COLLECTIVE COORDINATE QUANTIZATION OF THE CP^2 EXTENDED SKYRME-FADDEEV SOLITON

Y.AMARI[a], P.KLIMAS[b], N.SAWADO[a], Y.TAMAKI[a]

(a)**Tokyo University of Science**, (b)**Universidade Federal de Santa Catarina**

0.INTRODUCTION

- The Skyrme-Faddeev model is known as a low-effective theory of the $SU(2)$ Yang-Mills theory, and has vortex solutions and knot solitons.
  Glueballs !?
- The extended version of the Skyrme-Faddeev model on the target space $CP^N = \frac{SU(N+1)}{SU(N)\otimes U(1)}$, the CP^N extended Skyrme-Faddeev model, has been conjectured as a low-energy effective theory for the pure $SU(N+1)$ Yang-Mills theory.
- The CP^N extended Skyrme-Faddeev model possesses exact vortex solutions.
- For the case $N=2$, we regard the exact vortex solution as the glueball, and examine the mass spectrum of the vortex solutions by employing the collective coordinate quantization.

1.THE CP^N EXTENDED SKYRME-FADDEEV MODEL

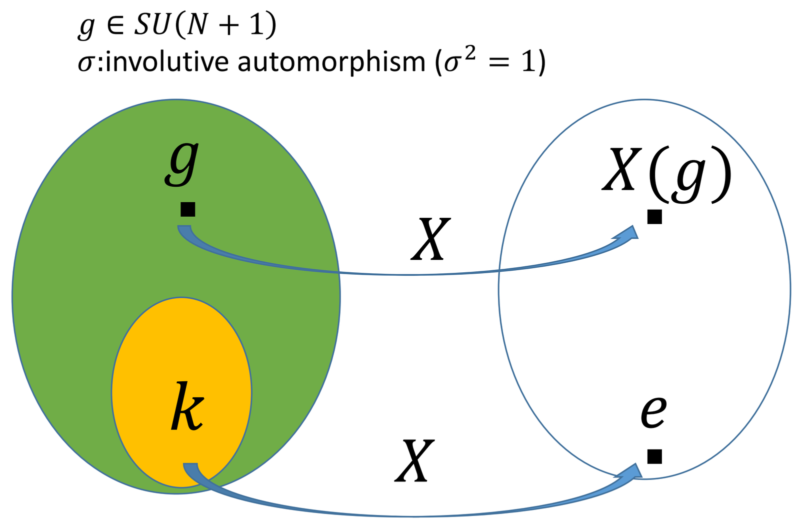
L.A. Ferreira,P.Klimas, JHEP1007.1667(2010)

The space CP^N can be naturally parametrized in terms of so called ***principal variable***.

Principal variable: $X(g) \equiv g\sigma(g)^{-1}$

$g \in SU(N+1)$
σ:involutive automorphism ($\sigma^2 = 1$)

$$\sigma(k) = k,\quad k \in SU(N)\otimes U(1)$$
$$X(k) = k\sigma(k)^{-1} = kk^{-1} = e$$
$$X(gk) = gk\sigma(gk)^{-1} = gk\sigma(k^{-1}g^{-1}) = gk\sigma(k^{-1})\sigma(g^{-1}) = g\sigma(g)^{-1} = X(g)$$



The Lagrangian density

$$\mathcal{L} = \frac{f^2}{2}\mathrm{Tr}(X^{-1}\partial_\mu X)^2 + \frac{1}{e^2}\mathrm{Tr}\left([X^{-1}\partial_\mu X, X^{-1}\partial_\nu X]\right)^2 + \frac{\beta}{2}\left[\mathrm{Tr}(X^{-1}\partial_\mu X)^2\right]^2 + \gamma\left[\mathrm{Tr}(X^{-1}\partial_\mu X\, X^{-1}\partial_\nu X)\right]^2$$

