Developments in String Theory and Quantum Field Theory @YITP 2015/11/10

Chaotic strings in a near Penrose limit of $AdS_5 \times T^{1,1}$

Daisuke Kawai (Kyoto Univ.)

In collaboration with Y.Asano (DIAS Dublin)、
H.Kyono (Kyoto Univ.) K.Yoshida (Kyoto Univ.)

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Motivation --- Why we consider chaos ? ---

Finding of the integrability behind the AdS/CFT correspondence

→ Developments in the study of Type IIB string/4D $\mathcal{N}=4$ SYM correspondenec

Ex.1) Heisenberg spin chain [J.A.Minahan, K.Zarembo, JHEP 0303(2003)013]

1-loop dilatation operator ⟷ Heisenberg spin chain Hamiltonian

Ex.2) quark anti-quark potential [J.Maldacena,arXiv:9803002], [S.Ray,J.Yee,arXiv 9803001]

Wilson loop ⟷ Minimal surface

Developments of study with integrability

→ Understanding of the dynamical aspects of the AdS/CFT correspondence

Remark : There are non-integrable systems in the AdS/CFT correspondence.

Non-integrable backgrounds

Complex beta-deformations [Giataganas-Pando Zayas-Zoubos, 1311.3241]

$AdS_5 \times T^{1,1}$ [Basu-Pando Zayas, 1103.4107] $AdS_5 \times Y^{p,q}$ [Basu-Pando Zayas, 1105.2540]

AdS BH [Pando Zayas-Terrero Escalante, 1007.0277] AdS solitons [Basu-Das-Ghosh, 1103.4101]

Klebanov-Strassler, Maldacena-Nunez [Basu-Das-Ghosh-Pando Zayas, 1201.5634]

Schrödinger spacetime with $z = 4, 5, 6$ [Giataganas-Sfetsos, 1403.2703]

Lifshitz space (with hyper-scaling violation) [Giataganas-Sfetsos, 1403.2703] [Bai-Chen-Lee-Moon, 1406.5816]

p-brane backgrounds [Stepanchuk-Tseytlin, 1211.3727] [Chervonyi-Lunin, 1311.1521]

Q. What measure is useful for the study of non-integrable systems?

Classical strings on non-integrable system usually have chaotic motions.

In the list of last slide, strings on

$AdS_5 \times T^{1,1}$ [Basu-Pando Zayas, 1103.4107]

AdS solitons [Basu-Das-Ghosh, 1103.4101]

AdS BH [Pando Zayas-Terrero Escalante, 1007.0277]

exhibit chaotic motion.

We study the way to uncover the dynamical aspects of non-integrable system with quantities specific to chaos.

- Kolmogorov-Sinai Entropy → Main topic of today's talk
- Fractal dimension

→ Systematic search for the dual operators of gauge theories.

Related Works

J. Maldacena, S.H.Shenker,D.Stanford “A bound on chaos”, 1503.01409

→ Thay claim the existence of the upper limit on the Lyapunov indices.

C.T.Asplund,D.Berenstein,

”Entanglement entropy converges to classical entropy around periodic orbits”, 1503.04857

D.Berenstein, A.M.Garcia-Garcia

“Universal quantum constraints on the butterfly effect”, 1510.08870

→ The growth rate of the entanglement entropy is limited by the Lyapunov indices.

Joseph Polchinski,

”Chaos in the black hole S-matrix”, 1505.08108

→ They discuss the relationship between the time evolution of Black Hole and chaos.

Characteristics of chaos

1. random trajectory (without noise)
2. Sensitivity to initial values
3. Boundedness

- System is deterministic, but information is produced in the time evolution.

∵) trajectory is governed by EOM without stochastic noise ➡ Uniquely determined.

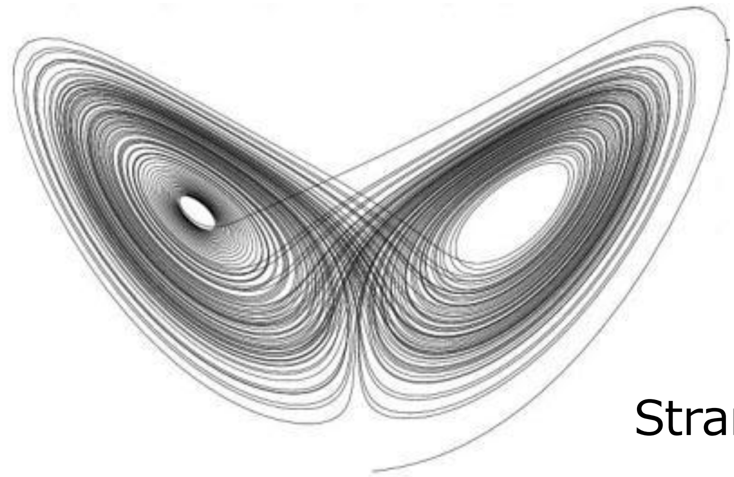
However, tiny difference in initial conditions grow exponentially.

➡ So, it makes significant difference in the late time.

It is reflected in possitive Kolmogorov-Sinai entropy.

