ADHM construction in R^8

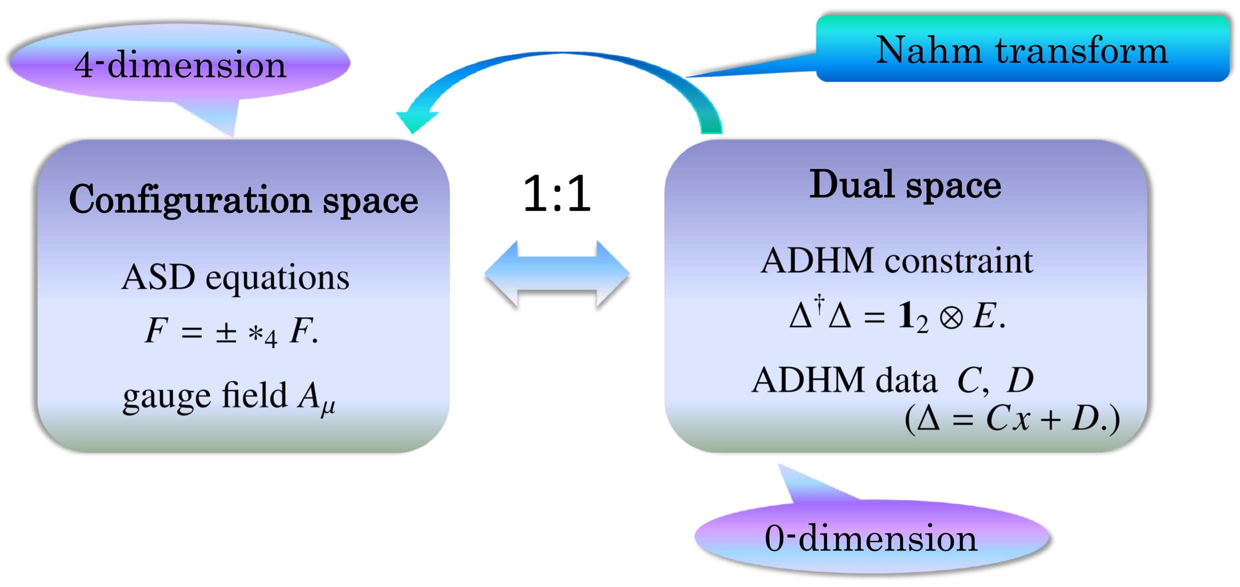
Koki Takesue, Shin Sasaki and Atsushi Nakamula

Department of Physics, School of Science, Kitasato University, Sagamihara, Kanagawa 252-0373, Japan

0: INTRODUCTION

It is well known that instantons in gauge theories play important roles in the study of non-perturbative effects. The gauge instantons in four dimensions are defined by configurations such that the gauge field strength 2-form F satisfies the self-duality relation $F = \pm *_4 F$, where $*_d$ is the Hodge dual operator in $\mathbb{R}^d$. A salient feature of the self-dual instantons in four dimensions is its systematic construction of solutions, known as ADHM construction.

ADHM construction in $\mathbb{R}^4$



In higher dimensions, these are two types of generalization of the ASD equations in $\mathbb{R}^4$. First type is called as ``secular type", and second type is called as ``self-dual type". In this poster, we define the eight dimensional ``instantons" such that the field strength satisfies the self-duality relation in $\mathbb{R}^8$. Furthermore, we expect that the eight dimensional instanton has non-zero topological charge given by the 4th Chern number $C^{(4)} = \int F \wedge F \wedge F \wedge F$. $F \wedge F = *_8 F \wedge F$

The generalization of the ASD equations in $\mathbb{R}^4$

$\mathbb{R}^4$

ASD equations

$$F = \pm * F$$

$$\Longleftrightarrow\ F_{\mu\nu} = \pm\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$$

$$\mu, \nu, \rho, \sigma = 1, 2, 3, 4$$

・ADHM construction

M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Yu.I. Manin, *"Construction of Instantons"*, Phys.Lett.A **65** (1978) 185

$\mathbb{R}^8$

I: Self-dual type

ASD equations

$$F \wedge F = \pm *_8 (F \wedge F)$$

・ADHM construction ?

II: Secular type

secular equations

$$F_{\mu\nu} = \pm\frac{1}{2}T_{\mu\nu\rho\sigma}F_{\rho\sigma}$$

$$\mu, \nu, \rho, \sigma = 1, 2 \ldots 8$$

・ADHM like construction

E. Corrigan, P. Goddard and A.Kent Commun. Math. Phys. **100** (1985) 1

We establish the general scheme to construct the self-dual gauge field configuration $F \wedge F = *_8 F \wedge F$, so we will call this scheme the eight dimensional ``ADHM construction". In this poster, we show that ADHM construction in $\mathbb{R}^8$ and reproduces the known one-instanton solution. Furthermore, we find explicit 't Hooft type solutions for topological charge $k = 2$ and $k = 3$.

