Random volumes from matrices

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based on works
[1] JHEP1507 (2015) 088 [arXiv:1503.08812]
[2] arXiv:1504.03532
with Masafumi Fukuma and Naoya Umeda

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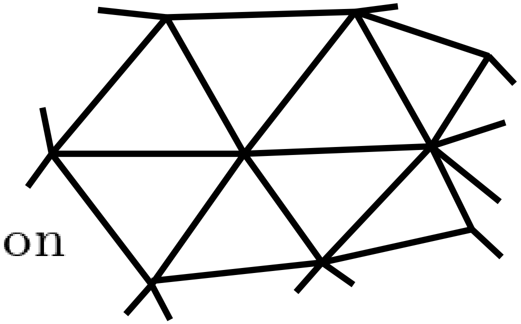
Introduction

Lattice approach to Quantum Gravity

$$\sum_{\text{topology}} \int \mathcal{D}g \ \sim \sum_{\text{random triangulation}}$$



This approach has achieved a success in 2D gravity.

Matrix models generate random surfaces as the Feynman diagrams.

- solvable
- a formulation of 2D quantum gravity and "(noncritical) string theory"

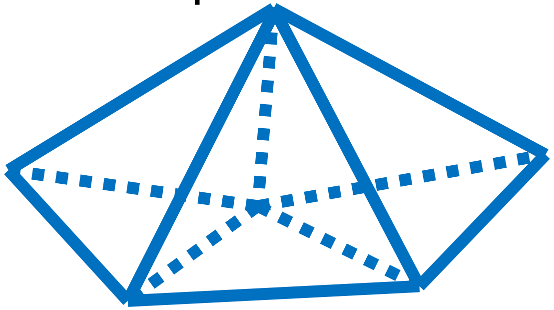
We expect that there are solvable models generating 3-dimensional random volumes.

This may lead to a formulation of membrane theory.

Natural generalizations of matrix models are tensor models.

[Ambjørn-Durhuus-Jonsson (1991), Sasakura (1991), Gross (1992)]

Tensor models generate random tetrahedral decomposition as the Feynman diagrams.



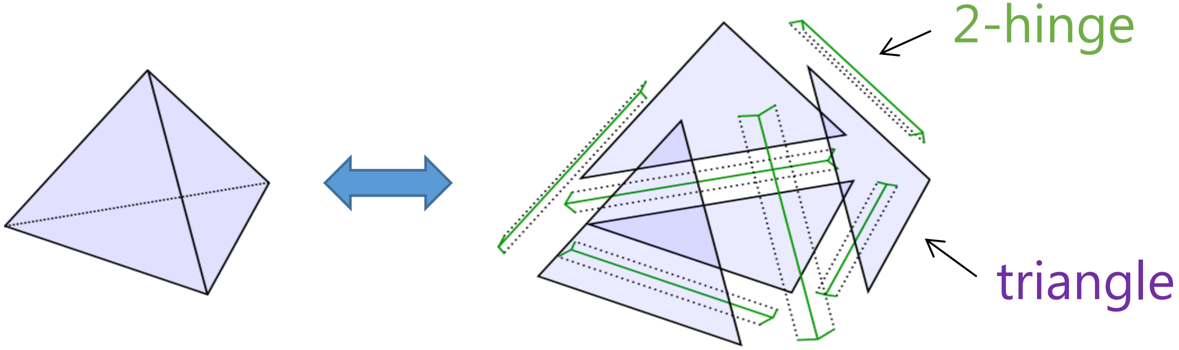
However, the models have not been solved.
(Recently, a special class of models, colored tensor models, have made a progress. [Gurau(2009-)])
We do not know how to take a continuum limit.

Triangle-hinge models [Fukuma, SS, Umeda, JHEP1507 (2015) 088]

A new class of models generating 3D random volumes as the Feynman diagrams

We call them triangle-hinge models.