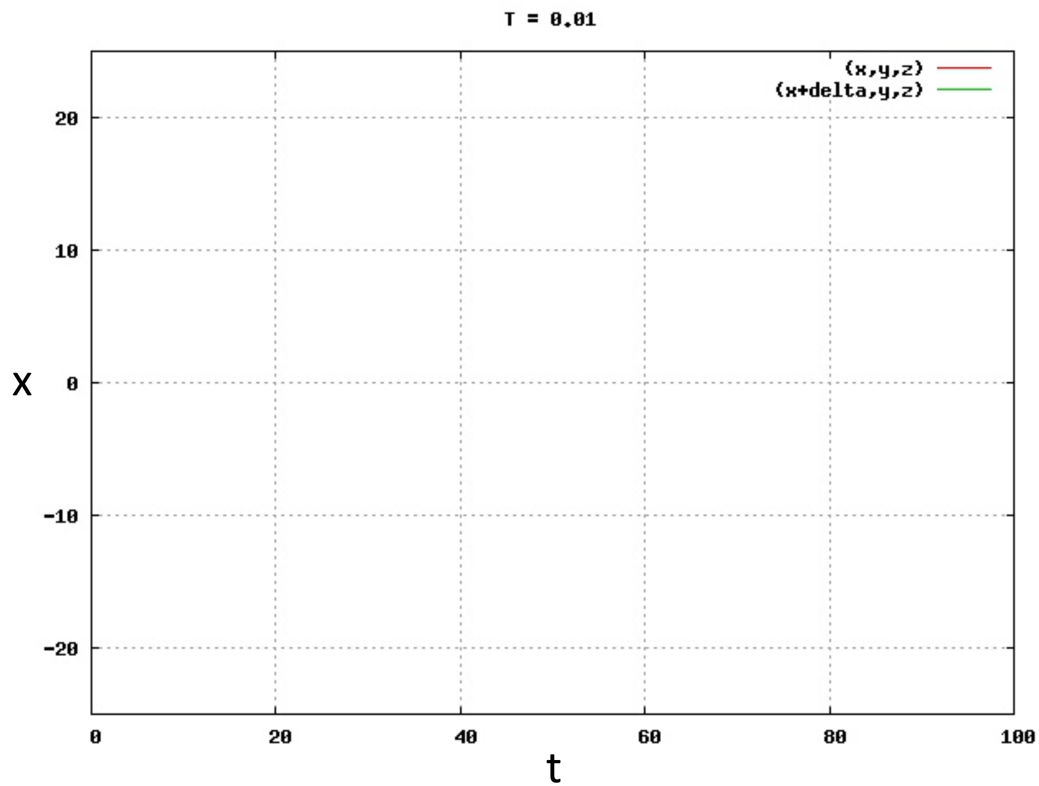
Lorenz model

$$\begin{cases} \dfrac{dx}{dt} = -px + py \\ \dfrac{dy}{dt} = -xz + rx - y \\ \dfrac{dz}{dt} = xy - bz \end{cases}$$



Strange Attractor

http://www.mathematik.uni-muenchen.de/~kremser/ODE_SoSe13.html



$r = 28\,,\ p = 10\,,\ b = \dfrac{8}{3}$

$(x, y, z) = (1.0, 2.0, 0.1)$

$(x + \delta x, y, z) = (1.0 + 1.0 \times 10^{-6}, 2.0, 0.1)$

2. Quantative measurement

1. Poincaré section
2. Lyapunov spectrum
3. Kolmogorov-Sinai Entropy

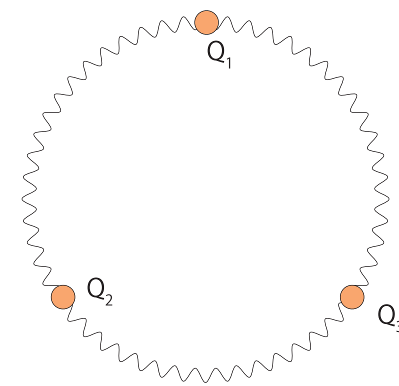
I introduce them by seeing the case of the Henon-Heiles system.

Henon-Heiles system --- a Hamilton system which exhibits chaos

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) + \lambda\left(q_1^2 q_2 - \frac{1}{3}q_2^3\right)$$

Non-linear interaction

Use the normal coordinates &
Remove the total momentum



3-body periodic lattice (with 3-order interaction)

$$\begin{aligned} H &= H_0 + \frac{1}{3}\alpha\left[(Q_1 - Q_2)^3 + (Q_2 - Q_3)^3 + (Q_3 - Q_1)^3\right] \\ H_0 &= \frac{1}{2}(P_1^2 + P_2^2 + P_3^3) + \frac{1}{2}\left[(Q_1 - Q_2)^2 + (Q_2 - Q_3)^2 + (Q_3 - Q_1)^2\right] \end{aligned}$$

c.f.) anti Henon-Hiles system is integrable.

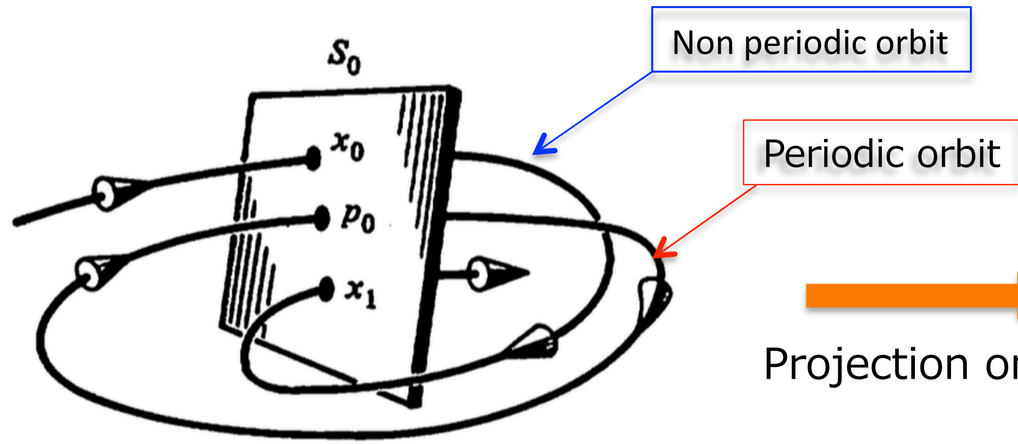
$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) + \lambda\left(q_1^2 q_2 + \frac{1}{3}q_2^3\right)$$

Poincaré section

We consider a section S_0 crossing with trajectories in phase space governed by EOM.

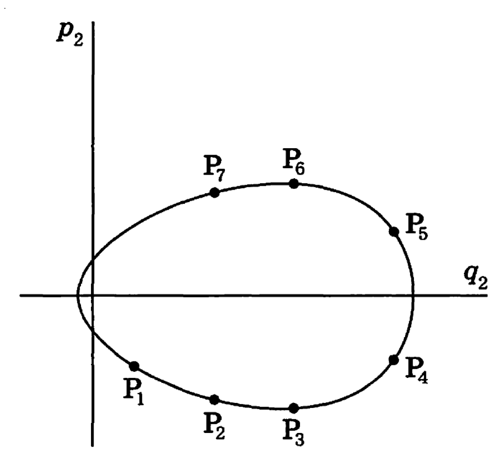
Then, we take a point on S_0 , $x_0 \in S_0$, and consider the time evolution of x_0 .