I: THE EIGHT DIMENSIONAL ADHM CONSTRUCTION

The first step of constructing the ADHM construction in $\mathbb{R}^8$ is to find appropriate algebra basis e_μ which constructing ``ASD tensor" $\Sigma^{(\pm)}_{\mu\nu}$ in $\mathbb{R}^8$. Here an ``ASD tensor" means that a tensor is satisfying the ASD equations, and "ASD algebra basis" means algebra basis which construct the ASD tensor.

$\mathbb{R}^4$

In four dimension, x^μ are the standard coordinates on $\mathbb{R}^4$ and indices $\mu, \nu, \cdots = 1, 2, 3, 4$ and $i, j, \cdots = 1, 2, 3$.

ASD equation in $\mathbb{R}^4$

$$F = \pm * F$$

$$\Longleftrightarrow\ F_{\mu\nu} = \pm\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$$

$\mathbb{R}^8$

In eight dimension, x^μ are the standard coordinates on $\mathbb{R}^8$ and indices $\mu, \nu, \cdots = 1, 2, \ldots, 8$ and $i, j, \cdots = 1, 2, \ldots, 7$.

ASD equation in $\mathbb{R}^8$

$$F \wedge F = \pm *_8 (F \wedge F),$$

$$\Longleftrightarrow\ F_{[\mu\nu}F_{\rho\sigma]} = \pm\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}.$$

where indices in square brackets [ ] are to be antisymmetrized(ex: $a_{[\mu\nu]} = \frac{1}{2}(a_{\mu\nu} - a_{\nu\mu})$).

I-i: ASD tensor

ASD tensor in $\mathbb{R}^4$ is

$$\text{'t Hooft tensor:}\quad \eta^{(+)}_{\mu\nu} := e^\dagger_\mu e_\nu - e^\dagger_\nu e_\mu,\quad \eta^{(-)}_{\mu\nu} := e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu$$

and 't Hooft tensor $\eta^{(\pm)}_{\mu\nu}$ satisfy the four dimensional ASD equations

$$\eta^{(\pm)}_{\mu\nu} = \pm\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\eta^{(\pm)}_{\rho\sigma}$$

By analogy of the ASD tensor in $\mathbb{R}^4$ (i.e. 't Hooft tensor), we define

$$\text{ASD tensor:}\quad \Sigma^{(+)}_{\mu\nu} := e^\dagger_\mu e_\nu - e^\dagger_\nu e_\mu,\quad \Sigma^{(-)}_{\mu\nu} := e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu \qquad (1)$$

and the ASD tensor $\Sigma^{(\pm)}_{\mu\nu}$ need satisfy the eight dimensional ASD equations

$$\Sigma^{(\pm)}_{[\mu\nu}\Sigma^{(\pm)}_{\rho\sigma]} = \pm\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}\Sigma^{(\pm)}_{\alpha\beta}\Sigma^{(\pm)}_{\gamma\delta}. \qquad (2)$$

It is well known that four dimensional ASD algebra is quaternion.

ASD algebra basis in $\mathbb{R}^4$:

$$e_\mu := \delta_{\mu 4}\mathbf{1}_2 - i\delta_{\mu j}\sigma_j,\quad e^\dagger_\mu := \delta_{\mu 4}\mathbf{1}_2 + i\delta_{\mu j}\sigma_j$$

where $\mathbf{1}_n$ is rank n identity matrix, and σ_j are Pauli matrices.

Some properties of ASD algebra basis e_μ,

$$e_\mu e^\dagger_\nu + e_\nu e^\dagger_\mu = e^\dagger_\mu e_\nu + e^\dagger_\nu e_\mu = 2\delta_{\mu\nu},$$

and ASD tensor $\eta^{(\pm)}_{\mu\nu}$,

$$\mathrm{Tr}\eta^{(\pm)}_{12}\eta^{(\pm)}_{34} = \pm 4\mathrm{Tr}\mathbf{1}_2 = \pm 8,$$

$$\eta^{(\pm)}_{\mu\nu}\eta^{(\pm)}_{\rho\sigma} = \varepsilon_{\mu\nu\rho\sigma}\eta^{(\pm)}_{12}\eta^{(\pm)}_{34}.$$

Now we can find eight dimensional ASD algebra basis with reference to construction of quaternion.

ASD algebra basis in $\mathbb{R}^8$:

$$e_\mu := \delta_{\mu 8}\mathbf{1}_8 - i\delta_{\mu j}\Gamma_j,\quad e^\dagger_\mu := \delta_{\mu 8}\mathbf{1}_8 + i\delta_{\mu j}\Gamma_j$$

where Γ_i are defined by

$$\Gamma_1 := \sigma_{112},\ \Gamma_2 := \sigma_{120},\ \Gamma_3 := -\sigma_{132},\ \Gamma_4 := -\sigma_{221},$$
$$\Gamma_5 := \sigma_{223},\ \Gamma_6 := -\sigma_{202},\ \Gamma_7 := \sigma_{300}.$$

where $\sigma_{ijk} := \sigma_i \otimes \sigma_j \otimes \sigma_k$ and $\sigma_0 := \mathbf{1}_2$.

