On skein relations in class S theory

towards $\mathcal{N}=4$ SYM, T_N & 4D-2D-1D applications

based on joint work arXiv:1504.00121 w/ Y.Tachikawa (Hongo, Univ. Tokyo) and other work in progress
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Introduction

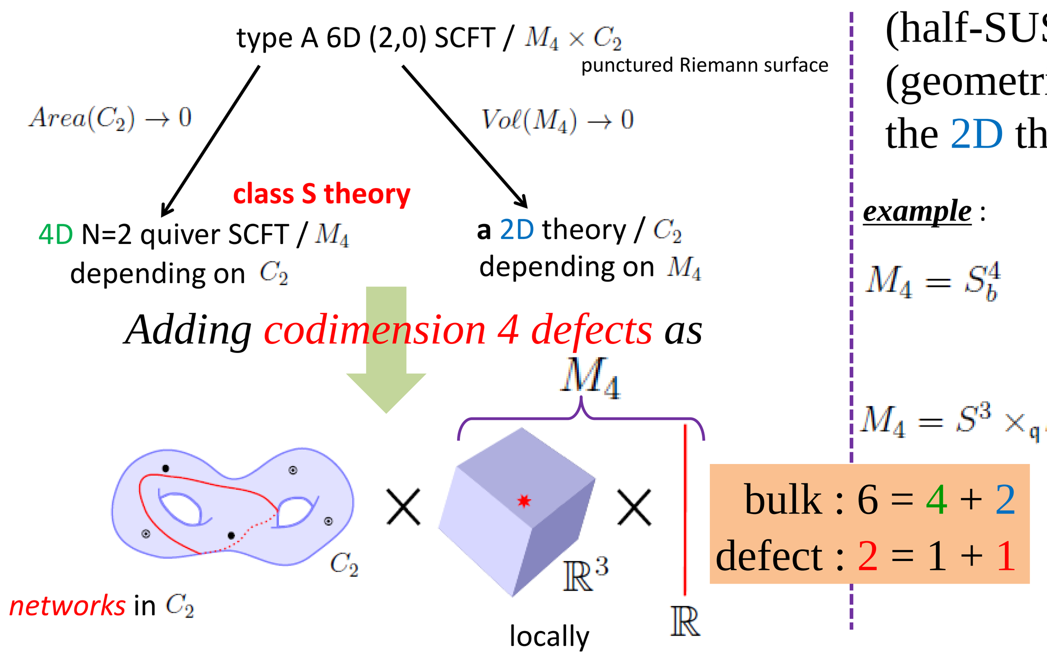
Motivation : Why defects ?

Defects in QFT naturally appear in brane systems.
(Ex. In 4D, local op., loop op., surface defect and domain wall…)

Here we focus on ***4D half-BPS Wilson-'t Hooft loops*** in SCQCD (at first) w/ 8SUSY's…

Physical meaning : Insert a heavy (non-dynamical) dyon and see a response as the result of gauge interactions with charged dynamical matters (sometimes play a role of order parameter)

Set-up : Class S Theory & 4D-2D duality



(half-SUSY-protected) "observables" in 4D are (geometrically) described by some quantities in the 2D theories and vice versa.
Alday-Gaiotto-Tachikawa '09 …
Gadde-Pomoni-Rastelli-Razamat '09 …

example :	4D gauge theory		2D "topological" theory
$M_4 = S^4_b$	S^4 partition function (Nekrasov hol. inst. p.f.)	=	Liouville /Toda CFT correlation functions (conformal block)
	loop operators	=	Verlinde operators
$M_4 = S^3 \times_q S^1$	superconformal index	=	topological qYM correlator (Schur limit)
	loop operators	=	Wilson operators

4D loop data : irreducible rep's of gauge type + "geometry" in C_2
label of codim. 4 op.
"geometry" = **network in the compactified C_2**

Key concepts : OPEs & Skein relations

4D dyonic loop operators : charges ⟶ value ("function")
What we want to establish : ↕ related — coincide ↕ [4D/2D says]
2D Verlinde/Wilson operators : networks ⟶ value ("function")

skein relations — network 1 ≀ network 2 ≀ network 3 ⟶ same charge
∪
crossing resolution (2D concept) = OPE of loops (4D concept)

the product of dyonic charges ⟷ the product of the loop operators (**OPE**)
↕
the product of "network" : what crossing resolutions say