➡ The set of points on S_0 reflects the integrablity structure of system.



From A.Jackson, “Perspectives of Nonlinear Dynamics”

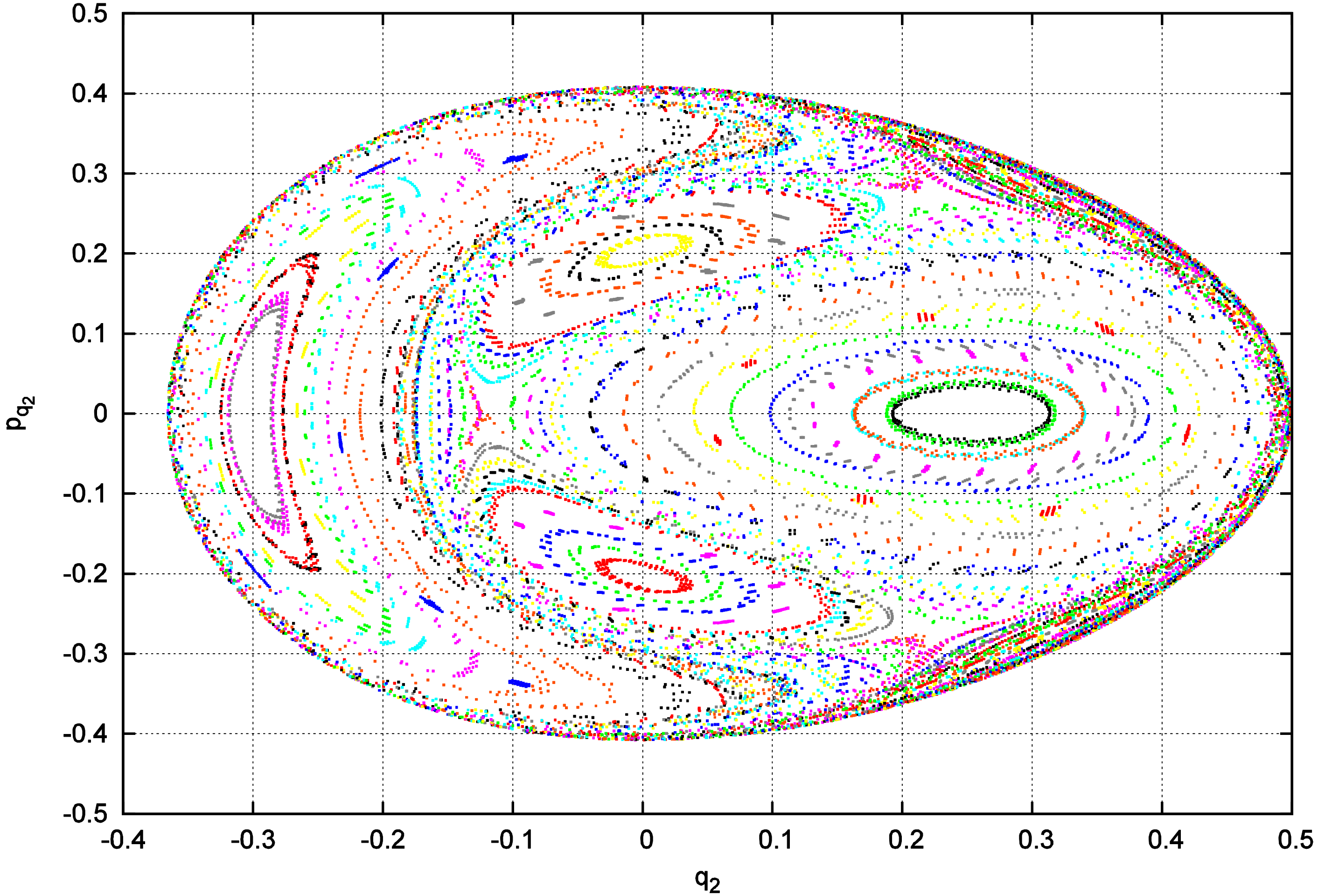
Projection on S_0



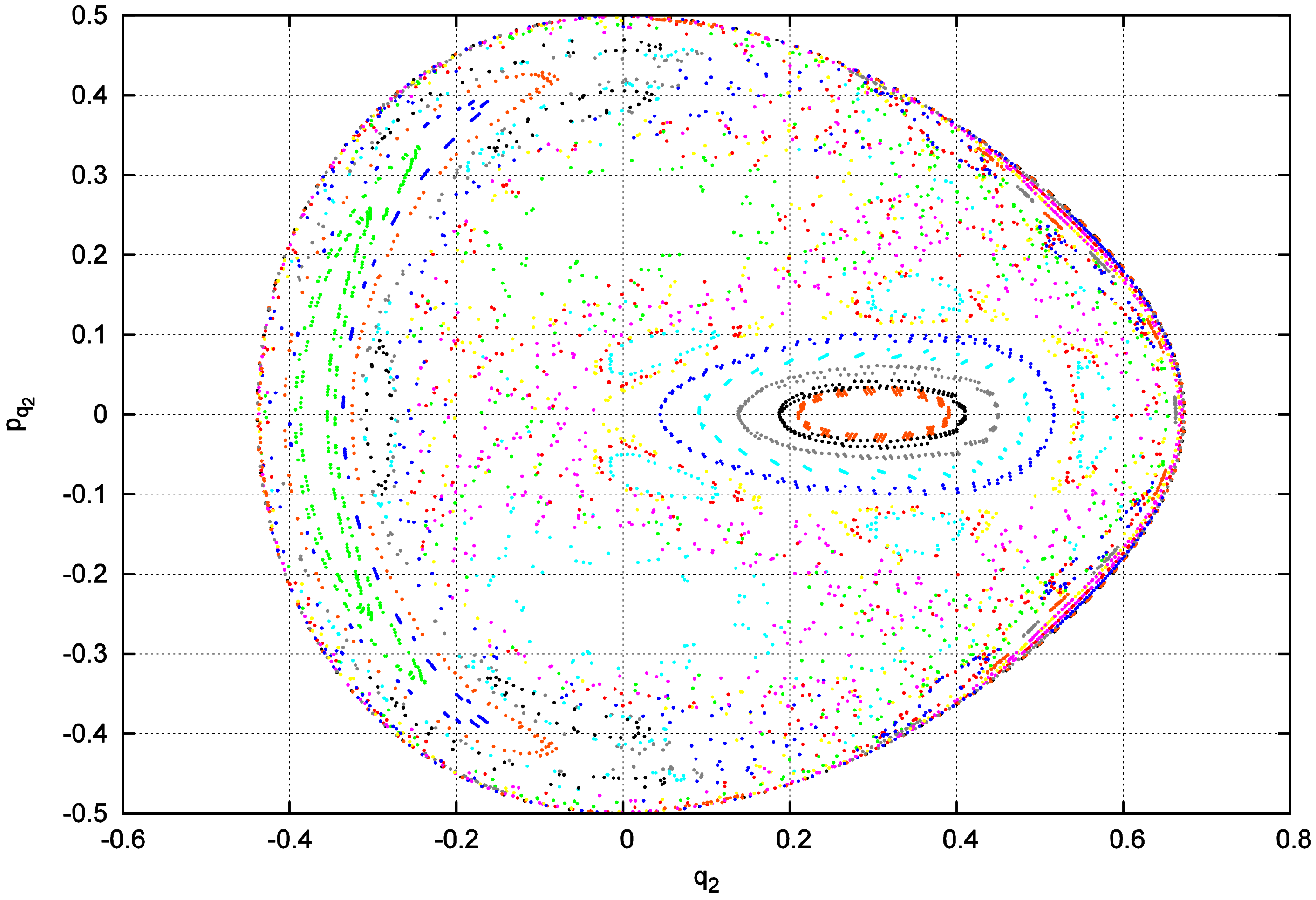
From M.Toda, “Non-linear waves and solitons”

(Integrable)	Periodic orbit	→	Periodically arranged points
	Quasi-periodic orbit	→	A closed orbit