Some properties of ASD algebra basis e_μ,

$$e_\mu e^\dagger_\nu + e_\nu e^\dagger_\mu = e^\dagger_\mu e_\nu + e^\dagger_\nu e_\mu = 2\delta_{\mu\nu},$$

and ASD tensor $\Sigma^{(\pm)}_{\mu\nu}$,

$$\mathrm{Tr}\Sigma^{(\pm)}_{12}\Sigma^{(\pm)}_{34}\Sigma^{(\pm)}_{56}\Sigma^{(\pm)}_{78} = \pm 16\mathrm{Tr}\mathbf{1}_8 = \pm 128,$$

$$\Sigma^{(\pm)}_{\mu\nu}\Sigma^{(\pm)}_{\rho\sigma}\Sigma^{(\pm)}_{\alpha\beta}\Sigma^{(\pm)}_{\gamma\delta} = \varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}\Sigma^{(\pm)}_{12}\Sigma^{(\pm)}_{34}\Sigma^{(\pm)}_{56}\Sigma^{(\pm)}_{78}.$$

Now we was got the eight dimensional ASD tensor $\Sigma^{(\pm)}_{\mu\nu}$. So next step is to construct eight dimensional ADHM construction and we introduce the eight dimensional ADHM constraints.

I-ii: ADHM construction in $\mathbb{R}^8$

First we introduce the eight dimensional Weyl operator

$$\text{Weyl operator}: \Delta := Cx + D,$$

(x means $x \otimes \mathbf{1}_k$. However usually $\mathbf{1}_k$ omite from equations.)

where C and D are $(k+1) \times k$ matrices with basis from e_μ, k is instanton charge and x is

$$x := x^\mu e_\mu \Rightarrow \Sigma^{(-)}_{\mu\nu} \text{ anti-self dual.} \quad (x := x^\mu e^\dagger_\mu \Rightarrow \Sigma^{(+)}_{\mu\nu} \text{ self dual.})$$

Nahm transform

A scheme of to obtain the eight dimensional instanton's gauge field from Weyl operator is analogy by four dimensional ones. So

$$\text{Weyl equations}: \Delta^\dagger V(x) = 0$$

where $V(x)$ is $(k+1)$ column vector with basis from e_μ. $V(x)$ satisfy

$$\text{normalization}: V^\dagger V = \mathbf{1}_8.$$

We obtain the gauge field $A_\mu(x)$ of eight dimensional instantons as

$$\text{gauge field}: A_\mu(x) = V^\dagger(x)\partial_\mu V(x) = -\partial_\mu V^\dagger(x)V(x). \qquad (3)$$

Introduce ADHM constraint in $\mathbb{R}^8$

Next we calculate field strength $F_{\mu\nu}$ from Eq.(3) to introduce the eight dimensional ADHM constraint.

$$\begin{aligned} F_{\mu\nu} &:= \partial_\mu A_\nu + A_\mu A_\nu - (\mu \leftrightarrow \nu) \\ &= \partial_\mu V^\dagger \partial_\nu V - \partial_\mu V^\dagger V V^\dagger \partial_\nu V - (\mu \leftrightarrow \nu) \qquad \because).\ V^\dagger\partial_\mu V = -\partial_\mu V^\dagger V \\ &= \partial_\mu V^\dagger\left(1 - VV^\dagger\right)\partial_\nu V - (\mu \leftrightarrow \nu) \end{aligned} \qquad (4)$$

Here we use the completeness relation $1 - VV^\dagger = \Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger$, Eq. (4) can write as

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu V^\dagger \Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger\partial_\nu V - (\mu \leftrightarrow \nu) \\ &= V^\dagger\partial_\mu\Delta(\Delta^\dagger\Delta)^{-1}\partial_\nu\Delta^\dagger V - (\mu \leftrightarrow \nu) \qquad \because).\ \Delta^\dagger V = 0 \\ &= V^\dagger C e_\mu(\Delta^\dagger\Delta)^{-1}e^\dagger_\nu C^\dagger V - (\mu \leftrightarrow \nu) \qquad \because).\ \Delta = Cx + D. \end{aligned} \qquad (5)$$

Here, we demand the next condition

$$e_\mu(\Delta^\dagger\Delta)^{-1} = (\Delta^\dagger\Delta)^{-1}e_\mu. \qquad (6)$$

Then Eq.(5) is

$$\begin{aligned} F_{\mu\nu} &= V^\dagger C(\Delta^\dagger\Delta)^{-1}(e_\mu e^\dagger_\nu - e_\nu e^\dagger_\mu)C^\dagger V \\ &= V^\dagger C(\Delta^\dagger\Delta)^{-1}\Sigma^{(-)}_{\mu\nu}C^\dagger V \qquad \because).(1) \end{aligned} \qquad (7)$$