The Lagrangian has

- a ***global left $SU(N+1)$ symmetry*** $g \to \bar{g}g$, with $\bar{g}, g \in SU(N+1)$
  $X \to \bar{g}X\sigma(\bar{g})^{-1}$ and so $X^{-1}\partial_\mu X \to \sigma(\bar{g})X^{-1}\partial_\mu X\sigma(\bar{g})^{-1}$
- a ***right local $SU(N)\otimes U(1)$ symmetry*** $g \to gk$, with $g \in SU(N+1), k \in SU(N)\otimes U(1)$

The Principal variable X is parametrized in terms of ***complex fields*** u_i, where $i = 1, \dots, N$.

$$X = \begin{pmatrix} \mathbb{1}_{N\times N} & 0 \\ 0 & -1 \end{pmatrix} + \frac{2}{\vartheta^2}\begin{pmatrix} -\boldsymbol{u}\otimes\boldsymbol{u}^\dagger & i\boldsymbol{u} \\ i\boldsymbol{u}^\dagger & 1 \end{pmatrix} \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, \qquad \vartheta = \sqrt{1 + \boldsymbol{u}\cdot\boldsymbol{u}^\dagger}$$

Dimensionless cylindrical coordinates (t, ρ, φ, z)

$$x^0 = r_0 t,\ x^1 = r_0\rho\cos\varphi,\ x^2 = r_0\rho\sin\varphi,\ x^3 = r_0 z$$
$$ds^2 = r_0^2(dt^2 - d\rho^2 - \rho^2 d\varphi^2 - dz^2)$$

The length scale

$$r_0^2 = -\frac{4}{f^2e^2}$$

We impose

- ***Zero curvature condition*** $\partial_\mu u_i \partial^\mu u_j = 0$ for any $i, j = 1, \dots, N$
- Additional constraint $\beta e^2 + \gamma e^2 = 2$.

The exact vortex solutions with the infinite conserved quantities

- ***Static planar vortex*** $u_j = c_j\rho^{n_j}e^{i\epsilon_1 n_j\varphi}$
  with ***the energy per unit length*** $E = 8\pi f^2(n_{max} + |n_{min}|)$
- ***Traveling wave vortex*** $u_j = c_j\rho^{n_j}e^{i\epsilon_1 n_j\varphi}e^{ik_j(z+\epsilon_2 t)}$

n_j:integers, c_j:complex constants, k_j:the inverse of a wave length $\epsilon_a = \pm1, a = 1,2$.
n_{max}:the highest positive integer in the set n_j, n_{min}:the lowest negative integer in the same set.

The topological charge $Q_{top} = n_{max} + |n_{min}|$

After this, we consider the case $N=2$ for discussing low energy phenomena of QCD, especially ***glueball*** properties, and choose $\beta e^2 = 4$, $\gamma e^2 = -2$ because the parameter set becomes available for the Hamiltonian picture with satisfying $\beta e^2 + \gamma e^2 = 2$.

2.COLLECTIVE COORDINATE QUANTIZATION

We identify the glueball with the planar vortex whose "*height*" is h.

The Lagrangian has the rotational degree of freedom.

Zero mode $X \to X(\boldsymbol{r}; A) = AX(\boldsymbol{r})A^\dagger \qquad A \in SU(3) \qquad A$:collective coordinate

Promote A to $A(t)$ to remove the classical degeneracy of static configuration.

Dynamical ansatz $X(\boldsymbol{r}; A(t)) = A(t)X(\boldsymbol{r})A^\dagger(t)$

Assumptions $\dot{X} = 0,\ A\dot{A} = \frac{i}{2}\lambda_P\Omega^P$. λ_P:Gell-mann matrices

The effective Lagrangian can be written as $L_{eff} = \frac{1}{2}I_{PQ}\Omega^P\Omega^Q - M_{cl}$

This Lagrangian has the same form as the rotating symmetrical top.

