Exact large N partition function of (non-)conformally deformed ABJM theory

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based on: [Moriyama-TN, 1410.4918]
[TN, 1511.xxxx]

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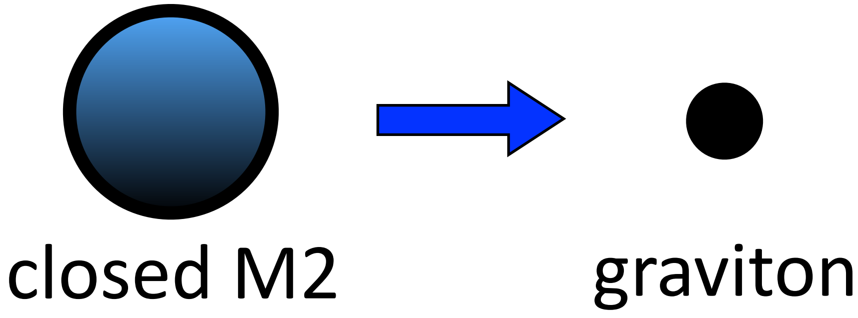
A motivation: Instantons in M-theory

- M-theory = Theory of Membranes



- A closed M2 brane can also **wind** compact cycles in space-time.

Winding on a 3-cycle = **"instanton"**

- Studied in worldvolume theory = probe approximation
[Becker-Becker-Strominger][Cagnazzo-Sorokin-Wulff][Drukker-Marino-Putrov]

- backreactions and interactions are difficult to study in gravity side.
[Naghdi][Ferreira]

- In gauge theory side of $\mathrm{AdS}_4/\mathrm{CFT}_3$

instanton ➡ $e^{-\mathrm{vol(M2)}} \sim e^{-R^3_{\mathrm{AdS}}} \sim e^{-\sqrt{N}}$

Fermi gas formalism gives such non-perturbative effects **exactly** & **directly**

Outline

1. U(N) circular quiver superconformal Chern-Simons theory

   (= discrete deformation of ABJM theory)

2. non-conformal deformation

   (= continuous deformation of ABJM theory)

3. Summary

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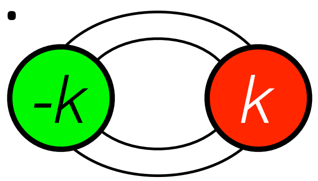
U(N) circular quiver SCCS

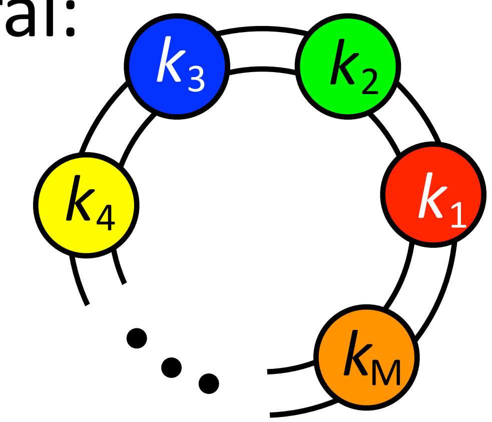
vertex → Chern-Simons vector multiplet with level k_i

edge → two bifundamental hypermultiplets

ABJM:



general:



- large N limit were studied exhaustively

  [Suyama][Drukker-Marino-Putrov][Herzog-Klebanov-Pufu-Tesileanu]…

- $\log Z \sim N^{3/2} \Longrightarrow \sum_i k_i = 0$ [Gulotta-Ang-Herzog]

  ➡ Numbers $\{s_i\}$ should be assigned on edges so that $k_i = \frac{k}{2}(s_i - s_{i-1})$

U(*N*) circular quiver SCCS

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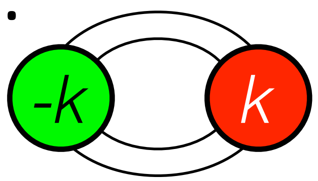
ABJM:



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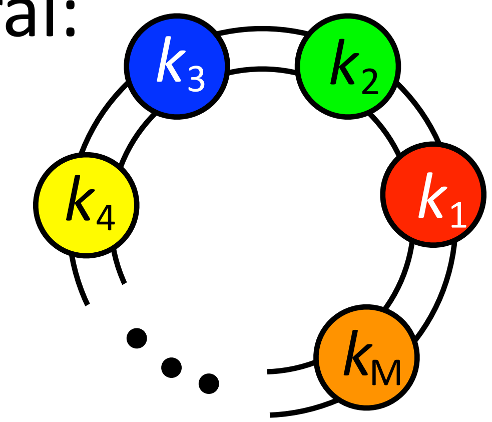