The OPE of loops is determined locally
⟶ Crossing resolution should be universal ! (or common both in ***CFT*** and in ***qYM***)

Review & Questions

Charge data of Wilson-'t Hooft loops for single gauge group

UV CFT definition : Kapustin '05

GNO magnetic charge $B \in \mathfrak{h} \subset \mathfrak{g}$ — (B, R_B) — *more useful way* ⟶ $[(\mu,\lambda)] \in (\Lambda^{\mathfrak{g}}_{mw} \times \Lambda^{\mathfrak{g}}_{wt})/\mathcal{W}_{\mathfrak{g}}$

Wilson loop's charge R_B : irr. rep. of $\mathfrak{g}_B$, $\mathfrak{g}_B := \{X \in \mathfrak{g} | [X,B]=0\}$

$(\mu,\lambda) \sim (w\cdot\mu, w\cdot\lambda)$ if $\exists\, w \in \mathcal{W}_{\mathfrak{g}}$

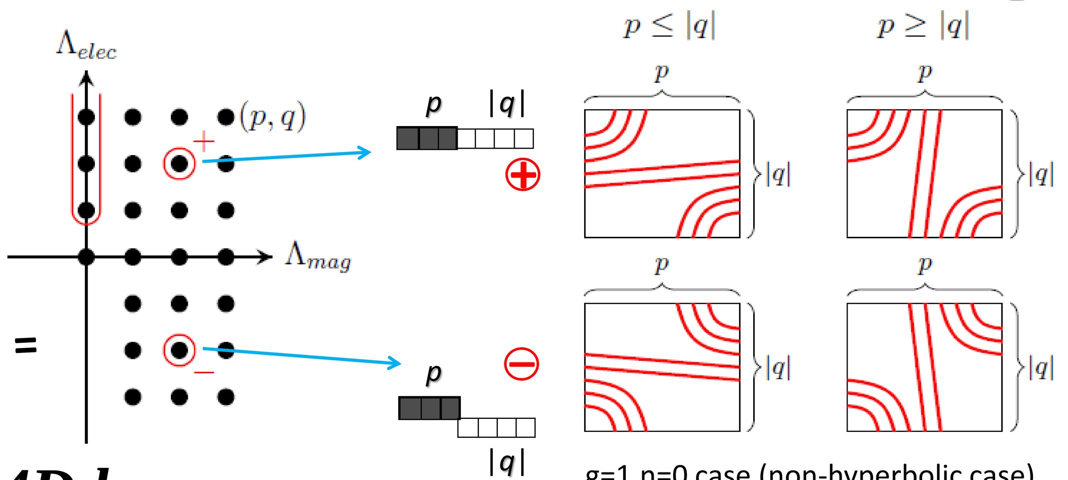
magnetic weight lattice $\Lambda^{\mathfrak{g}}_{mw} = \{B \in \mathfrak{h} \mid \lambda(B) \in \mathbb{Z} \text{ for } \forall \lambda \in \Lambda^{\mathfrak{g}}_{rt} \subset \mathfrak{h}^\vee\}$

Drukker-Morrison-Okuda's claim ('09)

There is a natural *one-to-one correspondence* between the charge lattice of Wilson-'t Hooft loops in A_1-type class S theories and non-intersecting unoriented loops (lamination) in C_2 .

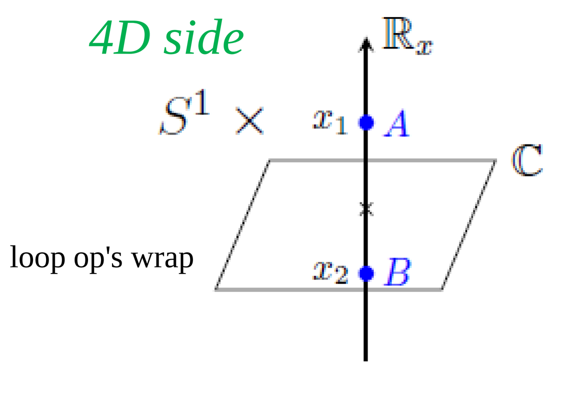
Ex : $\mathcal{N}$ = 4 su(2) SYM

Kapustin's charge set of $\mathcal{N}$ = 4: $\Lambda^{su(2)} \times \Lambda^{su(2)}/\mathcal{W}_{su(2)}$ =