(Non integrable)	Chaotic orbit	→	Randomly scattered points

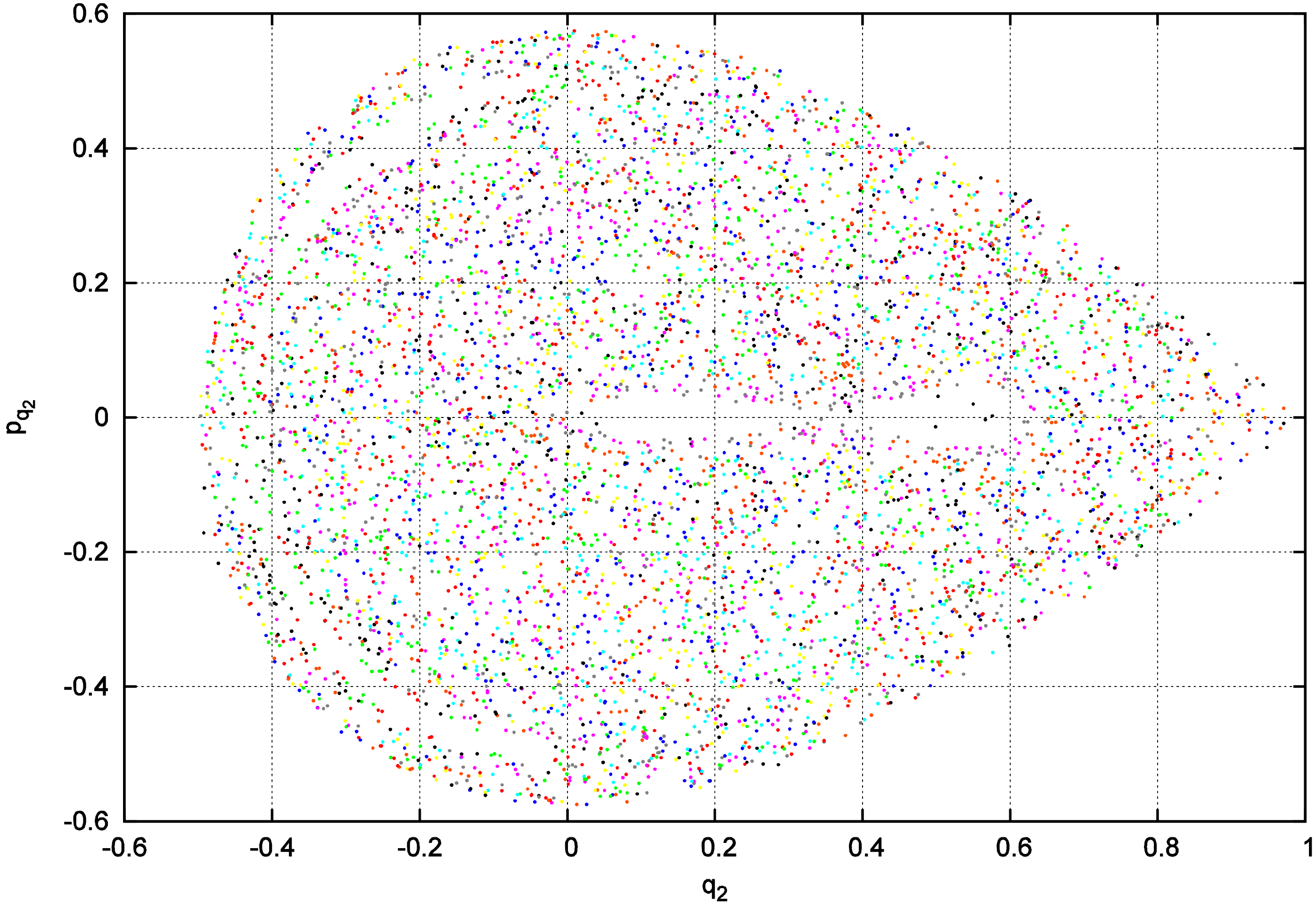
Numerical results E=0.0833



Numerical Results E=0.12500



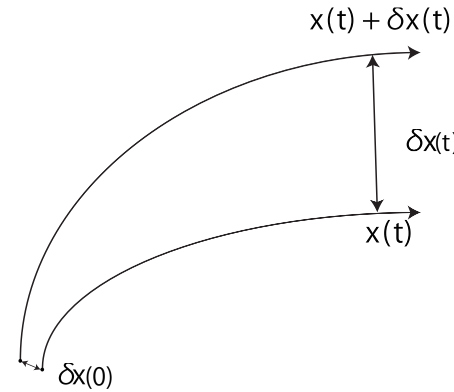
Numerical Results E=0.16667



Random points = Henon-Heiles system is chaotic

Lyapunov spectra

In general, the difference of trajectories grows exponentially.



