Dissipative Models and Nonequilibrium Statistical Approach

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Sec 1. Introduction: a.Boltzmann eq.

$$\frac{1}{h}\{f_n(x+h\ \ u_{n-1}(x),\ \ v)-f_{n-1}(x,v)\}\ =\ \Omega_n$$

Boltzmann Equation, 1872

2nd Law of Thermodynamics

Dynamical Origin: Einstein Theory (Geometry of "dynamics") ?

- $\mathbf{u}(\mathbf{x},\ 't')$: Velocity distribution of Fluid Matter
- Size of fluid-particles: L Atomic (10^{-10}m)$\ll$L$\leq$ Optical Microscope (10^{-6})m
- Temporal development of Distribution Function $f('t',\mathbf{x},\mathbf{v})$: probability of particle having velocity $\mathbf{v}$ at space $\mathbf{x}$ and time $'t'$

Sec 1. Introduction: b.Energy with Dissipation

Notion of Energy is obscure when Dissipation occurs.
Consider the movement of a particle under the influence of the friction force.
The emergent heat (energy) during the period [t_1, t_2] can **not** be written as.

$$\int_{x_1}^{x_2} F_{\text{friction}}\, dx = [E\{x(t), \dot{x}(t)\}]_{t_1}^{t_2} = E|_{t_2} - E|_{t_1}, \quad x_1 = x(t_1), x_2 = x(t_2) \qquad (1)$$

where $x(t)$: Orbit (path) of Particle.

Sec 1. Introduction: c.Discrete Morse Flow Theory(DMFT)

- Time should be re-considered, when dissipation occurs.
  $\rightarrow$ Step-Wise approach to time-development.
- Connection between step n and step $n-1$ is determined by the minimal energy principle.
- Time is "emergent" from the principle.
- Direction of flow (arrow of time) is built in from the beginning.

New approach to Statistical Fluctuation

Discrete Morse Flow Method(Kikuchi, '91)

Holography (AdS/CFT, '98)

Sec 2. Spring-Block Model a.Model Figure

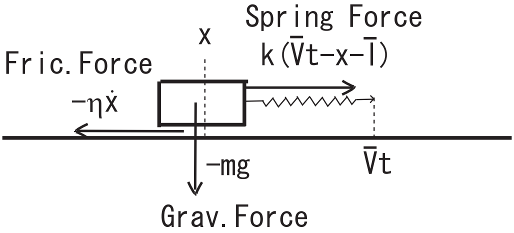

Figure: *The spring-block model, (4).*

Sec 2. Spring-Block Model b.Energy Functional

$$K_n(x) = V(x) - hnk\bar{V}x + \frac{\eta}{2h}(x - x_{n-1})^2$$
$$+\frac{m}{2h^2}(x - 2x_{n-1} + x_{n-2})^2 + K_n^0 \ , \ V(x) = \frac{kx^2}{2} + k\bar{\ell}x, \qquad (2)$$

Sec 2. Spring-Block Model c.Variat. Principle

Energy Minimal Principle

$$\left.\frac{\delta K_n(x)}{\delta x}\right|_{x=x_n} = 0 \quad .$$

$$\frac{k}{m}(x_n + \bar{\ell} - nh\bar{V}) + \frac{1}{h^2}(x_n - 2x_{n-1} + x_{n-2}) + \frac{\eta}{m}\frac{1}{h}(x_n - x_{n-1}) = 0 \ , \ \omega \equiv \sqrt{\frac{k}{m}} \ , \ \eta' \equiv \frac{\eta}{m}, \tag{3}$$

where $n = 2, 3, 4, \cdots$.

Sec 2. Spring-Block Model d.Continuous Limit

$$m\ddot{x} = k(\bar{V}t - x - \bar{\ell}) - \eta\dot{x} \quad . \tag{4}$$

This is the spring-block model. See Fig.1. The graph of movement (x_n, eq.(3)) is shown in Fig.2. Fig.3 shows the energy change as the step flows.