The inertia tensor

$$I_{PQ} = \frac{4h}{e^2}\int \rho d\rho d\theta\left\{-\mathrm{Tr}\left(X^{-1}\left[\frac{\lambda_P}{2}, X\right]X^{-1}\left[\frac{\lambda_Q}{2}, X\right]\right) + \mathrm{Tr}\left(\left[X^{-1}\left[\frac{\lambda_P}{2}, X\right], X^{-1}\partial_k X\right]\left[X^{-1}\left[\frac{\lambda_Q}{2}, X\right], X^{-1}\partial_k X\right]\right) + \frac{\beta e^2}{2}\mathrm{Tr}\left(X^{-1}\left[\frac{\lambda_P}{2}, X\right]X^{-1}\left[\frac{\lambda_Q}{2}, X\right]\right)\mathrm{Tr}(X^{-1}\partial_k X X^{-1}\partial_k X) + \gamma e^2\mathrm{Tr}\left(X^{-1}\left[\frac{\lambda_P}{2}, X\right]X^{-1}\partial_k X\right)\mathrm{Tr}\left(X^{-1}\left[\frac{\lambda_Q}{2}, X\right]X^{-1}\partial_k X\right)\right\}$$

The classical mass $M_{cl} = Eh = 8\pi f^2 h(n_{max} + |n_{min}|)$

The conserved quantities $\mathcal{J}_P = \frac{\partial L}{\partial\Omega^P} = I_{PQ}\Omega^Q$

For the static field configurations, we use the exact planar vortex solutions $u_j = \rho^{n_j}e^{in_j\varphi}$ where $j = 1,2$.

(n_1, n_2)	$(2,0)$	$(3,0)$	$(3,1)$	$(4,1)$
I_{33}	$-113.8h/e^2$	$-132.2h/e^2$	$-257.3h/e^2$	$-274.6h/e^2$
$I_{38} = I_{83}$	$124.6h/e^2$	$195.9h/e^2$	$191.7h/e^2$	$267.8h/e^2$
I_{88}	$-257.6h/e^2$	$-358.4h/e^2$	$-183.4h/e^2$	$-290.9h/e^2$
$I_{44} = I_{55}$	$-178.7h/e^2$	$-242.2h/e^2$	$-178.3h/e^2$	$-240.0h/e^2$

(n_1, n_2)	$(0,2)$	$(0,3)$	$(1,3)$	$(1,4)$
I_{33}	$-113.8h/e^2$	$-132.2h/e^2$	$-257.3h/e^2$	$-274.6h/e^2$
$I_{38} = I_{83}$	$-124.6h/e^2$	$-195.9h/e^2$	$-191.7h/e^2$	$-267.8h/e^2$
I_{88}	$-257.6h/e^2$	$-358.4h/e^2$	$-183.4h/e^2$	$-290.9h/e^2$
$I_{66} = I_{77}$	$-178.7h/e^2$	$-242.2h/e^2$	$-178.3h/e^2$	$-240.0h/e^2$

For $n_1 > n_2$, $I_{11}, I_{22}, I_{66}, I_{77}$ diverge. For $n_1 < n_2$, $I_{11}, I_{22}, I_{44}, I_{55}$ diverge.
The rotations around the axes with infinite moments of inertia correspond are forbidden.

The effective Lagrangian and the quantum Hamiltonian

For $n_1 > n_2$
$$L = \frac{1}{2}I_{33}\Omega^3\Omega^3 + I_{38}\Omega^3\Omega^8 + \frac{1}{2}I_{88}\Omega^8\Omega^8 + \frac{1}{2}I_{44}(\Omega^4\Omega^4 + \Omega^5\Omega^5) - M_{cl}$$
$$H = M_{cl} + \frac{1}{I_{33}I_{88} - I_{38}^2}\left(\frac{I_{88}}{2}\mathcal{J}_3^2 - I_{38}\mathcal{J}_3\mathcal{J}_8 + \frac{I_{33}}{2}\mathcal{J}_8^2\right) + \frac{1}{2I_{44}}(\mathcal{J}_4^2 + \mathcal{J}_5^2)$$