Main idea: interpret tetrahedral decmp as collection of triangles and multiple hinges



Outline

- **Triangle-hinge models**

  - Algebra
  - Free energy
  - Restriction to 3D manifolds with tetrahedral decomposition

- **Introducing matter fields**

Action:

$$S[A,B] = \frac{1}{2}A_{ij}B^{ji} - \underbrace{\frac{\lambda}{6}C^{ijklmn}A_{ij}A_{kl}A_{mn}}_{\text{triangle}} - \sum_{k\geq 2}\underbrace{\frac{\mu_k}{2k}B^{i_1 j_1}\cdots B^{i_k j_k}y_{i_1\ldots i_k}y_{j_k\ldots j_1}}_{\text{k-hinge}}$$

- dynamical variables are real symmetric matrices, $A_{ij} = A_{ji}\ ,\ B^{ij} = B^{ji}$
- C^{ijklmn} & $y_{i_1\ldots i_k}$ are real constant tensors assigned to triangle & k-hinge, which are characterized by algebra.

$$C^{ijklmn} = g^{ni}g^{jk}g^{lm}$$

“metric”

$$y_{i_1\ldots i_k} = y_{i_1 j_1}{}^{j_k}\, y_{i_2 j_2}{}^{j_1} \cdots y_{i_k j_k}{}^{j_{k-1}}$$

structure const

We expect that our models can be solvable since variables are matrices not tensors, although they have not been solved yet.

Algebra

- **Our models are characterized by semisimple associative algebra $\mathcal{A}$:**

  vector space $\mathcal{A}$ with multiplication ×
  satisfying associativity: $a \times (b \times c) = (a \times b) \times c,\ \ a, b, c \in \mathcal{A}$

- The size of matrices is given by the linear dim. of alg. $\mathcal{A}$ (dim$\mathcal{A}$ = N).

$$A_{ij}\,,\, B^{ij} \qquad (i, j = 1, \ldots, N)$$

- If we take a basis $\{e_i\}$ of $\mathcal{A}$ $(i = 1, \cdots, N)$,
  multiplication is expressed as $e_i \times e_j = \underbrace{y_{ij}{}^{k}}_{\text{structure const.}} e_k$.

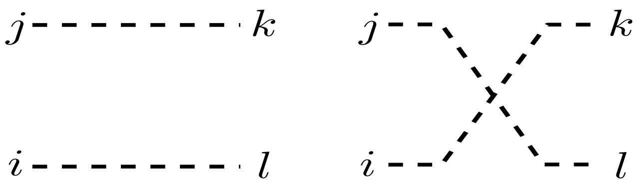
- Definition of "metric" g_{ij} : $\quad g_{ij} \equiv y_{ik}{}^{\ell}\, y_{j\ell}{}^{k}$

  g_{ij} has inverse g^{ij} $\Longleftrightarrow$ alg. $\mathcal{A}$ is semisimple

The Feynman diagrams

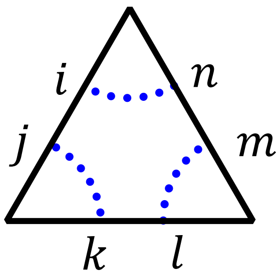
$$S[A,B] = \underline{\frac{1}{2}A_{ij}B^{ji}} - \frac{\lambda}{6}C^{ijklmn}A_{ij}A_{kl}A_{mn} - \sum_{k\geq 2}\frac{\mu_k}{2k}B^{i_1j_1}\cdots B^{i_kj_k}y_{i_1\ldots i_k}y_{j_k\ldots j_1}$$

- **propagator** $\langle A_{ij}B^{kl}\rangle = \delta_i^l\delta_j^k + \delta_i^k\delta_j^l$ $\Rightarrow$
  (Wick contraction)



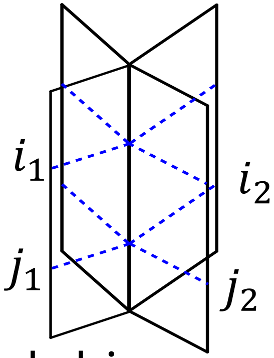
- **interaction terms**

$$C^{ijklmn} = g^{ni}g^{jk}g^{lm}$$



triangle

$$y_{i_1\ldots i_k}y_{j_k\ldots j_1}$$



k-hinge

◆ Each Feynman diagram can be interpreted as a diagram consisting of triangles which are glued together along multiple hinges.