$$\delta x(t) \sim \mathrm{e}^{\lambda t}\delta x(0)$$

$\delta x(0)$: initial difference (tiny)

$\delta x(t)$: difference at time t

➡ λ : Lyapunov index

$\lambda > 0$ = Sensitive to initial values ➡ The system is chaotic.

Pesin's equation [Y.B.Pesin,1977, Russ.Math.Surv. 32 55]

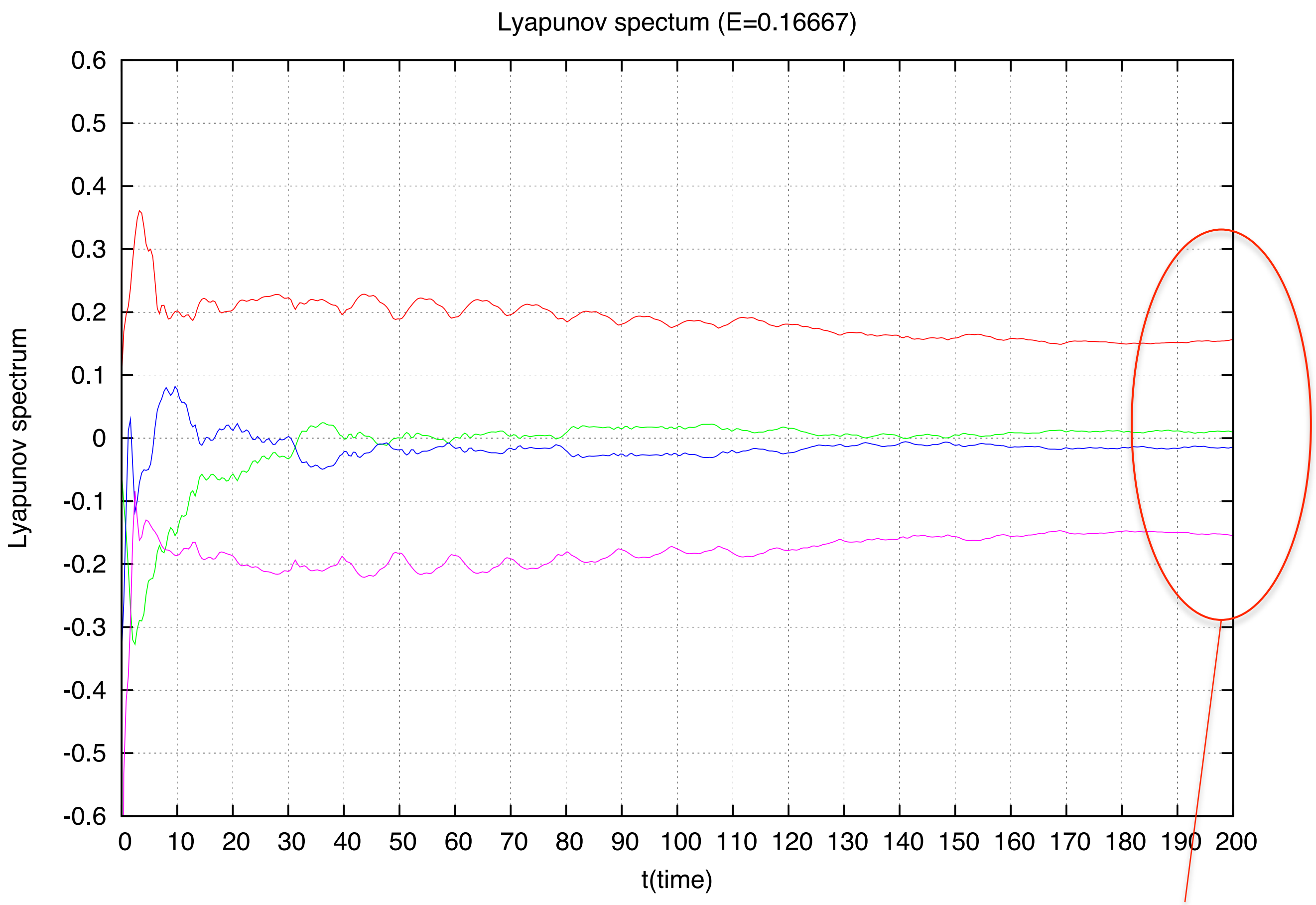
$$h = \sum_{i=1}^{M} \lambda_i$$ $\lambda_1, \lambda_2, \ldots, \lambda_M$ are positive Lyapunov spectra.

➡ Kolmogorov-Sinai Entropy and Lyapunov spectra are related by this equation.

Lyapunov spectra calculation reveals the existence of chaos and growth rate of information production.

Numerical result of Lyapunov spectra in the Henon-Heiles system

(with Shimada-Nagashima algorithm) [I. Shimada and T. Nagashima, Prog. Theor. Phys. 61 (1979) 1605]



Spectra can be read from long-time evolution.

The maximum index is positive ➡ The system has chaotic motion.