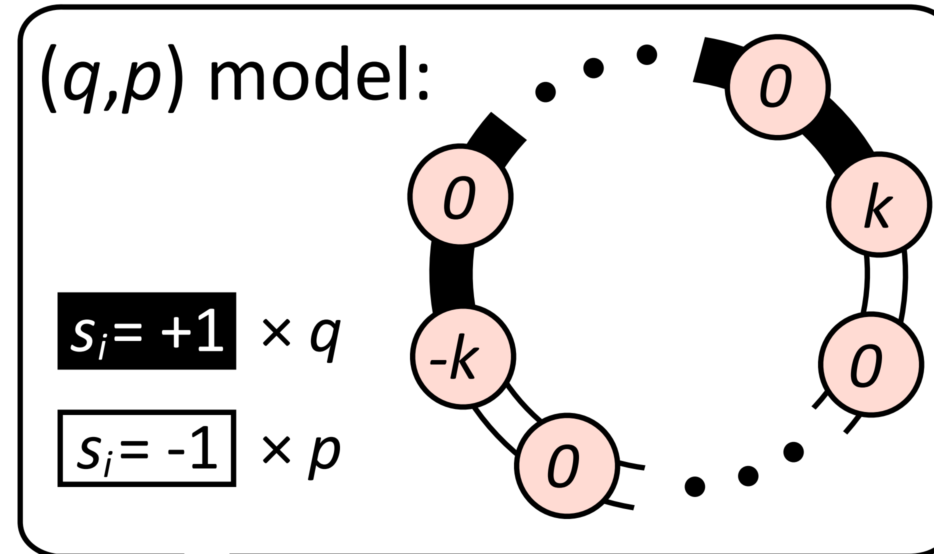

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- For simplificty we consider $\{s_i\}_{i=1}^{q+p} = \{(+1)^q, (-1)^p\}$ [Imamura-Kimura]

  ➡ dual to $\mathrm{AdS}_4 \times S^7/\Gamma_{q,p}$ $(\mathbb{C}^4/\Gamma_{q,p} = (\mathbb{C}^2/\mathbb{Z}_q \times \mathbb{C}^2/\mathbb{Z}_p)/\mathbb{Z}_k)$

Fermi Gas formalism for Partition Function [Marino-Putrov]

$$e^{J(\mu)} = \sum_{N\geq 0} e^{\mu N} Z(N) = \det(1 + e^{\mu - \widehat{H}})$$

- Inverse trsf.: $Z(N) = \int \frac{d\mu}{2\pi i} e^{J - \mu N}$

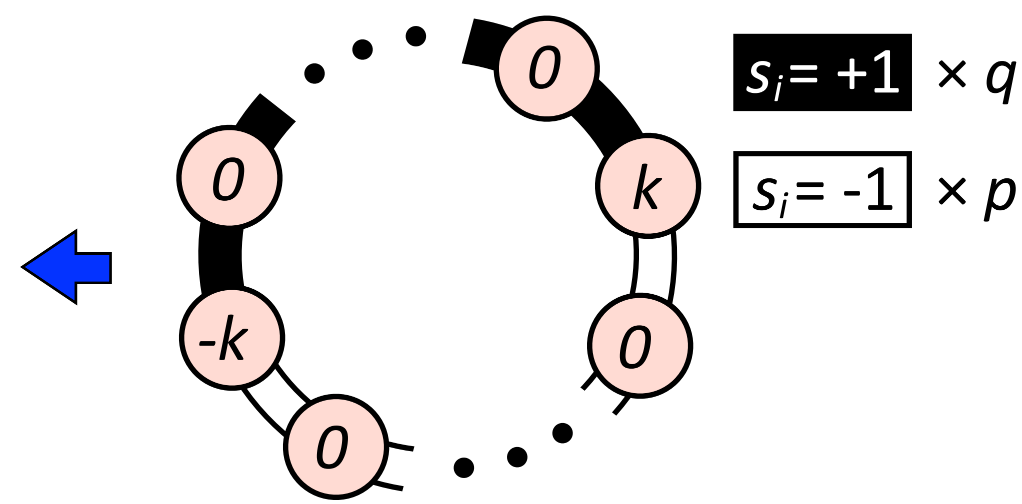
$\mu \sim \sqrt{N}$ so that $Z(N) \approx e^{-\bigcirc N^{3/2}}$