Free energy

The free energy is sum of contribution of connected diagrams γ

$$\log Z = \sum_{\gamma} \frac{1}{S(\gamma)} \lambda^{s_2(\gamma)} \Big(\prod_{k \geq 2} \mu_k^{s_1^k(\gamma)} \Big) \mathcal{F}(\gamma)$$

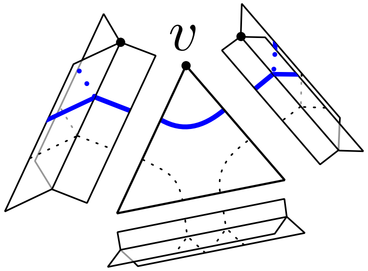
$S(\gamma)$: symmetry factor,

$s_2(\gamma)$: #(triangles), $s_1^k(\gamma)$: #(k-hinges),

$\mathcal{F}(\gamma)$: index function, which is given by contraction of indices

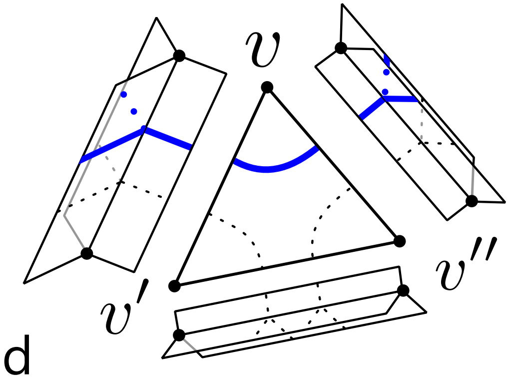
- Index function $\mathcal{F}(\gamma)$ is factorized into the contributions from vertices in diagram γ:

$$\mathcal{F}(\gamma) = \prod_{v \in \gamma} \zeta(v)$$

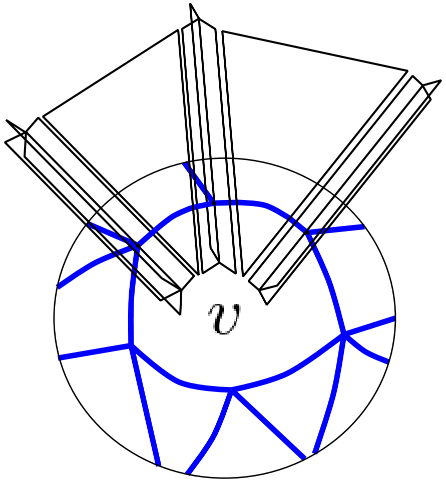


Index function and index network

- Factorization of index function: $\mathcal{F}(\gamma) = \prod_{v} \zeta(v)$



The index lines on two different hinges are connected through an intermediate triangle if and only if the hinges share the same vertex v.
The connected components of the index network have a 1 to 1 correspondence to the vertices in γ.



index network

Each index network can be regarded as a polygonal decomposition of a closed 2D surface Σ_v enclosing a vertex v. (Not necessarily 2D-sphere)
Due to the properties of associative algebra $\mathcal{A}$,
$\zeta(v)$ is topological invariant of 2D surface. [Fukuma-Hosono-Kawai (1992)]

Matrix ring

Here, we consider matrix ring.

$$\mathcal{A} = M_n(\mathbb{R}) = \{e_{ab}\}, \quad (a = 1, \ldots, n\,,\; N = n^2)$$

a basis: $$e_{ab} = \begin{pmatrix} 0 & & \cdots & & 0 \\ \vdots & & & & \vdots \\ & & & & \\ & 1 & & & \\ 0 & & \cdots & & 0 \end{pmatrix}$$

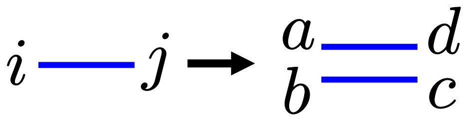
(a, b) componet

- multiplication: $e_{ab} \times e_{cd} = \delta_{bc} e_{ad}$

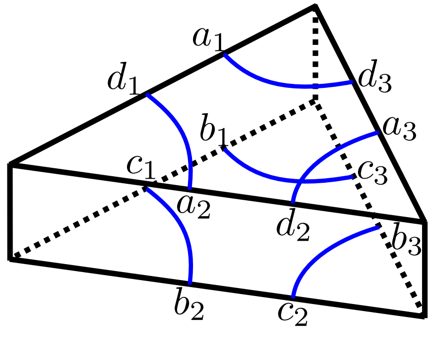
$$\Rightarrow \left\{ \begin{array}{l} g^{abcd} = \dfrac{1}{n}\delta^{ad}\delta^{bc} \\ y_{a_1 b_1 a_2 b_2 \ldots a_k b_k} = n\,\delta_{b_1 a_2}\delta_{b_2 a_3}\cdots\delta_{b_k a_1} \end{array} \right.$$

Note that index of algebra is expressed as double indices $i = (a, b)$.