Ordering of half-BPS 4D loops

- Two loop operators do not commute in general
- The origin of the non-cummutativity comes from half-SUSY preserving + Poynting vector



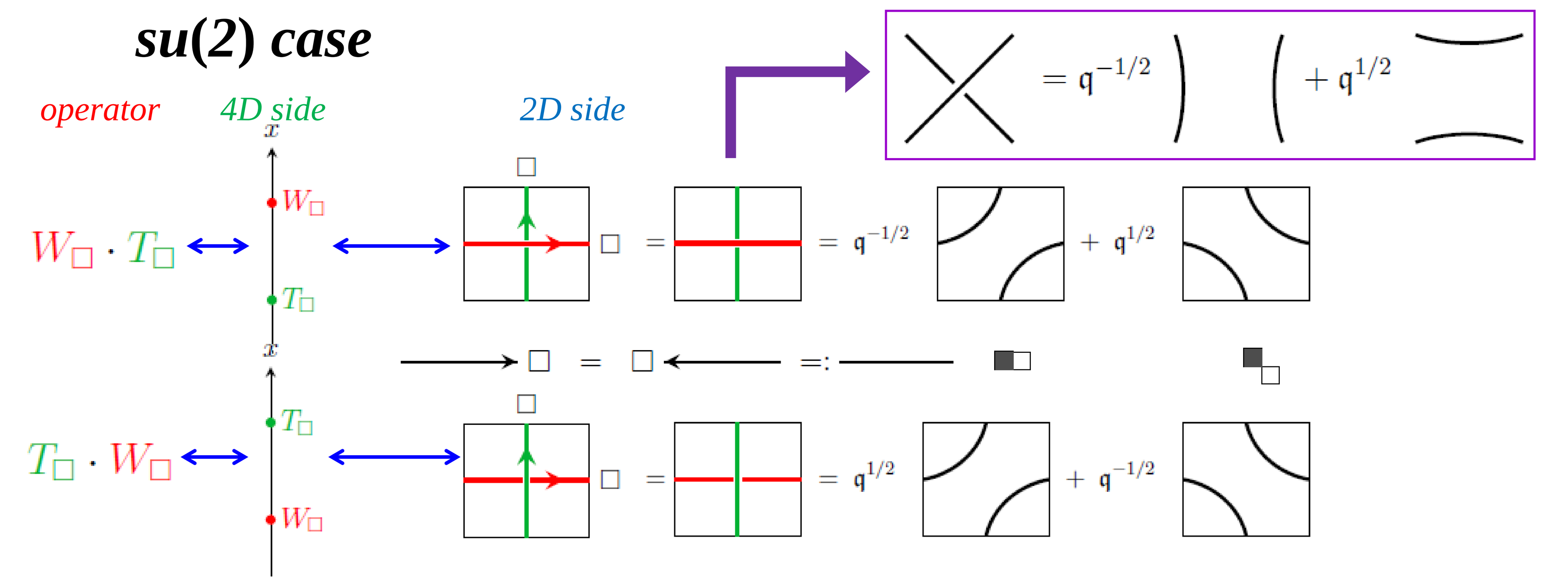
definition of ordering (locally, index with fugacity **q**)

$$\langle \ldots L_A(x_1) L_B(x_2)\rangle := \mathrm{Tr}_{\mathcal{H}^{BPS}_{\text{lines}}}\left[(-1)^F \mathsf{q}^{2\mathcal{J}}\right]$$