Substitute Eq.(7) into eight dimensional ASD equations

$$\begin{aligned} & F_{[\mu\nu}F_{\rho\sigma]} = -\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} \\ \Longleftrightarrow\ & \left(V^\dagger C(\Delta^\dagger\Delta)^{-1}\Sigma^{(-)}_{[\mu\nu}C^\dagger V\right)\left(V^\dagger C\Sigma^{(-)}_{\rho\sigma]}(\Delta^\dagger\Delta)^{-1}C^\dagger V\right) \\ & = -\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}\left(V^\dagger C(\Delta^\dagger\Delta)^{-1}\Sigma^{(-)}_{\alpha\beta}C^\dagger V\right)\left(V^\dagger C\Sigma^{(-)}_{\gamma\delta}(\Delta^\dagger\Delta)^{-1}C^\dagger V\right). \end{aligned} \qquad (8)$$

Example $\mathbb{R}^4$.

In four dimension, Eq.(8) is

$$\begin{aligned} & F_{\mu\nu} = -\frac{1}{2!}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma} \\ \Longleftrightarrow\ & \left(V^\dagger C(\Delta^\dagger\Delta)^{-1}\eta^{(-)}_{\mu\nu}C^\dagger V\right) = -\frac{1}{2!}\varepsilon_{\mu\nu\rho\sigma}\left(V^\dagger C(\Delta^\dagger\Delta)^{-1}\eta^{(-)}_{\rho\sigma}C^\dagger V\right). \end{aligned} \qquad (8')$$

So four dimensional ADHM constraint is

$$\Delta^\dagger\Delta = \mathbf{1}_2 \otimes E. \quad (\Longleftrightarrow\ e_\mu(\Delta^\dagger\Delta)^{-1} = (\Delta^\dagger\Delta)^{-1}e_\mu.)$$

Here we require that commutativity of $\Sigma^{(-)}_{\mu\nu}$ with $C^\dagger V V^\dagger C$, that is

$$e_\mu \left(C^\dagger V V^\dagger C\right) = \left(C^\dagger V V^\dagger C\right) e_\mu \qquad (9)$$

and then Eq.(8) is

$$F_{[\mu\nu}F_{\rho\sigma]} = -\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$$
$$\Longleftrightarrow \quad V^\dagger C(\Delta^\dagger\Delta)^{-1}\left(\Sigma^{(-)}_{[\mu\nu}\Sigma^{(-)}_{\rho\sigma]}\right)C^\dagger V V^\dagger C(\Delta^\dagger\Delta)^{-1}C^\dagger V$$
$$= -\frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}\, V^\dagger C(\Delta^\dagger\Delta)^{-1}\left(\Sigma^{(-)}_{\alpha\beta}\Sigma^{(-)}_{\gamma\delta}\right)C^\dagger V V^\dagger C(\Delta^\dagger\Delta)^{-1}C^\dagger V.$$

Since Eq.(2), $\Sigma^{(-)}_{[\mu\nu}\Sigma^{(-)}_{\rho\sigma]}$ is anti-self dual tensor. Therefore $F_{\mu\nu}$ which is constructed from the eight dimensional ADHM construction satisfies the ASD equation.

From the above, we was required two conditions Eq.(6) and Eq.(9). These conditions correspond with the four dimensional ADHM constraint, so we called these conditions ``the eight dimensional ADHM constraint". We are able to more simplify Eq.(9). Eq.(9) include V, so we rewrite Eq.(9) using completeness relation $1 - VV^\dagger = \Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger$,

$$\text{Eq.(9)} \Longleftrightarrow e_\mu\left(C^\dagger\left(1-\Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger\right)C\right) = \left(C^\dagger\left(1-\Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger\right)C\right)e_\mu,$$
$$\Longleftrightarrow \begin{cases} e_\mu C^\dagger C = C^\dagger C e_\mu, \\ e_\mu\left(C^\dagger\Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger C\right) = \left(C^\dagger\Delta(\Delta^\dagger\Delta)^{-1}\Delta^\dagger C\right)e_\mu. \end{cases}$$

The first condition $e_\mu C^\dagger C = C^\dagger C e_\mu$ is included condition Eq.(6), so we omit this condition. On the other hand, second condition is able to simplify using Eq.(6) to $e_\mu C^\dagger\Delta = C^\dagger\Delta e_\mu$. And this condition is also include condition Eq.(6).

$$e_\mu(\Delta^\dagger\Delta)^{-1} = (\Delta^\dagger\Delta)^{-1}e_\mu. \quad \Longleftrightarrow \quad \Delta^\dagger\Delta = \mathbf{1}_8 \otimes E$$

$$\Delta^\dagger\Delta = \left((x^\dagger\otimes\mathbf{1}_k)C^\dagger + D^\dagger\right)\left(C(x\otimes\mathbf{1}_k)+D\right)$$
$$= (x^\dagger\otimes\mathbf{1}_k)C^\dagger C(x\otimes\mathbf{1}_k) + (x^\dagger\otimes\mathbf{1}_k)C^\dagger D + D^\dagger C(x\otimes\mathbf{1}_k) + D^\dagger D$$