For $n_1 < n_2$
$$L = \frac{1}{2}I_{33}\Omega^3\Omega^3 + I_{38}\Omega^3\Omega^8 + \frac{1}{2}I_{88}\Omega^8\Omega^8 + \frac{1}{2}I_{66}(\Omega^6\Omega^6 + \Omega^7\Omega^7) - M_{cl}$$
$$H = M_{cl} + \frac{1}{I_{33}I_{88} - I_{38}^2}\left(\frac{I_{88}}{2}\mathcal{J}_3^2 - I_{38}\mathcal{J}_3\mathcal{J}_8 + \frac{I_{33}}{2}\mathcal{J}_8^2\right) + \frac{1}{2I_{66}}(\mathcal{J}_6^2 + \mathcal{J}_7^2)$$

Hereafter we consider for the case $n_1 > n_2$.
Taking the divergence of the moments of inertia into consideration, the rotation matrix is

$$A = e^{-i\alpha\lambda_3/2}e^{-i\beta\lambda_8/\sqrt{3}}e^{-i\gamma\lambda_4}e^{i\delta(\lambda_3+\sqrt{3}\lambda_8)/4}. \qquad \alpha, \beta, \gamma, \delta\text{: Eular angles}$$

Define the angular momentum operators as $[\mathcal{J}_3, A] = -\frac{\lambda_3}{2}A,\ [\mathcal{J}_8, A] = -\frac{\lambda_8}{\sqrt{3}}A,\ [\mathcal{J}_4, A] = -\lambda_4 A,\ [\mathcal{J}_5, A] = -\lambda_5 A.$

The angular momentum operators

$$\mathcal{J}_3 = -i\frac{\partial}{\partial\alpha},\quad \mathcal{J}_4 = i\frac{\sin\left(\frac{\alpha}{2}+\beta\right)}{\sin 2\gamma}\left\{\cos 2\gamma\frac{\partial}{\partial\alpha} + \frac{3}{2}\cos 2\gamma\frac{\partial}{\partial\beta} + 2\frac{\partial}{\partial\delta}\right\} - i\cos\left(\frac{\alpha}{2}+\beta\right)\frac{\partial}{\partial\gamma}$$

$$\mathcal{J}_8 = -i\frac{\partial}{\partial\beta},\quad \mathcal{J}_5 = -i\frac{\cos\left(\frac{\alpha}{2}+\beta\right)}{\sin 2\gamma}\left\{\cos 2\gamma\frac{\partial}{\partial\alpha} + \frac{3}{2}\cos 2\gamma\frac{\partial}{\partial\beta} + 2\frac{\partial}{\partial\delta}\right\} - i\sin\left(\frac{\alpha}{2}+\beta\right)\frac{\partial}{\partial\gamma}$$

Commutation relations

$$[\mathcal{J}_3, \mathcal{J}_4] = \frac{i}{2}\mathcal{J}_5,\quad [\mathcal{J}_3, \mathcal{J}_5] = -\frac{i}{2}\mathcal{J}_4,\quad [\mathcal{J}_3, \mathcal{J}_8] = 0,$$
$$[\mathcal{J}_8, \mathcal{J}_4] = i\mathcal{J}_5,\quad [\mathcal{J}_8, \mathcal{J}_5] = -i\mathcal{J}_4,\quad [\mathcal{J}_4, \mathcal{J}_5] = 2i\mathcal{J}_3 + 3i\mathcal{J}_8$$

The casimir operators $D^{(1)} \equiv 2\mathcal{J}_3 - \mathcal{J}_8,\ D^{(2)} \equiv \mathcal{J}_3^2 + \frac{3}{4}\mathcal{J}_8^2 + \frac{1}{4}\mathcal{J}_4^2 + \frac{1}{4}\mathcal{J}_5^2$