Sec 2. Spring-Block Model e.Model

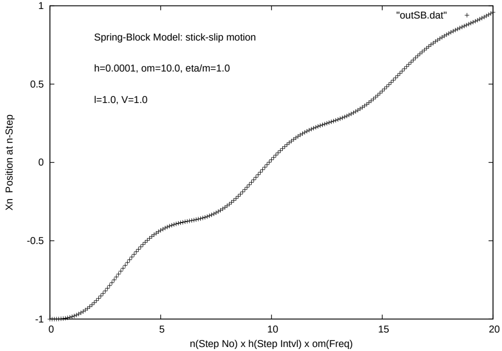

Figure: *Spring-Block Model, Movement,* $h{=}0.0001, \sqrt{k/m}{=}10.0$, $\eta/m{=}1.0$, $\bar{V}{=}1.0$, $\bar{\ell}{=}1.0$, total step no $=20000$. *The step-wise solution (3) correctly reproduces the analytic solution:* $x(t) = \mathrm{e}^{-\eta' t/2}\bar{V}\{(\eta'^2/2\omega^2 - 1)(\sin\Omega t)/\Omega + (\eta'/\omega^2)\cos\Omega t\} - \bar{\ell} + \bar{V}(t - \eta'/\omega^2)$ $\quad, \Omega = (1/2)\sqrt{4\omega^2 - \eta'^2} = 9.99$ $\quad, 0 \le t \le 2 \quad , \quad x(0) =$

Sec 2. Spring-Block Model f.Energy Change

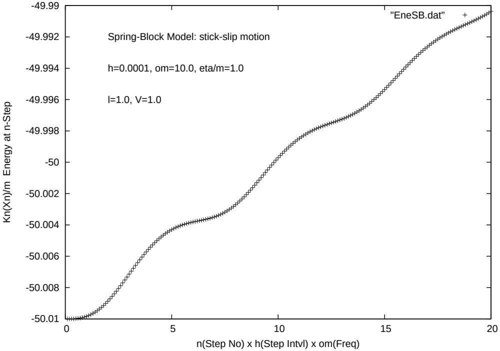

Figure: *Spring-Block Model, Energy Change,* h=0.0001,$\sqrt{k/m}$=10.0, η/m=1.0, $\bar{V}$=1.0, $\bar{\ell}$=1.0, *total step no* =20000.

Sec 2. Spring-Block Model : g.Bulk Metric

$$
\begin{aligned}
\Delta {s_n}^2 &\equiv 2h^2(K_n(x_n) - K_n^0) \\
&= 2\ dt^2 V_1(X_n) + (\Delta X_n)^2 + (\Delta P_n)^2, \\
V_1(X_n) &\equiv V(\frac{X_n}{\sqrt{\eta h}}) - nk\sqrt{\frac{h}{\eta}}\bar{V} X_n,\ dt \equiv h,
\end{aligned} \tag{5}
$$

where $X_n \equiv \sqrt{\eta h} x_n,\ P_n/\sqrt{m} \equiv h v_n = (x_n - x_{n-1}),.$

Sec 2. Spring-Block Model : h.Ensemble 1a

The first choice of the metric in the 3D (t,X,P) is the Dirac-type one:

$$(ds^2)_D \equiv 2V_1(X)dt^2 + dX^2 + dP^2$$
$$- \text{ on-path } (X = y(t), P = w(t)) \rightarrow$$
$$(2V_1(y) + \dot{y}^2 + \dot{w}^2)dt^2, \quad (6)$$

where $\{(y(t), w(t))|0 \leq t \leq \beta\}$ is a path (line) in the 3D space. See Fig.4.

Sec 2. Spring-Block Model i.Path in 3D

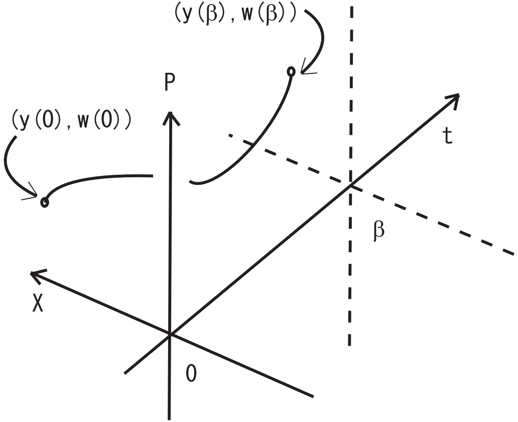

Figure: *The path $\{(y(t), w(t), t)|0 \leq t \leq \beta\}$ of line in 3D bulk space* (X,P,t).