✓ index line becomes double lines:

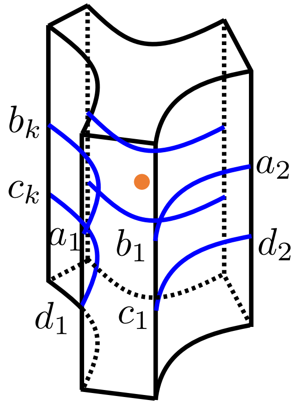


◆ index lines of triangles and hinges



triangle

$$\frac{\lambda}{n^3}\,\delta^{d_1 a_2}\delta^{c_1 b_2}\ldots\delta^{d_3 a_1}\delta^{c_3 b_1}$$



k-hinge

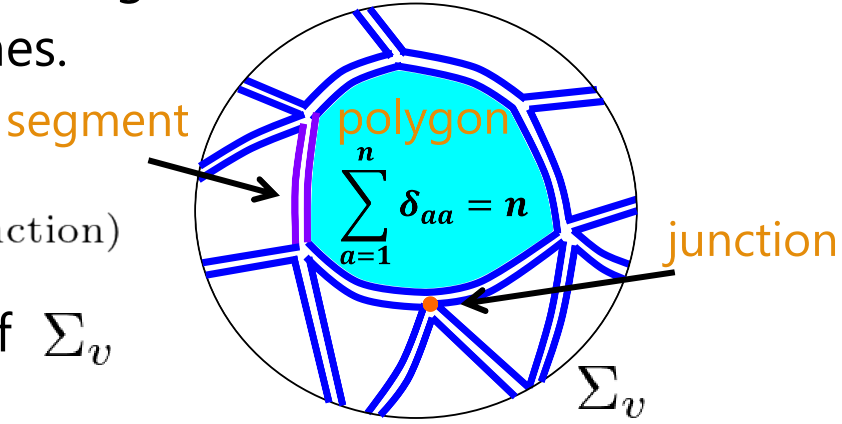
$$n^2\mu_k\,\delta_{b_1 a_2}\delta_{c_1 d_2}\ldots\delta_{b_k a_1}\delta_{c_k d_1}$$

In the case of matrix ring, index network gives a polygonal decomp with double lines. Each contribution is given by

$$\zeta(v) = n^{\#(\text{polygon})-\#(\text{segment})+\#(\text{junction})}$$

$$= n^{2-2g(v)}$$

$g(v)$: genus of Σ_v

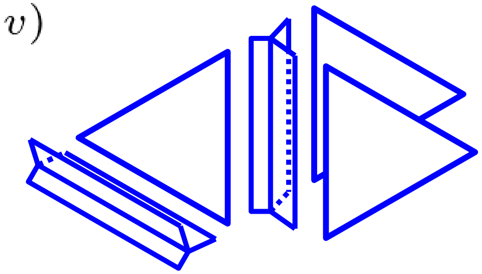


Similarly, in the case of $\mathcal{A} = \underbrace{M_n(\mathbb{R}) \oplus \cdots \oplus M_n(\mathbb{R})}_{K}$,

$\zeta(v) = K\, n^{2-2g(v)}$.

In this case, the free energy is given by

$$\log Z = \sum \frac{1}{S} \lambda^{s_2} \left(\prod_{k \geq 2} \mu_k^{s_1^k} \right) \prod_{v:\,\mathrm{vertex}} K\, n^{2-2g(v)}$$



General diagrams does not represent
3D manifolds because triangles and hinges are glued randomly.