The simultaneous eigenfunction of $\mathcal{J}_3, \mathcal{J}_8, D^{(2)}$ and H

$$\psi^l_{Y,m,m'} = e^{i\alpha m}e^{i\beta Y}d^l_{\frac{3Y+2m}{4},m'}(2\gamma)e^{i\delta m'}$$

d:Wigner small d-function
m:z-projection of an isospin, Y:Hyper charge
m':z-projection of a spin, l:spin

The eigenevalue equations

$$\mathcal{J}_3\psi = m\psi,\quad \mathcal{J}_8\psi = Y\psi,\quad D^{(2)}\psi = \left\{l(l+1) + \frac{3(Y-2m)^2}{16}\right\}\psi$$

$$H\psi = \left[M_{cl} + \frac{1}{I_{33}I_{88} - I_{38}^2}\left(\frac{I_{88}}{2}\mathcal{J}_3^2 - I_{38}\mathcal{J}_3\mathcal{J}_8 + \frac{I_{33}}{2}\mathcal{J}_8^2\right) + \frac{1}{2I_{44}}\left(4D^{(2)} - 4\mathcal{J}_3^2 - 3\mathcal{J}_8^2\right)\right]\psi$$
$$= \left[M_{cl} + \frac{1}{I_{33}I_{88} - I_{38}^2}\left(\frac{I_{88}}{2}m^2 - I_{38}mY + \frac{I_{33}}{2}Y^2\right) + \frac{2}{I_{44}}\left\{l(l+1) - \left(\frac{3Y+2m}{4}\right)^2\right\}\right]\psi$$

Mass spectrum of the quantum vortex

Define the parity operator and the parity eigenvalue as $\hat{P}AX(\boldsymbol{r})A^{-1}\hat{P}^{-1} = X(-\boldsymbol{r}), \hat{P}\psi = P\psi.$

parity

$$\hat{P} = \exp\left[-\frac{\pi}{4}\left\{(n_1 - 2n_2)iD^{(1)} + 4n_1\frac{\partial}{\partial\delta}\right\}\right],\quad P = \exp\left[-\frac{\pi i}{4}\{(n_1 - 2n_2)(2m - Y) + 4n_1 m'\}\right]$$

3.PHYSICAL INTERPRETATION AS A GLUEBALL

The glueball is isosinglet particle labeled by the total angular momentum J, the parity P, and the C-parity C.

For the isosinglet particle, we obtain the mass and the parity

$$M = M_{cl} + \frac{2l(l+1)}{I_{44}},\quad P = (-1)^{n_1 m'}$$

Glueball masses in lattice for pure spin states

	J^{PC}	Mass(MeV)
2-gluon state	0^{++}	1730
	2^{++}	2400
3-gluon state	1^{--}	3850
	3^{--}	4130

C.J.Morningstar et.al. Phys.Rev.D60,034509(1999)

We identify number of component gluons with the winding numbers.

2-gluon state $\leftrightarrow$ $(n_1, n_2) = (2,0)$, 3-gluon state $\leftrightarrow$ $(n_1, n_2) = (3,0)$

A gluon in a massless representation has only two spin projections. **C.N.Yang phys.Rev.77 242(1950)**

For 2-gluon state, m' is even and $P = +$. For 3-gluon state, m' is odd and $P = -$.

The vortex height h should depend on number of component gluons, so we describe it as h_{n_1} and the corresponding moment of inertia I_{44} as I_{n_1}.

0^{++}	$M = 16\pi f^2 h_2$	2130MeV	Input	Input	Input
2^{++}	$M = 16\pi f^2 h_2 + 12/I_2$	Input	2064MeV	Input	Input
1^{--}	$M = 24\pi f^2 h_3 + 4/I_3$	Input	Input	3495MeV	Input
3^{--}	$M = 24\pi f^2 h_3 + 24/I_3$	Input	Input	Input	4422MeV
	h_3/h_2	1.19	1.46	1.30	1.44

We can derive the glueball mass which agree with the values in lattice gauge theory within 23%.