non-commutativity

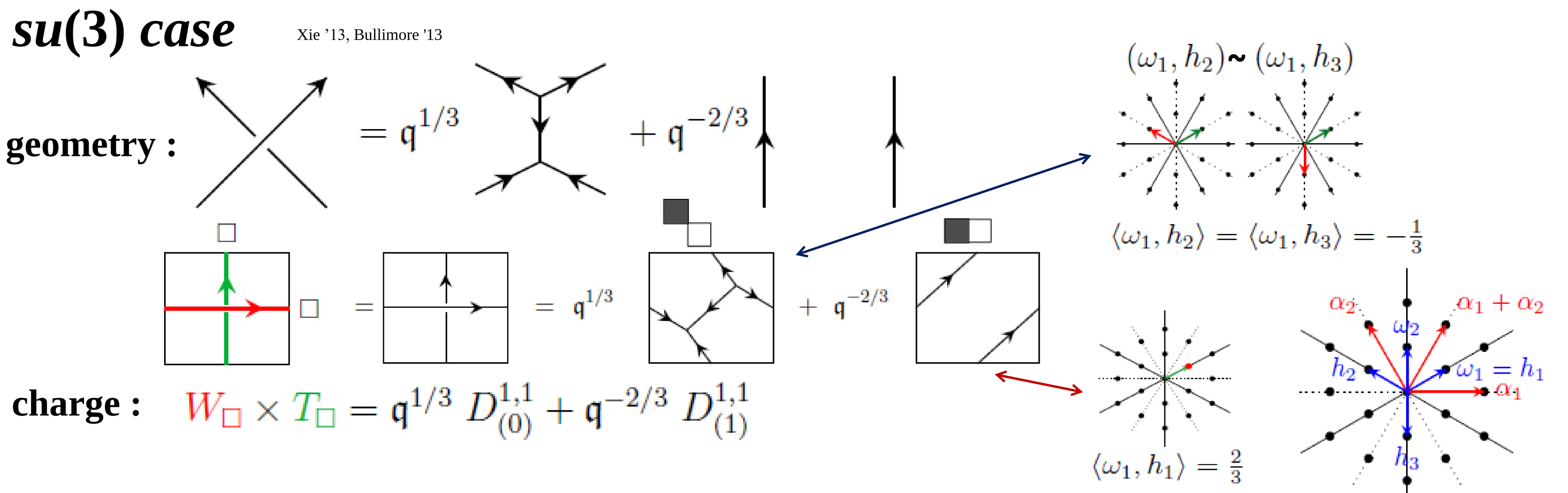
$$\langle \ldots L_A(x_1) L_B(x_2)\rangle \neq \langle \ldots L_B(x_1) L_A(x_2)\rangle$$

su(2) case



su(3) case

Xie '13, Bullimore '13



Q

✓ *How many skein relations exist in general rank ?*
✓ *In particular , what are the crossing resolutions ?*

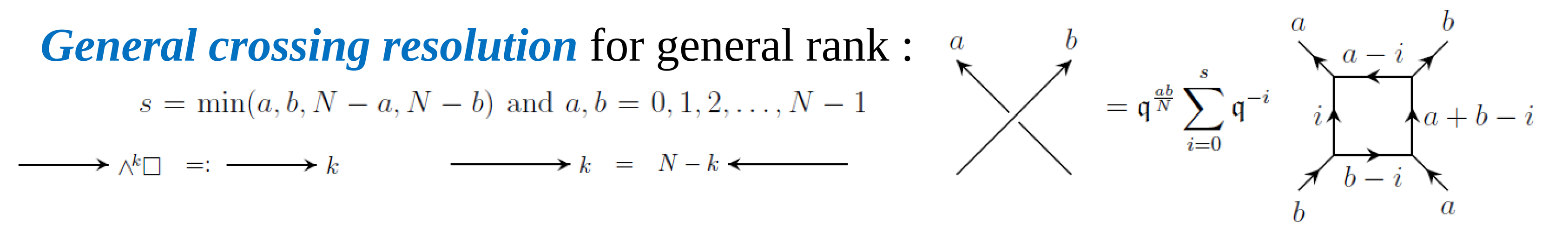
Proposals (Generalization & Consistency)

Notation

① Some *junctions* are allowed but only *trivalent*
② All *crossings* should be resolved into junctions w/o any crossings
③ Each edge carries one *fundamental representation* i.e. one of (0,)1,2,…,N-1