4. Chaotic strings in the $AdS_5 \times T^{1,1}$

String theory on $AdS_5 \times T^{1,1}$

The metric of $AdS_5 \times T^{1,1}$

$$ds^2 = R^2 (ds^2_{AdS_5} + ds^2_{T^{1,1}})$$

R : AdS radius

$$ds^2_{AdS_5} = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2$$

(AdS_5 part)

$$ds^2_{T^{1,1}} = \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \sum_{i=1}^{2}(d\theta_i^2 + \sin^2\theta_i d\phi_i^2)$$

($T^{1,1}$ part)

Polyakov Action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma G_{\mu\nu} \partial_a X^\mu \partial^a X^\nu$$

Virasoro constraint

$$G_{\mu\nu}\left(\partial_\tau X^\mu \partial_\tau X^\nu + \partial_\sigma X^\mu \partial_\sigma X^\nu\right) = 0, \quad G_{\mu\nu}\partial_\tau X^\mu \partial_\sigma X^\nu = 0$$

Type IIB super string on $AdS_5 \times T^{1,1}$ ⟷ (dual) 4D N=1 SCFT

[I. R. Klebanov and E. Witten, 9807080]

The work by Basu and Pando Zayas [arXiv:1103.4107]

→ It proves the existence of chaotic strings on $AdS_5 \times T^{1,1}$.

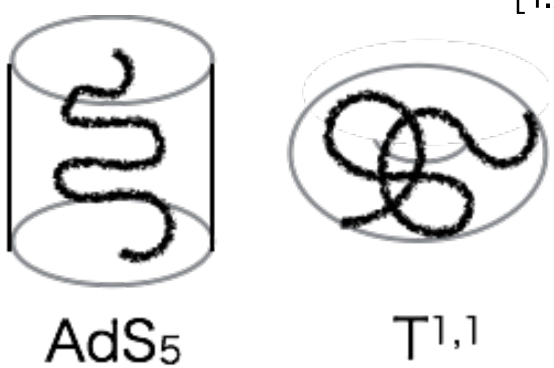
Question

Are chaotic srings persist in the near Penrose limit of $AdS_5 \times T^{1,1}$?

If chaotics srings exist, it means there exist dual near BPS operators in the dual gauge theory.

We revealed the existence of chaotic strings in the near Penrose limit.

[Y.Asano, DK, H.Kyono, K.YoshidaJHEP:08(2015)060, arXiv:1505.07583]



↔ dual

Unknown operator

= chaotic

Near Penrose limit of AdS$_5$×T^{1,1}

We redefine coordinates by

$$\widetilde{x}^+ \equiv t,\ \widetilde{x}^- \equiv -t + \frac{1}{3}(\psi + \phi_1 + \phi_2),\ \Phi_1 \equiv \phi_1 - t,\ \Phi_2 \equiv \phi_2 - t.$$

Rescale

$$\widetilde{x}^+ = x^+,\ \widetilde{x}^- = \frac{x^-}{R^2},\ \rho = \frac{r}{R},\ \theta_i = \sqrt{6}\frac{r_i}{R}.$$

Expand metric by order R^{-2} (near Penrose limit)

$$ds^2 = ds_0^2 + \frac{1}{R^2} ds_2^2 + \mathcal{O}\left(\frac{1}{R^4}\right)$$

ds_0^2 : pp-wave background ds_2^2 : correction term

$$ds_0^2 = 2dx^+dx^- - (r^2 + r_1^2 + r_2^r)(dx^+)^2 + dr^2 + r^2 d\Omega_3^2 + dr_1^2 + r_1^2 d\Phi_1^2 + dr_2^2 + r_2^2 d\Phi_2^2$$

$$ds_2^2 = \left(-\frac{1}{3}r^4 + 2r_1^2 r_2^r\right)(dx^+)^2 - 2(r_1^2 + r_2^2)dx^+dx^- + (dx^-)^2 + \frac{1}{3}r^4 d\Omega_3^2 + r_1^2(-r_1^2 + 2r_2^2)dx^+ d\Phi_1$$
$$+ r_2^2(-r_1^2 + 2r_2^2)dx^+ d\Phi_2 - 2r_1^2 dx^- d\Phi_1 - 2r_2^2 dx^- d\Phi_1 + 2r_1^2 r_2^2 d\Phi_1 d\Phi_2 - r^4 d\Phi_1^2 - r_2^4 d\Phi_2^2$$

Light-cone gauge Hamiltonian

Light-cone gauge $x^+ = \tau,\ p_- = const.$

$$H_{lc} \equiv -p_+ = -\frac{p_1 g^{+-}}{g^{++}} - \frac{1}{g^{++}}\sqrt{p_-^2 g - g^{++}\left(g^{--}\left(\frac{p_I x'^I}{p_-^2}\right)^2 + p_I p_J g^{IJ} + x'^I x'^J g_{IJ}\right)}$$

$$= \mathcal{H}_0 + \frac{1}{R^2}\mathcal{H}_{int} + \mathcal{O}\left(\frac{1}{R^4}\right)$$

[I.Swanson, arXiv:0505028]

Ansatz : $r = 0,\ p_r = 0,\ r_1 = r_1(\tau),\ p_{r1} = p_{r1}(\tau),\ r_2 = r_2(\tau),\ p_{r2} = p_{r2}(\tau),$