$\mathbf{1}_8\otimes E^{(1)}$ $\quad$ $\mathbf{1}_8\otimes E^{(2)}$ $\quad$ $\mathbf{1}_8\otimes E^{(2)\,\dagger}$ $\quad$ $\mathbf{1}_8\otimes E^{(3)}$

$$\to e_\mu C^\dagger\Delta = C^\dagger\Delta e_\mu$$
$$\to e_\mu C^\dagger C = C^\dagger C e_\mu$$

Therefore, we find that it is enough to just demand only one condition Eq.(6).

The eight dimensional ADHM constraint:

ADHM constraint in $\mathbb{R}^8$: $\Delta^\dagger\Delta = (\mathbf{1}_2\otimes\mathbf{1}_2\otimes\mathbf{1}_2)\otimes E = \mathbf{1}_8\otimes E$.

where E is $k\times k$ matrix.

The eight dimensional ADHM equations with canonical form

The eight dimensional ADHM data C,D are canonical form, such that

$$C = \begin{pmatrix} 0_{[8]\times[8k]} \\ \mathbf{1}_{8k} \end{pmatrix}_{[8+8k]\times[8k]}$$

where [ ] means matrix size.

$$D = \begin{pmatrix} S_{[8]\times[8k]} \\ T_{[8k]\times[8k]} \end{pmatrix}_{[8+8k]\times[8k]} = \begin{pmatrix} S_{[8]\times[8k]} \\ e_{\mu\,[8]}\otimes T^\mu_{[k]} \end{pmatrix}$$
$$= \begin{pmatrix}
S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 \\
T^8 - iT^7 & 0 & 0 & 0 & 0 & T^3 - iT^6 & -T^2 + iT^5 & -T^1 - iT^4 \\
0 & T^8 - iT^7 & 0 & 0 & -T^3 + iT^6 & 0 & T^1 - iT^4 & -T^2 - iT^5 \\
0 & 0 & T^8 - iT^7 & 0 & T^2 - iT^5 & -T^1 + iT^4 & 0 & -T^3 - iT^6 \\
0 & 0 & 0 & T^8 - iT^7 & T^1 + iT^4 & T^2 + iT^5 & T^3 + iT^6 & 0 \\
0 & T^3 + iT^6 & -T^2 - iT^5 & -T^1 + iT^4 & T^8 + iT^7 & 0 & 0 & 0 \\
-T^3 - iT^6 & 0 & T^1 + iT^4 & -T^2 + iT^5 & 0 & T^8 + iT^7 & 0 & 0 \\
T^2 + iT^5 & -T^1 - iT^4 & 0 & -T^3 + iT^6 & 0 & 0 & T^8 + iT^7 & 0 \\
T^1 - iT^4 & T^2 - iT^5 & T^3 - iT^6 & 0 & 0 & 0 & 0 & T^8 + iT^7
\end{pmatrix}$$

We assume that $S_{[8]\times[8k]} = e_\mu\otimes\tilde{S}^\mu_{[1]\times[k]}$ then,

$\mathbb{R}^8$ ADHM-eq. : $\Delta^\dagger\Delta = \mathbf{1}_8\otimes E_{[k]}$ $\Longleftrightarrow$

