat{\mu}_2 \widehat{\sigma}_L \ ,\ \sigma_L : y_2 \leftrightarrow y_3 \text{ , for } \varphi = L := ST^{-1}S^{-1}$$

$$\hat{\varphi} := \hat{\mu}_1 \widehat{\sigma}_R \ ,\ \sigma_R : y_1 \leftrightarrow y_3 \text{ , for } \varphi = R := T$$

CS ptn on mapping torus

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Use FG Coordinates for $\rho_\alpha =$ `maximal'

- Inclusion of Wilson loop operator

$$Z[SL(N)\,CS\ \ (\Sigma_{g,h}\times S^1)_\varphi+\text{Wilson loop along }\gamma\in\pi_1(\Sigma_{\mathrm{g,h}})\ ,(\rho_a,\overrightarrow{\mathfrak{M}_\alpha})]$$

$$=\mathrm{Tr}_{\mathcal{H}_N(\Sigma_{g,h},(\rho_a,\overrightarrow{\mathfrak{M}_\alpha}))}\widehat{\mathrm{Tr}_R\mathrm{Hol}(\gamma)}\,\hat{\varphi}$$

$$\mathrm{Tr}_R\mathrm{Hol}(\gamma)=\sum_k c_k e^{a_i^{(k)}Y_i}\xrightarrow{\text{Quantization}}\widehat{\mathrm{Tr}_R\mathrm{Hol}(\gamma)}=\sum_k \widehat{c_k}e^{a_i^{(k)}\mathsf{Y}_i}\ \ ,\ \ c_k\in\mathbb{N}\bigcup\{0\}$$

$$\hat{1}=1\ ,\hat{2}=q^a+q^{-a}\ ,.....\text{ambiguity in quantization unless }c_k=1$$

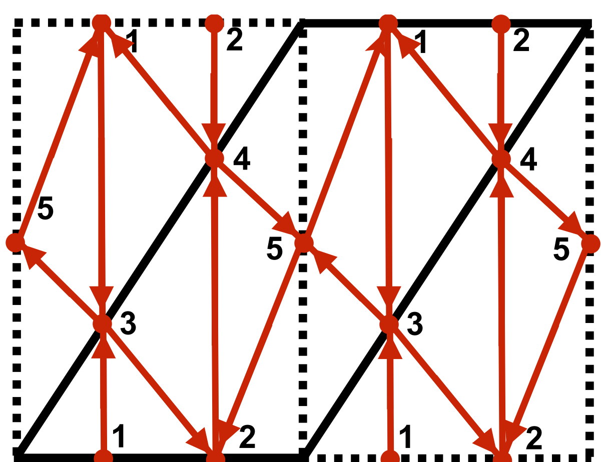
Ex) $\Sigma_{1,1}$ with $N=2$, $\mathrm{Tr}(\mathrm{Hol}(\gamma_x))=y_1^{1/2}y_3^{-1/2}+y_3^{1/2}y_1^{-1/2}+y_1^{-1/2}y_3^{-1/2}$

$$\widehat{\mathrm{Tr}(\mathrm{Hol}(\gamma_x))}=e^{1/2\mathsf{Y}_1-1/2\mathsf{Y}_3}+e^{-1/2\mathsf{Y}_1+1/2\mathsf{Y}_3}+e^{-1/2\mathsf{Y}_1-1/2\mathsf{Y}_3}$$

CS ptn on mapping torus

- For Non-maximal case, cluster coordinates?

  The answer seems to be "yes"



$Q_{[2,1]}$

Quiver for $\Sigma_{1,1}$, N=3 and $\rho = [2,1]$.

$$\hat{\varphi} = \hat{\mu}_5 \hat{\sigma}_S \ , \ \sigma_S : (y_1, y_2, y_3, y_4) \to (y_4, y_3, y_1, y_2) \text{ for } \varphi = S$$

$$\hat{\varphi} = \hat{\mu}_1 \hat{\mu}_2 \hat{\sigma}_T \ , \sigma_T : (y_1, y_2, y_3, y_4) \to (y_3, y_4, y_1, y_2) \text{ for } \varphi = T$$