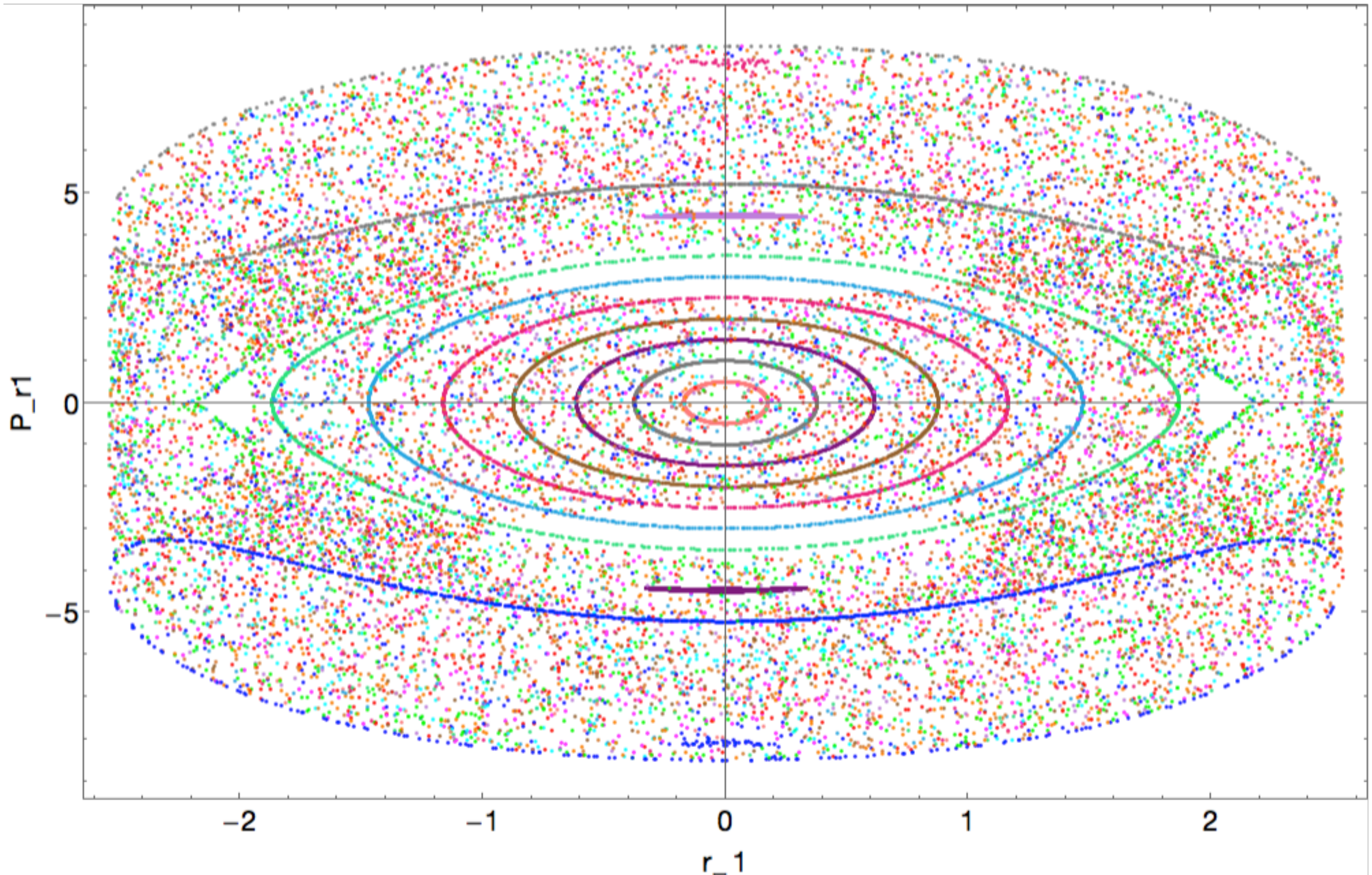
$\Phi_1 = \alpha_1 \sigma,\ p_{\Phi_1} = 0,\ \Phi_2 = \alpha_2 \sigma,\ p_{\Phi_2} = 0$

$$\mathcal{H}_0 = \frac{1}{2}\left[p_{r1}^2 + p_{r2}^2 + (1+\alpha_1^2) r_1^2 + (1+\alpha_2^2) r_2^2\right]$$

$$\mathcal{H}_{int} = -\frac{1}{8}\left[p_{r1}^2 + p_{r2}^2 + (1+\alpha_1^2) r_1^2 + (1+\alpha_2^2) r_2^2\right]^2 - \frac{1}{2}\left(\alpha_1 r_1^2 - \alpha_2 r_2^2\right)^2 + \frac{1}{2}\left(r_1^4 + r_2^4\right)$$

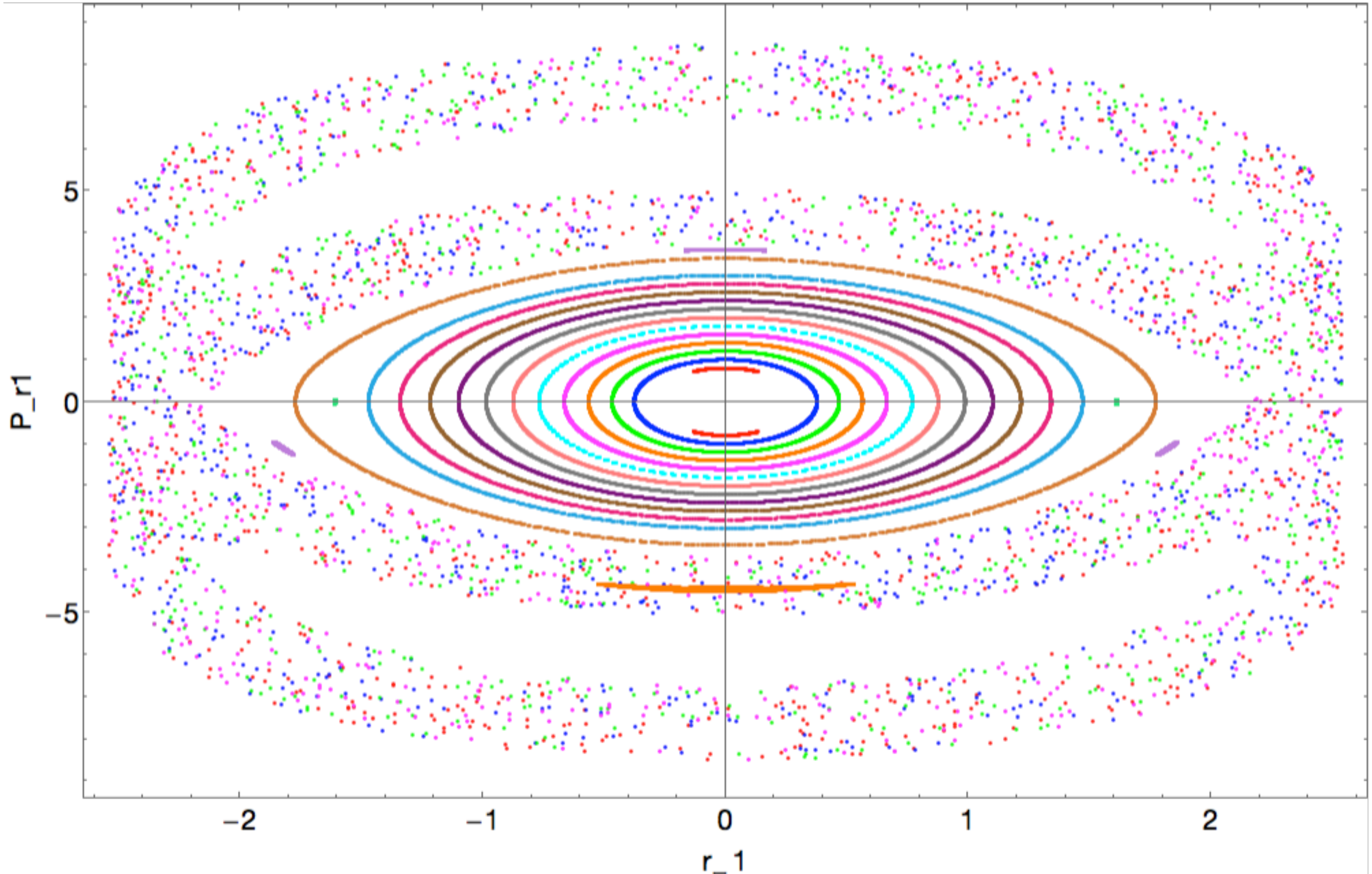
$\mathcal{H}_{int}$ have an influence as a source of chaotic strings.