$$[T^2,T^5] - [T^3,T^6] + \frac{i}{2}\left(S_4^\dagger S_4 - S_1^\dagger S_1\right) = 0,$$
$$[T^3,T^6] - [T^1,T^4] + \frac{i}{2}\left(S_4^\dagger S_4 - S_2^\dagger S_2\right) = 0, \quad [T^1,T^4] - [T^2,T^5] + \frac{i}{2}\left(S_4^\dagger S_4 - S_3^\dagger S_3\right) = 0,$$
$$[T^1,T^2] + [T^4,T^5] + \frac{1}{2}\left(S_1^\dagger S_2 - S_2^\dagger S_1\right) = 0, \quad [T^1,T^5] - [T^4,T^2] + \frac{i}{2}\left(S_1^\dagger S_2 + S_2^\dagger S_1\right) = 0,$$
$$[T^1,T^3] + [T^4,T^6] + \frac{1}{2}\left(S_1^\dagger S_3 - S_3^\dagger S_1\right) = 0, \quad [T^1,T^6] - [T^4,T^3] + \frac{i}{2}\left(S_1^\dagger S_3 + S_3^\dagger S_1\right) = 0,$$
$$[T^2,T^3] + [T^5,T^6] + \frac{1}{2}\left(S_2^\dagger S_3 - S_3^\dagger S_2\right) = 0, \quad [T^2,T^6] - [T^5,T^3] + \frac{i}{2}\left(S_2^\dagger S_3 + S_3^\dagger S_2\right) = 0,$$
$$[T^1,T^2] - [T^4,T^5] + \frac{1}{2}\left(S_4^\dagger S_3 - S_3^\dagger S_4\right) = 0, \quad [T^1,T^5] + [T^4,T^2] - \frac{i}{2}\left(S_4^\dagger S_3 + S_3^\dagger S_4\right) = 0,$$
$$[T^2,T^3] - [T^5,T^6] + \frac{1}{2}\left(S_4^\dagger S_1 - S_1^\dagger S_4\right) = 0, \quad [T^2,T^6] + [T^5,T^3] - \frac{i}{2}\left(S_4^\dagger S_1 + S_1^\dagger S_4\right) = 0,$$
$$[T^3,T^1] - [T^6,T^4] + \frac{1}{2}\left(S_4^\dagger S_2 - S_2^\dagger S_4\right) = 0, \quad [T^3,T^4] + [T^6,T^1] - \frac{i}{2}\left(S_4^\dagger S_2 + S_2^\dagger S_4\right) = 0,$$
$$[T^8,T^1] + [T^7,T^4] + \frac{1}{2}\left(S_2^\dagger S_7 - S_7^\dagger S_2\right) = 0, \quad [T^8,T^4] - [T^7,T^1] + \frac{i}{2}\left(S_2^\dagger S_7 + S_7^\dagger S_2\right) = 0,$$
$$[T^8,T^2] + [T^7,T^5] + \frac{1}{2}\left(S_3^\dagger S_5 - S_5^\dagger S_3\right) = 0, \quad [T^8,T^5] - [T^7,T^2] + \frac{i}{2}\left(S_3^\dagger S_5 + S_5^\dagger S_3\right) = 0,$$
$$[T^8,T^3] + [T^7,T^6] + \frac{1}{2}\left(S_1^\dagger S_6 - S_6^\dagger S_1\right) = 0, \quad [T^8,T^6] - [T^7,T^3] + \frac{i}{2}\left(S_1^\dagger S_6 + S_6^\dagger S_1\right) = 0,$$
$$[T^8,T^1] - [T^7,T^4] + \frac{1}{2}\left(S_4^\dagger S_5 - S_5^\dagger S_4\right) = 0, \quad [T^8,T^4] + [T^7,T^1] - \frac{i}{2}\left(S_4^\dagger S_5 + S_5^\dagger S_4\right) = 0,$$
$$[T^8,T^2] - [T^7,T^5] + \frac{1}{2}\left(S_4^\dagger S_6 - S_6^\dagger S_4\right) = 0, \quad [T^8,T^5] + [T^7,T^2] - \frac{i}{2}\left(S_4^\dagger S_6 + S_6^\dagger S_4\right) = 0,$$
$$[T^8,T^3] - [T^7,T^6] + \frac{1}{2}\left(S_4^\dagger S_7 - S_7^\dagger S_4\right) = 0, \quad [T^8,T^6] + [T^7,T^3] - \frac{i}{2}\left(S_4^\dagger S_7 + S_7^\dagger S_4\right) = 0.$$
$$-[T^1,T^4] - [T^2,T^5] - [T^3,T^6] - [T^8,T^7] - \frac{i}{2}\left(S_4^\dagger S_4 - S_8^\dagger S_8\right) = 0.$$

II: THE EIGHT DIMENSIONAL ADHM DATA

In the eight dimensions, instanton charge Q is 4th Chern number

$$Q = N\int_{\mathbb{R}^8}\mathrm{Tr}\,(F\wedge F\wedge F\wedge F) = N\int_{\mathbb{R}^8}d^8x\,\mathrm{Tr}\left(\frac{1}{8!}\varepsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}F_{\mu\nu}F_{\rho\sigma}F_{\alpha\beta}F_{\gamma\delta}\right) =: N\int_{\mathbb{R}^8}d^8x\,Q.$$

where N is normalization constant. However it is difficult to use this equations directly for calculation of charge density. So we use formula to calculate the charge density form ADHM data.

Formula to calculate charge density : $Q = \pm 16\,\mathrm{Tr}\left(V^\dagger C(\Delta^\dagger\Delta)^{-1}C^\dagger V\right)^4$.

II-i: BPST type 1-instanton

We extend the four dimensional BPST instanton ADHM data to the eight dimensional ones

$$C = \begin{pmatrix} 0 \\ \mathbf{1}_8 \end{pmatrix}, \qquad D = \begin{pmatrix} \lambda\mathbf{1}_8 \\ -a^\mu e_\mu \end{pmatrix}, \qquad \therefore\ \Delta = \begin{pmatrix} \lambda\mathbf{1}_8 \\ \tilde{x} \end{pmatrix}$$

where $\lambda\in\mathbb{R}$ is size moduli, $\tilde{x} := (x^\mu - a^\mu)e_\mu$ and $a^\mu\in\mathbb{R}$ is position moduli.