Sec 2. Spring-Block Model : j.1st Geometry

$$L_D = \int_0^\beta ds|_{on-path} = \int_0^\beta \sqrt{2V_1(y) + \dot{y}^2 + \dot{w}^2}dt$$

$$= h \sum_{n=0}^{\beta/h} \sqrt{2V_1(y_n) + \dot{y}_n^2 + \dot{w}_n^2},\ d\mu = \mathrm{e}^{-\frac{1}{\alpha}L_D} \prod_t \mathcal{D}y\mathcal{D}w,$$

$$\mathrm{e}^{-\beta F} = \int \prod_n dy_n dw_n \mathrm{e}^{-\frac{1}{\alpha}L_D}, \qquad (7)$$

where the free energy F is defined.

Sec 2. Spring-Block Model : k.Ensemble 1b

The second choice of the metric is the standard type:

$$(ds^2)_S \equiv \frac{1}{dt^2}[(ds^2)_D]^2 \quad -\text{on-path}\rightarrow \\ (2V_1(y)+\dot{y}^2+\dot{w}^2)^2 dt^2. \qquad (8)$$

Sec 2. Spring-Block Model : I.2nd Geometry

$$L_S = \int_0^\beta ds|_{on-path} = \int_0^\beta (2V_1(y) + \dot{y}^2 + \dot{w}^2)dt = h\sum_{n=0}^{\beta/h}(2V_1(y_n) + \dot{y}_n^2 + \dot{w}_n^2),$$

$$d\mu = \mathrm{e}^{-\frac{1}{\alpha}L_S}\mathcal{D}y\mathcal{D}w,\ \ \mathrm{e}^{-\beta F} = \int\prod_n dy_n dw_n \mathrm{e}^{-\frac{1}{\alpha}L_S} = (\mathrm{const})\int\prod_{n=0}^{\beta/h} dy_n \mathrm{e}^{-\frac{h}{\alpha}(2V_1(y_n)+\dot{y}_n^2)}. \quad (9)$$

Sec 2. Spring-Block Model : m.Minimal Path

The minimal path of (9), by changing $y_n \rightarrow y,\ nh \rightarrow t$ and using the variation $y \rightarrow y + \delta y$, we obtain

$$-\eta h \ddot{x} = k(\bar{V}t - x - \bar{\ell}), \quad x = \frac{y}{\sqrt{\eta h}} \quad . \tag{10}$$

比較 $$m\ddot{x} = k(\bar{V}t - x - \bar{\ell}), \quad \text{(4) with } \eta = 0 \quad . \tag{11}$$

Sec 2. Spring-Block Model : n.Comp. w. (4)

1) the viscous term disappeared;
2) the mass parameter m is replaced by ηh;
3) the sign in front of the acceleration-term (inertial-term) is different.

By changing to the Euclidean time $\tau = it$, the above equation reduces to the harmonic oscillator when we take $\bar{V} = 0, \ \bar{\ell} = 0$.

Sec 2. Spring-Block Model : o.Ensemble 2

$$
\begin{aligned}
(ds^2)_D \equiv 2V_1(X)dt^2 + dX^2 + dP^2 \equiv e_1 G_{IJ}(\tilde{X}) d\tilde{X}^I d\tilde{X}^J, \\
I, J = 0, 1, 2;\ (\tilde{X}^0, \tilde{X}^1, \tilde{X}^2) \equiv (t/d_0, X/d_1, P/d_2) \\
e_1 = m\bar{\ell}^2,\ d_0 = \sqrt{\frac{k}{m}}, d_1 = d_2 = \sqrt{m}\bar{\ell}, \\
(G_{IJ}) = \begin{pmatrix} 2{d_0}^2 V_1(d_1\tilde{X}^1) & 0 & 0 \\ 0 & {d_1}^2 & 0 \\ 0 & 0 & {d_2}^2, \end{pmatrix} \qquad (12)
\end{aligned}
$$

where we have introduced the *dimensionless* coordinates $\tilde{X}^I$.

Sec 2. Spring-Block Model : p.Surface in 3D

$$\frac{X^2}{{d_1}^2}+\frac{P^2}{{d_2}^2}=\frac{r(t)^2}{{d_1}^2},\quad 0\leq t\leq\beta, \tag{13}$$

where the radius parameter r is chosen to have the dimension of $\sqrt{M}L$. See Fig.5.