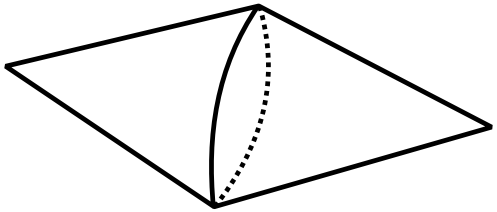
In 3D manifolds, each neighborhood around vertex is 3D ball.

Thus, all $g(v)$ should be zero.

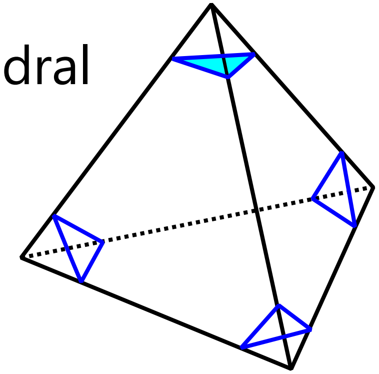
Diagrams whose all $g(v) = 0$ dominate in the large n limit.

Restriction to tetrahedral decomposition 1

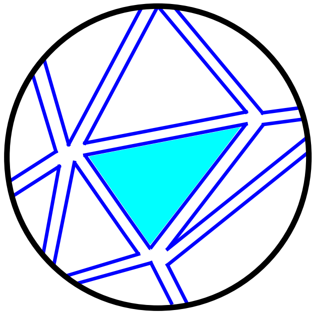
There are objects which are not tetrahedral decompositions.
It is not suitable to assign 3D volume.



All index networks of the objects which represent tetrahedral decompositions are always triangular decompositions.



Restriction to tetrahedral decomposition can be done by slightly modifying the triangle tensor C^{ijklmn} such that all index polygons are triangles.



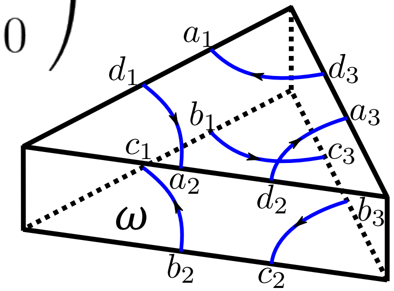
Restriction to tetrahedral decomposition 2

- Set the size of matrix ring as $n = 3m$. $M_{3m}(\mathbb{R})$
- Change the form of tensor C^{ijklmn}.

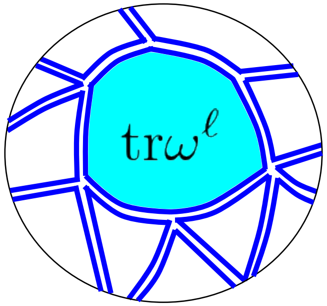
$$C^{a_1b_1c_1d_1a_2b_2c_2d_2a_3b_3c_3d_3} = \frac{1}{n^3}\delta^{d_1a_2}\delta^{b_2c_1}\delta^{d_2a_3}\delta^{b_3c_2}\delta^{d_3a_1}\delta^{b_1c_3}$$

$$\to \frac{1}{n^3}\omega^{d_1a_2}\omega^{b_2c_1}\omega^{d_2a_3}\omega^{b_3c_2}\omega^{d_3a_1}\omega^{b_1c_3},$$

where ω is a permutation matrix: $\omega = \begin{pmatrix} 0 & 1_m & 0 \\ 0 & 0 & 1_m \\ 1_m & 0 & 0 \end{pmatrix}$



- ➢ This means that each index line in a triangle has ω.



Each index polygon with ℓ segments gets a factor tr ω^ℓ.

Only $3k$-gons can appear in index networks.

Restriction to tetrahedral decomposition 3

Furthermore, we can take a limit where only triangles remain.

Each weight can be rewritten as

$$\frac{1}{S}\lambda^{s_2}\left(\prod_{k\geq 2}\mu_k^{s_1^k}\right)\prod_{v:\,\mathrm{vertex}} K\,n^{2-2g(v)}$$

$$=\frac{1}{S}\prod_{v:\,\mathrm{vertex}}\left[K\left[\prod_{k\geq 2}(\lambda^2\mu_k)^{\frac{1}{2}t_0^k(v)}\right]\left(\frac{n}{\lambda}\right)^{2-2g(v)}\underline{\left(\frac{1}{\lambda}\right)^{\frac{1}{3}d(v)}}\right]$$

where $d(v)=\sum_{\ell}(\ell-3)\,t_2^{\ell}(v)$ and $t_2^{\ell}(v)$ = #(ℓ-gons in index network).