Since $\Delta^\dagger = (\lambda\mathbf{1}_8 \ \ \tilde{x}^\dagger)$, indeed this ADHM data satisfy the eight dimensional ADHM constraints

$$\Delta^\dagger\Delta = \left(\lambda\mathbf{1}_8 \ \ \tilde{x}^\dagger\right)\begin{pmatrix} \lambda\mathbf{1}_8 \\ \tilde{x} \end{pmatrix} = \mathbf{1}_8\otimes\left(\lambda^2 + \|\tilde{x}\|^2\right)$$

where $\|\tilde{x}\|^2 := \tilde{x}\tilde{x}^\dagger = \tilde{x}^\dagger\tilde{x} = (x^\mu - a^\mu)(x_\mu - a_\mu)$ and $\tilde{x}^\dagger := (x^\mu - a^\mu)e^\dagger_\mu$.

Weyl-eq. : $\Delta^\dagger V = 0 \Longleftrightarrow \left(\lambda\mathbf{1}_8 \ \ \tilde{x}^\dagger\right)V = 0$

$$\Longrightarrow \text{zero mode} : V = \frac{1}{\sqrt{\rho}}\begin{pmatrix} \tilde{x}^\dagger \\ -\lambda\mathbf{1}_8 \end{pmatrix}, \quad \text{where } \rho = \lambda^2 + \|\tilde{x}\|^2.$$

$$\therefore \ \text{gauge field} : A_\mu = -\frac{1}{2}\frac{x^\nu - a^\nu}{\lambda^2 + \|\tilde{x}\|^2}\Sigma^{(-)}_{\mu\nu}. \quad \text{field strength} : F_{\mu\nu} = \frac{\lambda^2}{(\lambda^2 + \|\tilde{x}\|^2)^2}\Sigma^{(-)}_{\mu\nu}.$$

This gauge field and field strength are identified as Grossman's 1-instanton [1].

[1] B. Grossman, T. W. Kephart and J. D. Stasheff, ``*Solutions to Yang-Mills Field Equations in Eight-dimensions and the Last Hopf Map*," Commun. Math. Phys. **96** (1984) 431 [Erratum-ibid. **100** (1985) 311].

Instanton charge is

$$Q = N\int_{\mathbb{R}^8}d^8x\left(\frac{\lambda^2}{(\lambda^2+\tilde{x}^2)^2}\right)^4\mathrm{Tr}\left(\Sigma^{(-)}_{[12}\Sigma^{(-)}_{34}\Sigma^{(-)}_{56}\Sigma^{(-)}_{78]}\right)$$
$$= N\int_{S^7}d\Omega\int_0^\infty dx\,\frac{x^7}{(1+x^2)^8}\mathrm{Tr}\left(\Sigma^{(-)}_{12}\Sigma^{(-)}_{34}\Sigma^{(-)}_{56}\Sigma^{(-)}_{78}\right)$$
$$= N\cdot\frac{2\pi^4}{\Gamma(4)}\frac{1}{280}\cdot(-128) = -N\frac{16\pi^4}{105}.$$

Therefore the normalization constant N is $\quad N := \dfrac{105}{16\pi^4} \Longrightarrow Q = -1.$

II-ii: 't Hooft type k-instanton

't Hooft type ADHM data are

$$C = \begin{pmatrix} 0 \\ \mathbf{1}_8\otimes\mathbf{1}_k \end{pmatrix}, \qquad D = \begin{pmatrix} S \\ e_\mu\otimes T^\mu \end{pmatrix}, \qquad \text{where} \quad \begin{aligned} S &= \mathbf{1}_8\otimes\left(\lambda_1 \ \ \lambda_2 \ \ \ldots \ \ \lambda_k\right), \\ T^\mu &= \mathrm{diag}^k_{p=1}\left(-a^\mu_p\right) \end{aligned}$$

Here, k is instanton charge, $a^\mu_p\in\mathbb{R}$ are position moduli and $\lambda_p\in\mathbb{R}$ are size moduli.

Therefore zero mode V is

$$V = \frac{1}{\sqrt{\phi}}\begin{pmatrix} -\mathbf{1}_8 \\ \left(\left(e^\dagger_\mu\otimes(x^\mu\mathbf{1}_k + T^\mu)\right)^{-1}S^\dagger\right) \end{pmatrix} \quad \text{where } \phi := 1 + \sum_{p=1}^k\frac{\lambda_p^2}{\|\tilde{x}_p\|^2} \text{ and } \tilde{x}_p := (x^\mu - a^\mu_p)e_\mu.$$

't Hooft k-instanton's gauge field is

$$A_\mu = -\frac{1}{2\phi}\sum_{p=1}^k\frac{\lambda_p^2(x^\nu - a^\nu_p)}{\|\tilde{x}_p\|^4}\Sigma^{(+)}_{\mu\nu} = \frac{1}{4}\Sigma^{(+)}_{\mu\nu}\partial_\nu\ln\phi.$$