Sec 2. Spring-Block Model : q.Surface in 3D

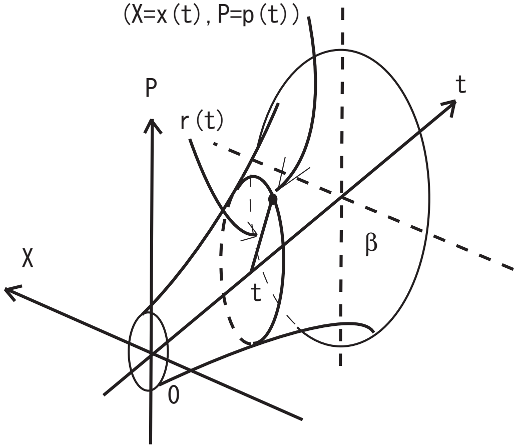

Figure: *The two dimensional surface, (13), in 3D bulk space (X,P,t).*

Sec 2. Spring-Block Model : s.3rd Geometry

$$
\begin{aligned}
(ds^2)_D\big|_{\text{on-path}} &= 2V_1(X)dt^2 + dX^2 + dP^2\big|_{\text{on-path}} \\
&= e_1 \sum_{i,j=1}^{2} g_{ij}(\tilde{X})d\tilde{X}^i d\tilde{X}^j \quad , \quad e_1 = m\bar{\ell}^2 \quad , \\
(g_{ij}) &= \begin{pmatrix} 1 + \frac{e_1}{{d_1}^2}\frac{2V_1}{r^2\dot{r}^2}X^2 & \frac{e_1}{d_1 d_2}\frac{2V_1}{r^2\dot{r}^2}XP \\ \frac{e_1}{d_1 d_2}\frac{2V_1}{r^2\dot{r}^2}PX & 1 + \frac{e_1}{{d_2}^2}\frac{2V_1}{r^2\dot{r}^2}P^2 \end{pmatrix} , \qquad (14)
\end{aligned}
$$

Sec 2. Spring-Block Model : t.3rd Distribution

The third partition function $\mathrm{e}^{-\beta F}$ is given by

$$A = \int \sqrt{\det g_{ij}} d^2 \tilde{X} = \frac{1}{d_1 d_2} \int \sqrt{1 + \frac{2V_1}{\dot{r}^2}} dX dP,$$
$$\mathrm{e}^{-\beta F} = \int_0^\infty d\rho \int_{\substack{r(0)=\rho \\ r(\beta)=\rho}} \prod_t \mathcal{D}X(t)\mathcal{D}P(t) \mathrm{e}^{-\frac{1}{\alpha}A}, \qquad (15)$$

where α is the (dimensionless) "string" constant and here is a model parameter.

Sec 3. Burridge-Knopoff Model a. Model Figure

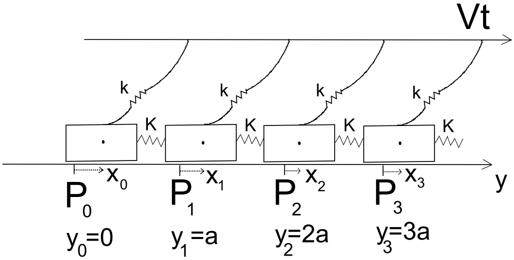

Figure: *Burridge-Knopoff Model (17)*

Sec 3. Burridge-Knopoff Model b.Energy Function

n-th energy function to define Burridge-Knopoff (BK) model in the step(n) flow method.

$$I_n(x) = -xF(\dot{x}_{n-1}) + G(\dot{x}_{n-1})\frac{1}{a}(x - x_{n-1})(\dot{x}_{n-1} - \dot{x}_{n-2})$$
$$+\frac{m}{2}(\frac{dx}{dt})^2 - \frac{k}{2}(x - Vt)^2 + \frac{K}{2a^2}(x - 2x_{n-1} + x_{n-2})^2 + I_n^0, \qquad (16)$$

where $\dot{x}_n = dx_n(t)/dt$. t is the time variable.