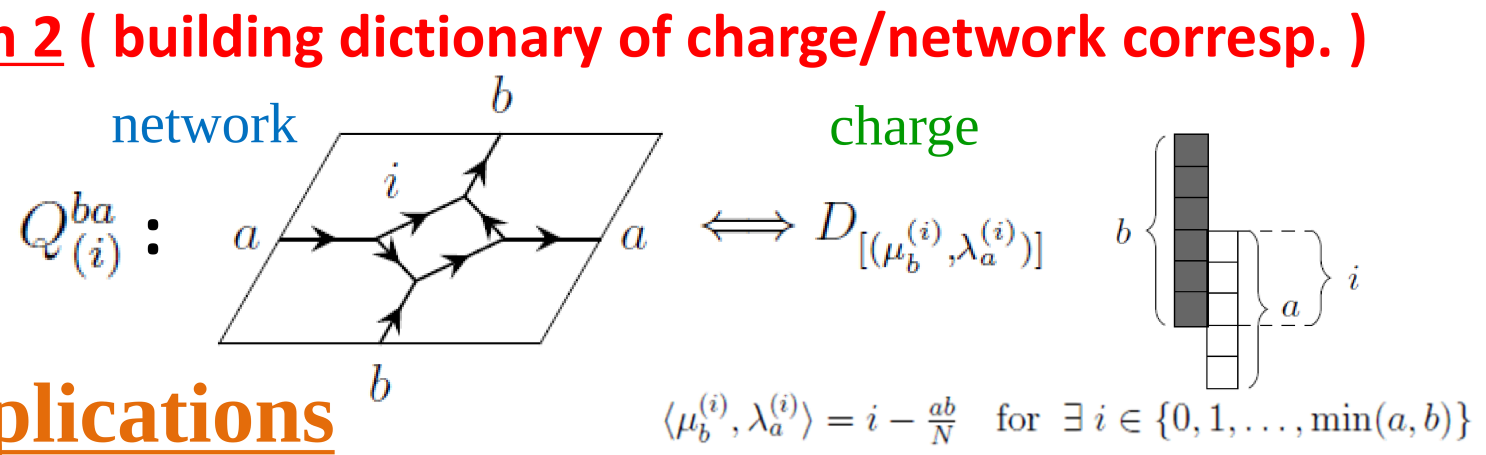
claim 1 (*OPE* = MOY rel.)

General crossing resolution for general rank :



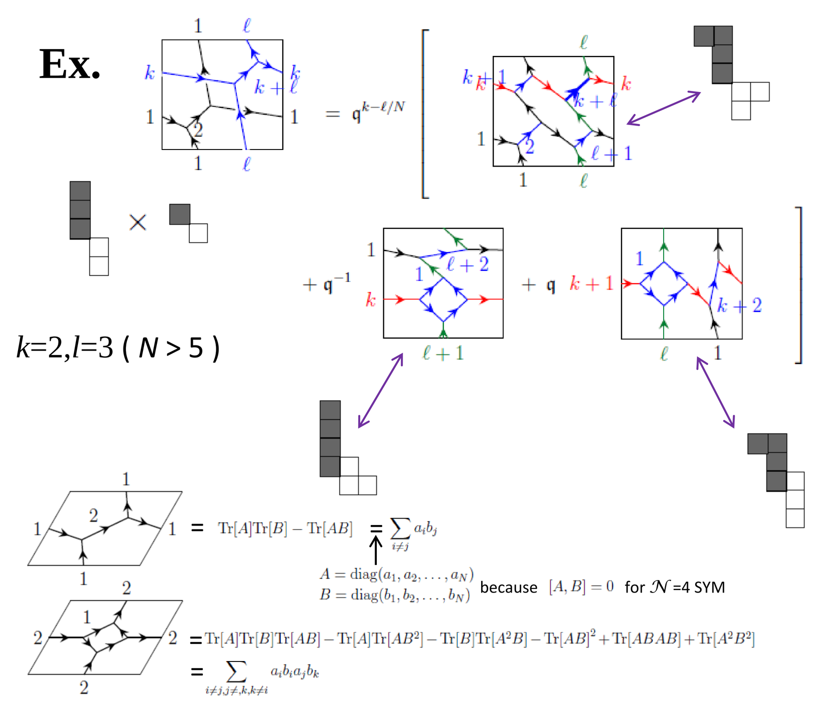
We find that the general resolution representing OPEs is same as the Murakami-Ohtsuki-Yamada's relation.

claim 2 (building dictionary of charge/network corresp.)