➡ instantons = $\mathcal{O}(e^{-\mu})$ effects in large μ expansion

- $e^{-\widehat{H}}$: one-particle density matrix

$$e^{-\widehat{H}} = \left(2\cosh\frac{\widehat{Q}}{2}\right)^{-q}\left(2\cosh\frac{\widehat{P}}{2}\right)^{-p}$$

$$[\widehat{Q}, \widehat{P}] = i\hbar \quad (\hbar = 2\pi k)$$



➡ $J(\mu)$ can be computed by small *k*-expansion

$k \to 0$: "classical limit"

$$J(\mu) = \mathrm{Tr} \log \left[1 + e^{\mu} \frac{1}{\left(2 \cosh \frac{\widehat{Q}}{2}\right)^{q} \left(\cosh \frac{\widehat{P}}{2}\right)^{p}} \right]$$

$$\widehat{Q}, \widehat{P} \Rightarrow Q, P$$

$$\mathrm{Tr} \Rightarrow \int \frac{dQdP}{2\pi\hbar}$$

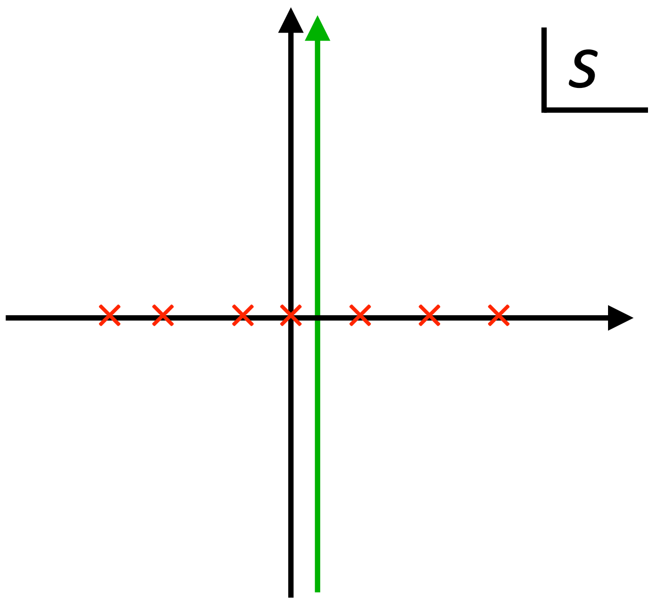
$$J(\mu) = \frac{1}{2\pi\hbar} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{n\mu} \frac{\Gamma\left(\frac{nq}{2}\right)^2 \Gamma\left(\frac{np}{2}\right)^2}{\Gamma(nq)\Gamma(np)}$$

- We have to rewrite small e^{μ} expansion into large e^{μ} expansion.

How to get large μ expansion

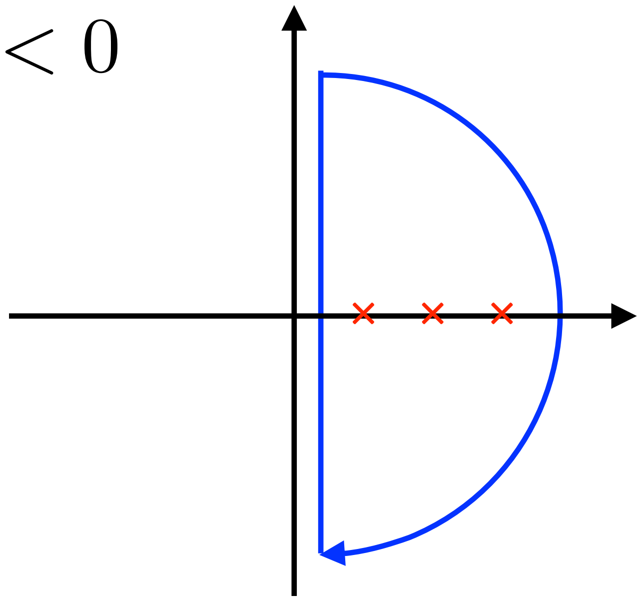
- Mellin-Barnes representation [Hatsuda]

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{n\mu}\mathcal{Z}(n) = -\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{ds}{2\pi i}\Gamma(s)\Gamma(-s)e^{s\mu}\mathcal{Z}(s)$$