$d(v)=\sum_{\ell}(\ell-3)\,t_2^{\ell}(v)\geq 0$ ⟹ In the limit $\lambda\to\infty$, the leading contri. are diagrams s.t. $d(v)=0$

${}^{\forall}d(v)=0$ ⟺ all index networks represent triangular decompositions.

⟺ diagram represents a tetrahedral decomposition

Restriction to manifolds with tetrahedral decomposition

$$\log Z = \sum_{\gamma} \frac{1}{S} \prod_{v:\,\mathrm{vertex}} \left[K \Big[\prod_{k\geq 2} (\lambda^2 \mu_k)^{\frac{1}{2} t_0^k(v)} \Big] \Big(\underbrace{\frac{n}{\lambda}}_{\searrow\ \infty} \Big)^{2-2g(v)} \Big(\underbrace{\frac{1}{\lambda}}_{\searrow\ 0} \Big)^{\frac{1}{3} d(v)} \right]$$

manifoldness tetra decomp

The leading contributions represent 3D manifolds with tetrahedral decomposition

$$\boxed{\sum_{\gamma'} \frac{1}{S} (\mu K n^2)^{s_0} (\lambda^2 \mu)^{s_3} \qquad [\mu_k = \mu\ (k \geq 2)]}$$

s_0 = #(vertices in γ'), s_3 = #(tetrahedra in γ')

The models correspond to pure gravity with CC.

Introducing matter to triangle-hinge models

[Fukuma, SS, Umeda (arXiv:1504.03532)]

We can introduce **matter degrees of freedom**.

General prescriptions

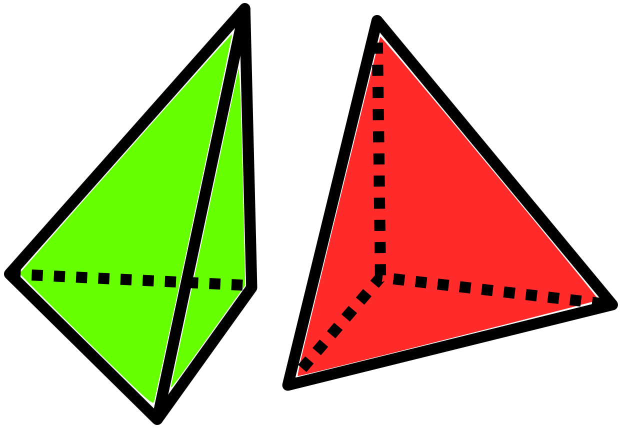
- Take algebra as $\mathcal{A} = \underline{\mathcal{A}_{\mathrm{grav}}} \otimes \underline{\mathcal{A}_{\mathrm{mat}}}$
- Assume a factorized form $C = \underline{C_{\mathrm{grav}}} \otimes \underline{C_{\mathrm{mat}}}$

⇨ Then, index functions factorize as $\mathcal{F}(\gamma) = \mathcal{F}_{\mathrm{grav}}(\gamma)\,\mathcal{F}_{\mathrm{mat}}(\gamma)$

The "gravity" part restricts diagrams to 3D manifolds as explained above.

The "matter" part gives various matter d.o.f.

Matter fields in triangle-hinge models



◆ We can assign q colors to tetrahedra.

In the case of $q = 2$,

the model realizes the Ising model on random volumes.

◆ We can formally take the set of colors to be $\mathbb{R}^D$: $\{1, \ldots, q\} \to \mathbb{R}^D$

This gives 3dim gravity coupled to D scalars.

⟷ membrane in $\mathbb{R}^D$

We do not know whether the models actually describe membrane.

We need to take continuum limits. (future work)

Summary

- We proposed a new class of models (triangle-hinge models) which generate 3D random volumes.

✓ The fundamental building blocks are triangles and multiple hinges.

✓ The dynamical variables are symmetric matrices.
Thus, there is a possibility that we can solve models analytically by using the techniques of matrix models.

- We can introduce matter dof. to models.

✓ We expect that models can describe membrane theory.