Applications

➢ *Geometrical Computation of OPEs*



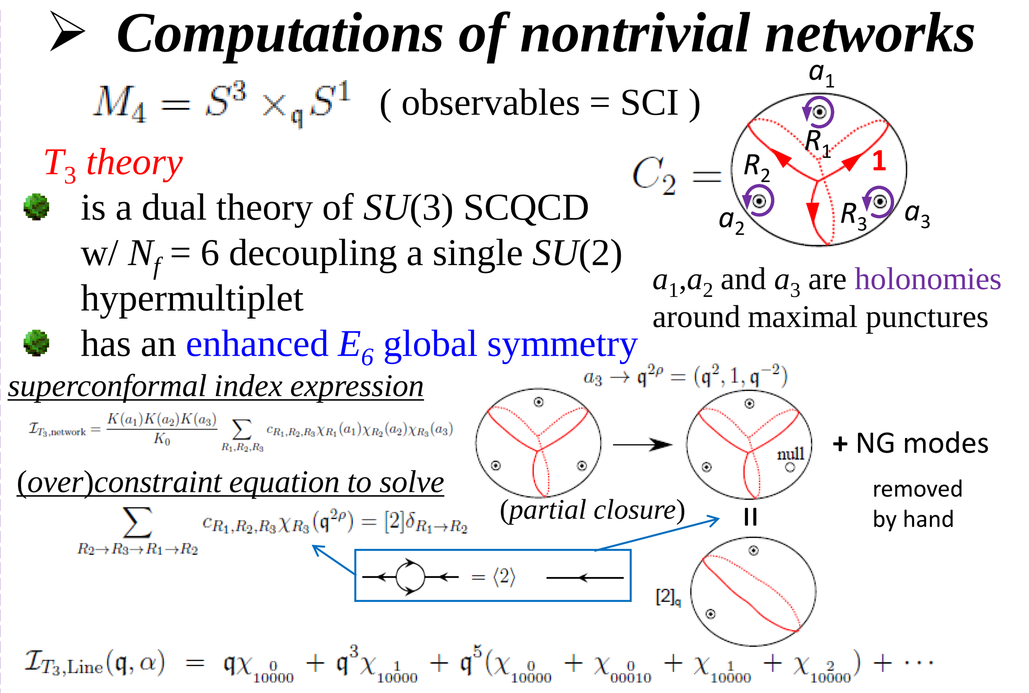
➢ *Computations of nontrivial networks*

$M_4 = S^3 \times_q S^1$ (observables = SCI)

T_3 theory
- is a dual theory of $SU(3)$ SCQCD w/ N_f = 6 decoupling a single $SU(2)$ hypermultiplet
- has an enhanced E_6 global symmetry

a_1, a_2 and a_3 are holonomies around maximal punctures

superconformal index expression
(over)constraint equation to solve (partial closure)
+ NG modes removed by hand



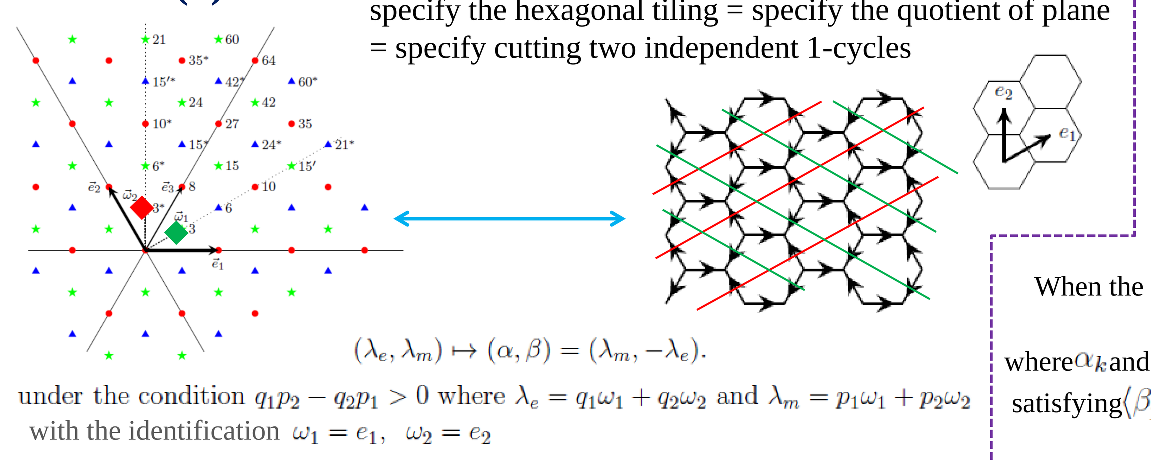
$$\mathcal{I}_{T_3,\text{Line}}(\mathfrak{q},\alpha) = \mathfrak{q}\chi_{10000} + \mathfrak{q}^3\chi_{10000} + \mathfrak{q}^5(\chi_{10000} + \chi_{00010} + \chi_{10000} + \chi_{10000}) + \cdots$$

the characters of E_6 appear again !