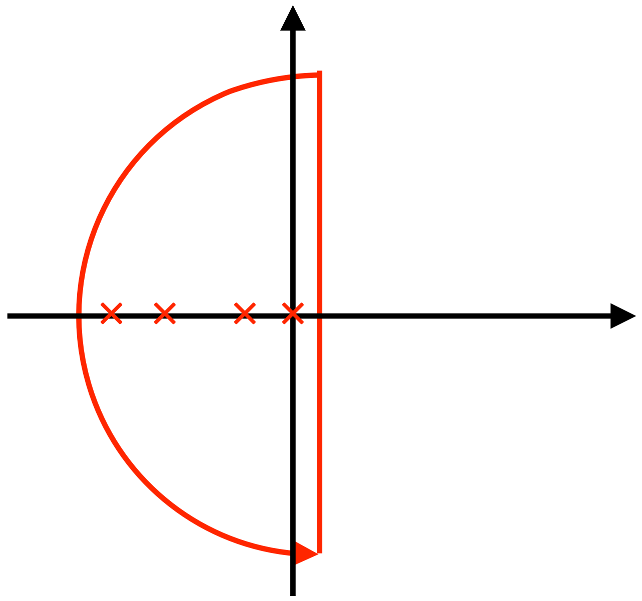
$$\mathcal{Z}(s) = \frac{1}{2\pi\hbar}\frac{\Gamma(\frac{sq}{2})^2\Gamma(\frac{sp}{2})^2}{\Gamma(sq)\Gamma(sp)}$$

① $\mu < 0$



- $s = 1, 2, \cdots$ ➡ l.h.s.

② $\mu > 0$



- $s = 0$ ➡ polynomial in μ

($N^{3/2}$ + perturbative corrections)

- $s < 0$ ➡ $e^{-\bigcirc\mu}$: instantons

Results [Moriyama-TN]

winding on $S^7/\Gamma_{q,p}$:

$$J(\mu) = J^{\text{pert}} + \frac{1}{\hbar}\Bigg[\sum_{m_1\ge 1} \binom{2m_1}{m_1} \frac{1}{m_1 \sin\frac{2\pi m_1}{q}} \frac{\Gamma\left(-\frac{pm_1}{q}\right)^2}{\Gamma\left(-\frac{2bm_1}{q}\right)} e^{-\frac{2m_1\mu}{q}}$$

$S^1/\mathbb{Z}_q$

$$+ \sum_{m_2\ge 1} \binom{2m_2}{m_2} \frac{1}{m_2 \sin\frac{2\pi m_2}{p}} \frac{\Gamma\left(-\frac{qm_2}{p}\right)^2}{\Gamma\left(-\frac{2qm_2}{p}\right)} e^{-\frac{2m_2\mu}{p}}$$

$S^1/\mathbb{Z}_p$

$$+ \sum_{m_3\ge 1} \frac{(-1)^{m_3-1}}{2\pi m_3} \frac{\Gamma\left(-\frac{qm_3}{2}\right)^2 \Gamma\left(-\frac{pm_3}{2}\right)^2}{\Gamma(-qm_3)\Gamma(-pm_3)} e^{-m_3\mu} \Bigg]$$

???

$(\mathbb{C}^4/\Gamma_{q,p} = (\mathbb{C}^2/\mathbb{Z}_q \times \mathbb{C}^2/\mathbb{Z}_p)/\mathbb{Z}_k)$

Higher order in semiclassical expansion: $\mathcal{Z}(s) \to \frac{f(s)}{g(s)} \mathcal{Z}(s)$ (*f*,*g*: polynomials)

➡ This structure persist for **finite *k***

General properties of instanton effects:

- exponents modded by orbifold
- coefficients have ***poles*** at $q, p \in \mathbb{N}$ ➡ New hints for instantons in gravity?

Does "divergence" really meaningful ?

$$J(\mu) = J^{\rm pert} + \frac{1}{\hbar}\Bigg[\sum_{m_1 \geq 1} \binom{2m_1}{m_1} \frac{1}{m_1 \sin\frac{2\pi m_1}{q}} \frac{\Gamma\left(-\frac{pm_1}{q}\right)^2}{\Gamma\left(-\frac{2pm_1}{q}\right)} e^{-\frac{2m_1\mu}{q}} + \sum_{m_2 \geq 1} \binom{2m_2}{m_2} \frac{1}{m_2 \sin\frac{2\pi m_2}{p}} \frac{\Gamma\left(-\frac{qm_2}{p}\right)^2}{\Gamma\left(-\frac{2qm_2}{p}\right)} e^{-\frac{2m_2\mu}{p}} + \sum_{m_3 \geq 1} \frac{(-1)^{m_3-1}}{2\pi m_3} \frac{\Gamma\left(-\frac{qm_3}{2}\right)^2 \Gamma\left(-\frac{pm_3}{2}\right)^2}{\Gamma(-qm_3)\Gamma(-pm_3)} e^{-m_3\mu} \Bigg]$$