't Hooft 2,3-instanton's charge density are

$$Q^{(k=2)} = -128\left(\frac{\lambda_1^2\|\tilde{x}_2\|^4 + \lambda_2^2\|\tilde{x}_1\|^4 + \lambda_1^2\lambda_2^2\left(\|\tilde{x}_1\|^2 + \|\tilde{x}_2\|^2 - 2\tilde{x}^\mu_1\tilde{x}^\mu_2\right)}{\left(\lambda_1^2\|\tilde{x}_2\|^2 + \lambda_2^2\|\tilde{x}_1\|^2 + \|\tilde{x}_1\|^2\|\tilde{x}_2\|^2\right)^2}\right)^4$$

$$Q^{(k=3)} = -128\Big[\gamma\Big(\lambda_1^2\|\tilde{x}_2\|^4\|\tilde{x}_3\|^4 + \lambda_2^2\lambda_3^2\|\tilde{x}_1\|^4\left(\|\tilde{x}_2\|^2 + \|\tilde{x}_3\|^2 - 2\tilde{x}^\mu_2\tilde{x}^\mu_3\right)$$
$$+ \lambda_2^2\|\tilde{x}_1\|^4\|\tilde{x}_3\|^4 + \lambda_1^2\lambda_3^2\|\tilde{x}_2\|^4\left(\|\tilde{x}_1\|^2 + \|\tilde{x}_3\|^2 - 2\tilde{x}^\mu_1\tilde{x}^\mu_3\right) + \lambda_3^2\|\tilde{x}_1\|^4\|\tilde{x}_2\|^4 + \lambda_1^2\lambda_2^2\|\tilde{x}_3\|^4\left(\|\tilde{x}_1\|^2 + \|\tilde{x}_2\|^2 - 2\tilde{x}^\mu_1\tilde{x}^\mu_2\right)\Big)\Big]^4.$$

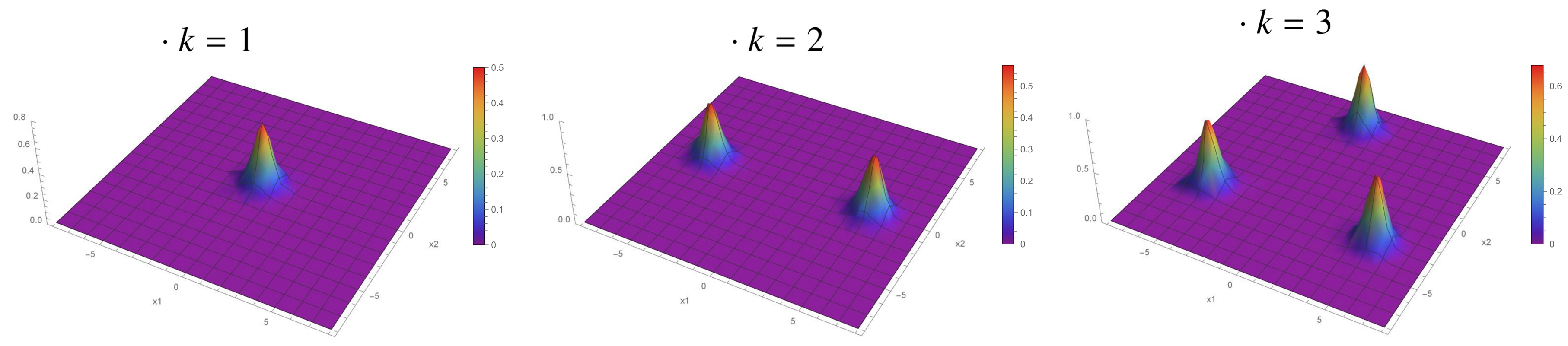
$$\text{where } \gamma := \frac{1}{\left(\lambda_1^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2 + \lambda_2^2\|\tilde{x}_1\|^2\|\tilde{x}_3\|^2 + \lambda_3^2\|\tilde{x}_1\|^2\|\tilde{x}_2\|^2 + \|\tilde{x}_1\|^2\|\tilde{x}_2\|^2\|\tilde{x}_3\|^2\right)^2}.$$

2,3-instanton's charge (numerical calculations)

$$Q^{(k=2)} = \int d^8x\,Q^{(k=2)} \simeq 2,$$
$$Q^{(k=3)} = \int d^8x\,Q^{(k=3)} \simeq 3.$$

Visualization of the charge densities. (x^1-x^2 plane)

· $k = 1$ $\qquad$ · $k = 2$ $\qquad$ · $k = 3$



III: SUMMARY AND FUTHER WORKS

- We have established the general scheme to construct the eight dimensional instantons, i.e. eight dimensional ADHM construction.
- We have also shown the explicit form of the higher charge solutions based on the 't Hooft ansatz.
- Is there the eight dimensional generalization of the Osborn's formula?
- Can we establish the eight dimensional noncommutative ADHM construction?
- Can we establish the seven/eight dimensional Nahm construction of monopole/caloron?
- Can we establish more general dimensional ADHM construction?
- How to relate between the eight dimensional ADHM and D-brane systems? etc…