➢ *Hints for descriptions of 4D charges and 2D interface*

$\mathcal{N}$ = 4 su(3) SYM

specify the hexagonal tiling = specify the quotient of plane = specify cutting two independent 1-cycles



$\mathcal{N}$ = 4 su(N) SYM

the charge is the leading one in the network like

$Q^{p_1q_1}_{(i_{11})}$	$Q^{p_2q_1}_{(i_{21})}$	$Q^{p_3q_1}_{(i_{31})}$
$Q^{p_1q_2}_{(i_{12})}$	$Q^{p_2q_2}_{(i_{22})}$	$Q^{p_3q_2}_{(i_{32})}$
$Q^{p_1q_3}_{(i_{13})}$	$Q^{p_2q_3}_{(i_{23})}$	$Q^{p_3q_3}_{(i_{33})}$

When the dyonic charge is given by $\lambda_e = \alpha_{q_1} + \alpha_{q_2} + \alpha_{q_3}$, $\mu_m = \beta_{p_1} + \beta_{p_2} + \beta_{p_3}$

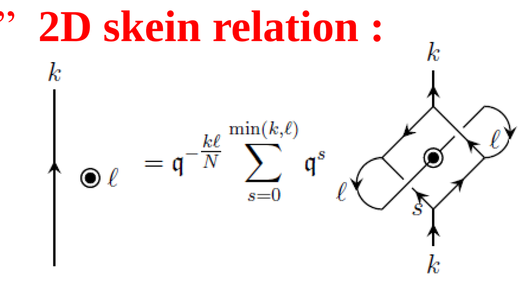
where α_k and β_l are weights of k-th and l-th antisym. rep's respectively satisfying $\langle \beta_{p_u}, \alpha_{q_v}\rangle = i_{uv} - \frac{p_u q_v}{N}$

extension of skein relations to include codim 4. origin surface operator

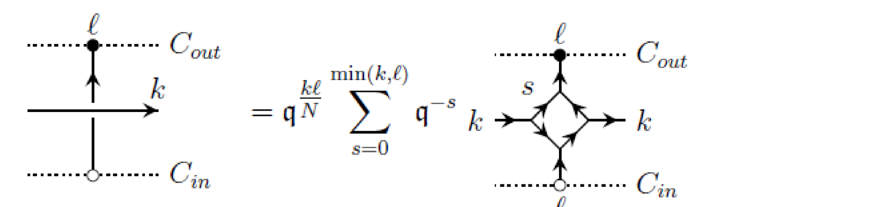
Our claim : 4D-2D coupled systems with loop operators are geometrically described by networks in 3D consisting of 2D C_2 and 1D in 4D M_4

2D bulk > networks in "3D" = open networks on 2D
↕ *extension of 4D/2D duality*
4D bulk > 2D vortex > 1D loop in 4D = interface on 2D

"New" 2D skein relation :



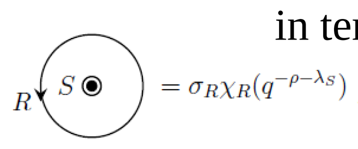
✓ This is the same as above if we apply it like



✓ *qYM operator expression when the network wraps :*

$$\hat{\mathcal{O}}^{(k,\ell)}_s := \sum_{(\lambda,\mu)\in\Pi(\wedge^k\square, \wedge^\ell\square)_s} a^{\lambda/2}\Delta_{-\mu} a^{\lambda/2}$$

✓ reproduce the following rel. in terms of skein rel's



Open questions

✓ The above applications are far from completeness …
- the precise dictionary between charges and networks
- dyonic version of irreducible rep.'s (dyon screening ?)
- general computations of indices (based on quantum groups)
- How to define such interfaces on 2D $\mathcal{N}$ = (2,2) SQCD on surface defects ?

✓ How to define open networks and classify them… ?
✓ Relations to stringy descriptions (MQCD + M2's) …