- **Actual** instanton coefficients are always finite

$$q = 2, p = 3 \;\Rightarrow\; \frac{4\Gamma\left(\frac{1}{3}\right)^2}{\sqrt{3}\Gamma\left(\frac{2}{3}\right)} e^{-\frac{2\mu}{3}} - 64 e^{-\mu} + \frac{30\sqrt{3}\Gamma\left(\frac{2}{3}\right)^2}{\Gamma\left(\frac{1}{3}\right)} e^{-\frac{4\mu}{3}} + \left(\frac{-120\mu^2 + 604\mu + 469}{3\pi} + 30\pi \right) e^{-2\mu} + \cdots$$

: due to the cancellation of pairwise divergences

- Poles are visible only after the deformation $q \to q + \epsilon \notin \mathbb{N}$

➡ nonsense in gravity side !

> Can we continuously deform theory,
> keeping this instanton structure (& tractability)?

➡ R-charge deformation

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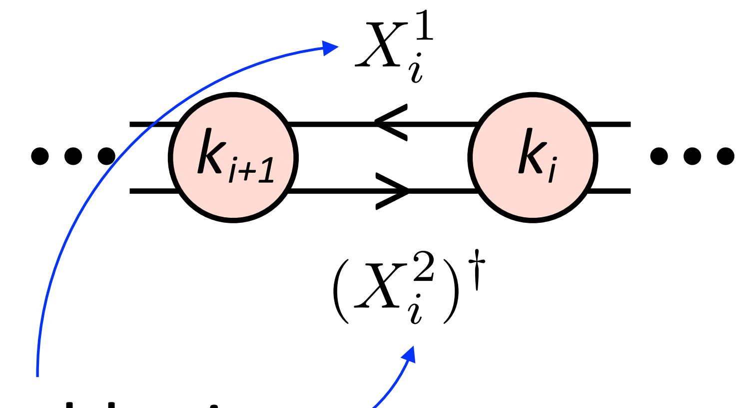
Example: general R-charge assignment



- Turn on background gauge multiplet for SU(2) $_i$

$$(A'_{i,\mu}, \sigma'_i, D'_i, \lambda'_i) = \left(0, -\frac{i\xi_i}{r}\sigma^3, -\frac{\xi_i}{r^2}\sigma^3, 0\right)$$

$$[\delta, \widetilde{\delta}]X_i^\alpha = -(i\epsilon\gamma^\mu\widetilde{\epsilon})D_\mu X_i^\alpha + \epsilon\widetilde{\epsilon}\left(i\sigma + \frac{1}{2r}\right)X_i^\alpha$$

$$\Rightarrow \begin{cases} [\delta, \widetilde{\delta}]X_i^1 & = -(i\epsilon\gamma^\mu\widetilde{\epsilon})D_\mu X_i^1 + \epsilon\widetilde{\epsilon}\left[i\sigma + \frac{1}{r}\left(\frac{1}{2}+\xi_i\right)\right]X_i^1 \\ [\delta, \widetilde{\delta}]X_i^2 & = -(i\epsilon\gamma^\mu\widetilde{\epsilon})D_\mu X_i^2 + \epsilon\widetilde{\epsilon}\left[i\sigma + \frac{1}{r}\left(\frac{1}{2}-\xi_i\right)\right]X_i^2 \end{cases}$$

[Festuccia-Seiberg]

- continuous deformation
- Z(N) (localization) do not change drastically

$$Z^{\text{1-loop}}_{\text{hyper},i} = \prod_{a,b=1}^{N} \frac{1}{2\cosh\frac{\sigma_i^a - \sigma_{i+1}^b - \pi i \xi_i}{2}}$$

⇒ Fermi gas formalism still exist

Results [TN]