Poincare section (E = 10)



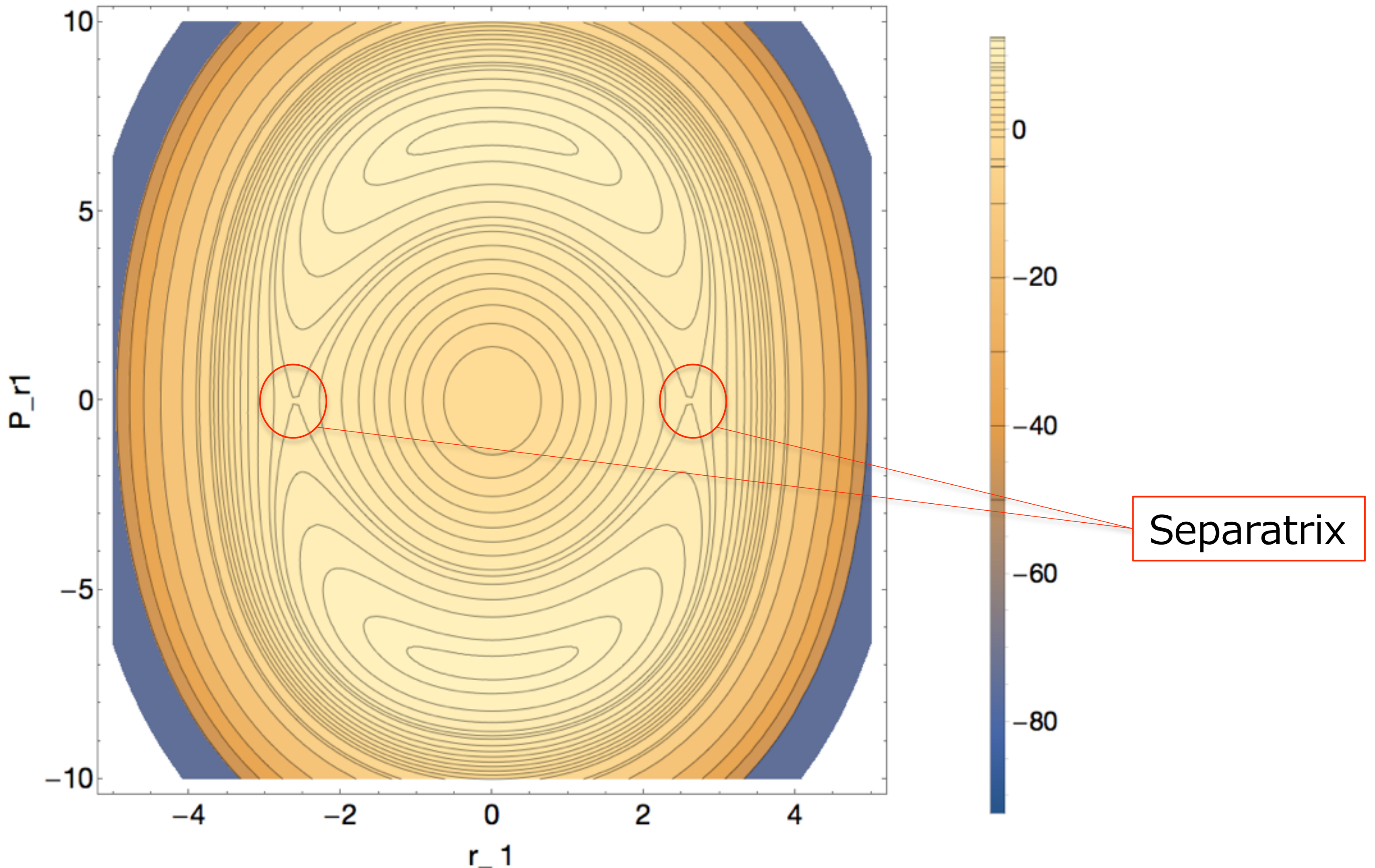
⁕ By quartic term, periodic orbits and chaotic one are mixed.

Poincare section (E = 10, $p_{r2} < 5.2$)



Orbital pattern is collapsed. ➡ Chaotics strings exist

Energy contour of AdS$_5$×T^{1,1} $(r_2 = 0, p_{r_2} = 0)$



There exist separatrices → They are the source of chaotic stings.

Summary

- We proved the existence of chaotic strings in near Penrose limit of $AdS_5 \times T^{1,1}$.

→ Separatrices serves as a source of chaos.

Discussion & Future works

- Understanding of chaos in dual gauge theories with

quantities specific to chaos
1. Kolmogorov-Sinai Entropy
2. Fractarl dimension

→ By these ones, we pursue the determination of dual operators.

The study of AdS/CFT by chaos → Understanding of non-integrable system