For $k = 0$ and $4\pi i / \sigma = \hbar = \log q$

$$Z[SL(3)_{(k,\sigma)} \ CS \ \ (\Sigma_{1,1} \times S^1)_{\varphi = STS^{-1}T} \ , (\rho = [2,1], \mathfrak{M} = \frac{\hbar}{2} m + \log \eta)]$$

$$= \mathrm{Tr}_{\mathcal{H}_{N=3}(\Sigma_{1,1}, (\rho = [2,1], \mathfrak{M} = \frac{\hbar}{2} m + \log \eta))} (\hat{\mu}_5 \hat{\sigma}_S \hat{\mu}_1 \hat{\mu}_2 \hat{\sigma}_T \hat{\sigma}_S{}^{-1} \hat{\mu}_5 \hat{\mu}_1 \hat{\mu}_2)$$

$$= 1 + \left(2\eta + \frac{2}{\eta} \right) q^{\frac{3}{2}} + \left(8 + 2\eta^2 + \frac{2}{\eta^2} \right) q^2 + \left(6\eta + \frac{6}{\eta} \right) q^{\frac{5}{2}} + \left(2 - 3\eta^2 - \frac{3}{\eta^2} \right) q^3 + \ldots$$

It matches the index computed using localization on $\mathrm{Tr}(T[SU(3), \varphi = STS^{-1}T])$!!

Cluster partition function

- Generalizing the previous computation, we consider

$$\mathrm{Tr}^{(k,\sigma)}_{Q,\vec{m},\vec{\sigma}}(\overrightarrow{\mathfrak{M}}) = \mathrm{Tr}_{\mathcal{H}^{(k,\sigma)}(Q,\overrightarrow{\mathfrak{M}})}(\hat{\mu}_{m_1}\widehat{\sigma}_1\hat{\mu}_{m_2}\widehat{\sigma}_2\ldots\hat{\mu}_{m_\sharp}\hat{\sigma}_\sharp)$$

- After explicit computation, the ptn can be written as

$$\mathrm{Tr}^{(k,\sigma)}_{Q,\vec{m},\vec{\sigma}}(\overrightarrow{\mathfrak{M}}) = \left\langle C_I = 0, \mathfrak{M}_\alpha \middle| \diamondsuit^{\otimes\sharp} \right\rangle = \int d^\sharp \vec{X}\, \langle C_I = 0, \mathfrak{M}_\alpha | \vec{X} \rangle\langle \vec{X} | \diamondsuit^{\otimes\sharp} \rangle$$

$$= \int d^\sharp \vec{X} \left\langle C_I = 0, \mathfrak{M}_\alpha \middle| \vec{X} \right\rangle \prod_{i=1}^{\sharp} \psi_{\hbar,\tilde{\hbar}}(X_i)$$

$$\left| \diamondsuit^{\otimes\sharp} \right\rangle := \left(\left| \diamondsuit \right\rangle \right)^{\otimes\sharp},\ \left| C_I, \mathfrak{M}_\alpha \right\rangle \in (\mathcal{H}^{(k,\sigma)})^{\otimes\sharp} \quad , \quad \left\langle X \middle| \diamondsuit \right\rangle = \prod_{r=0}^{\infty} \frac{1 - q^{r+1} e^{-X}}{1 - \tilde{q}^{-r} e^{-\bar{X}}}$$

$\mathcal{H}^{(k,\sigma)} = \{\text{spanned by position basis} \left| X \right\rangle\}$, Hilber-space obtained by quantizing

a phase-space $P = \{(x := e^X, p := e^P)\} = (\mathbb{C}^*)^2$, $\omega = \frac{1}{\hbar} dX \wedge dP + \frac{1}{\tilde{\hbar}} d\bar{X} \wedge d\bar{P}$

$$\begin{pmatrix} C_I \\ \mathfrak{M}_\alpha \end{pmatrix}_\sharp = A_{\sharp\times\sharp} \cdot X_\sharp + B_{\sharp\times\sharp} \cdot P_\sharp \qquad (A,B)_{\sharp\times(2\sharp)} \text{ form upper block of } \mathrm{Sp}(2\sharp,\mathbb{Q})\ i.e., \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2\sharp,\mathbb{Q})$$