- Fermi gas formalism is still varid, with $e^{-\widehat{H}} = \frac{e^{\frac{\xi \widehat{Q}}{4}}}{(2\cosh\frac{\widehat{Q}}{2})^{q/2}} \frac{e^{\frac{\eta \widehat{P}}{4}}}{(2\cosh\frac{\widehat{P}}{2})^{p/2}}$

$$J(\mu) = J^{\rm pert} + \frac{1}{\hbar}\left[\sum_{n=1}^{\infty} c_n^{(1)} e^{-\frac{2n}{q+\xi}} + \sum_{n=1}^{\infty} c_n^{(2)} e^{-\frac{2n}{q-\xi}} + \sum_{n=1}^{\infty} c_n^{(3)} e^{-\frac{2n}{q+\eta}} + \sum_{n=1}^{\infty} c_n^{(4)} e^{-\frac{2n}{q-\eta}} + \sum_{n=1}^{\infty} c_n^{(5)} e^{-n\mu}\right]$$

$$c_n^{(1)} = \frac{(-1)^{n-1}}{\pi n!(q+\xi)} \frac{\Gamma\left(-\frac{2n}{q+\xi}\right)\Gamma\left(\frac{2n}{q+\xi}\right)\Gamma\left(-\frac{q-\xi}{q+\xi}n\right)\Gamma\left(-\frac{p+\eta}{q+\xi}n\right)\Gamma\left(-\frac{p-\eta}{q+\xi}n\right)}{\Gamma\left(-\frac{2q}{q+\xi}n\right)\Gamma\left(-\frac{2p}{q+\xi}n\right)}$$

$$c_n^{(2)} = \frac{(-1)^{n-1}}{\pi n!(q-\xi)} \frac{\Gamma\left(-\frac{2n}{q-\xi}\right)\Gamma\left(\frac{2n}{q-\xi}\right)\Gamma\left(-\frac{q+\xi}{q-\xi}n\right)\Gamma\left(-\frac{p+\eta}{q-\xi}n\right)\Gamma\left(-\frac{p-\eta}{q-\xi}n\right)}{\Gamma\left(-\frac{2q}{q-\xi}n\right)\Gamma\left(-\frac{2p}{q-\xi}n\right)}$$

$$c_n^{(3)} = \frac{(-1)^{n-1}}{\pi n!(p+\eta)} \frac{\Gamma\left(-\frac{2n}{p+\eta}\right)\Gamma\left(\frac{2n}{p+\eta}\right)\Gamma\left(-\frac{p-\eta}{p+\eta}n\right)\Gamma\left(-\frac{q+\xi}{p+\eta}n\right)\Gamma\left(-\frac{q-\xi}{q+\eta}n\right)}{\Gamma\left(-\frac{2p}{p+\eta}n\right)\Gamma\left(-\frac{2q}{p+\eta}n\right)}$$

$$c_n^{(4)} = \frac{(-1)^{n-1}}{\pi n!(p-\eta)} \frac{\Gamma\left(-\frac{2n}{p-\eta}\right)\Gamma\left(\frac{2n}{p-\eta}\right)\Gamma\left(-\frac{p+\eta}{p-\eta}n\right)\Gamma\left(-\frac{q+\xi}{p-\eta}n\right)\Gamma\left(-\frac{q-\xi}{q-\eta}n\right)}{\Gamma\left(-\frac{2p}{p-\eta}n\right)\Gamma\left(-\frac{2q}{p-\eta}n\right)}$$

$$c_n^{(5)} = -\frac{(-1)^{n-1}}{2\pi n} \frac{\Gamma\left(-\frac{q+\xi}{2}n\right)\Gamma\left(-\frac{q-\xi}{2}n\right)\Gamma\left(-\frac{p+\eta}{2}n\right)\Gamma\left(-\frac{p-\eta}{2}n\right)}{\Gamma(-qn)\Gamma(-pn)}$$

- Divergence are visible as we move ξ, η **continuously**

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- General property of instantons discovered: **singular coefficients (& cancellations)**
- continuous deformation ➡ poles will be visible also in gravity side

New hints for instantons ?

Problems:

- 11d gravity dual for continuous deformation?

  non-canonical R-charges $\xi_i \neq 0$ break conformal symmetry [Jafferis]

  ➡ holographic RG flow [Freedman-Pufu][Pilch-Tyukov-Warner]

- Finite *k* $(k \in \mathbb{N})$ ➡ $e^{-\bigcirc \mu / k}$

  instanton coeffs at finite *k*

Other directions:

- closed expression for full instanton series for $k \in \mathbb{R}$

  (for ABJ(M), *q*=*p*=2, OSp ➡ refined topological string)

- mass deformation $\xi \in i\mathbb{R}$ (see poster by K. Shimizu)