Sec 3. Burridge-Knopoff Model c.Model Parameters

I_n^0 : a constant term, not depend on $x(t)$.
The system: N particles (blocks) distributing over the (1-dim) space $\{y\}$. y is periodic: $y \rightarrow y + 2L$.
The particles are moving around the equilibrium points $\{P_n \mid n = 1, 2, \cdots, n-1, N\}$ where $P_N \equiv P_0$.
The point P_n is located at $y = y_n \equiv na$ $(Na = 2L)$ where a is the 'lattice-spacing'.
$N(= 2L/a)$ is a huge number and the present system constitutes the statistical ensemble.
The n-th particle's position at t, $x_n(t)$ (deviation from the equilibrium point P_n) is determined by the energy minimal principle $\delta I_n(x)|_{x=x_n} = 0$ with the pre-known movement of the (n-1)-th particle, $x_{n-1}(t)$, and that of the (n-2)-th, $x_{n-2}(t)$.

Sec 3. Burridge-Knopoff Model d.Recurs. Relation

$$-m\frac{d^2x_n}{dt^2} - F(\dot{x}_{n-1}) + G(\dot{x}_{n-1})\ \frac{\dot{x}_{n-1} - \dot{x}_{n-2}}{a}$$
$$-k\ (x_n - Vt) + \frac{K}{a^2}\ (x_n - 2x_{n-1} + x_{n-2}) = 0, \qquad (17)$$

where $0 \leq t \leq \beta$, and $F(\dot{x}_{n-1})$ and $G(\dot{x}_{n-1})$ are some functions of $\dot{x}_{n-1}$.

Sec 3. Burridge-Knopoff Model e.Conti. Space Limit

In the continuous space limit, the step flow equation (17) reduces to

$$-m\frac{\partial^2 x}{\partial t^2} - F(\dot{x}) + G(\dot{x})\frac{\partial^2 x}{\partial y \partial t} - k(x - Vt) + K\frac{\partial^2 x}{\partial y^2} = 0,$$
$$x = x(t,y) \quad , \quad \dot{x} = \frac{\partial x(t,y)}{\partial t} \quad . \qquad (18)$$

Sec 3. Burridge-Knopoff Model f. Metric'

$$
\begin{aligned}
\Delta s_n{}^2 \equiv 2a^2(I_n(x_n) - I_n^0) = \\
\{-2x_n F(\dot{x}_{n-1}) + m\dot{x}_n^2 - k(x_n - Vt)^2\}dy^2 \\
-a\frac{\partial G(\dot{x}_{n-1})}{\partial t}\ \Delta x_n{}^2 + Ka^2\ \Delta \tilde{v}_n{}^2 \quad , \quad dy \equiv a, \\
\Delta x_n \equiv x_n - x_{n-1},\ \frac{x_n - x_{n-1}}{a} \equiv \tilde{v}_n,\ \tilde{v}_n - \tilde{v}_{n-1} = \Delta \tilde{v}_n,
\end{aligned}
\qquad (19)
$$

where we assume $\Delta \dot{x}_{n-1} = \Delta \dot{x}_n$. $\tilde{v}_n$ is the *longitudinal strain*.

Sec 3. Burridge-Knopoff Model g.Metric

$$
\begin{aligned}
\widetilde{ds}^2 &= \{-2xF(v) + mv^2 - k(x-Vt)^2\}(dy^2 - dt^2) \\
&\quad + ma^2 dv^2 - a\frac{\partial G(v)}{\partial t}\, dx^2 + Ka^2\,(\frac{\partial v}{\partial y})^2 dt^2 \\
&= e_1 G_{IJ}(X) dX^I dX^J, e_1 = Ka^2 \text{ or } ma^2V^2, v \equiv \dot{x} = \frac{\partial x}{\partial t}, \\
&(X^I) = (X^0, X^1, X^2, X^3) = (t/d_0, y/d_1, x/d_2, v/d_3), \\
&d_0 = \sqrt{\frac{m}{k}}, d_1 = V\sqrt{\frac{m}{k}}, d_2 = \sqrt{\frac{K}{k}}, d_3 = \sqrt{\frac{K}{m}},
\end{aligned}
\qquad (20)
$$

where we use $d\tilde{v} = d(\partial x/\partial y) = (\partial v/\partial y)dt$.

Sec 3. Burridge-Knopoff Model h.Map

The map: 2D space $\{(t, y)|\ 0 \leq t \leq \beta,\ 0 \leq y \leq 2L\}$ —> 4D space (t, y, x, v).

$$x = \bar{x}(t, y),\ v = \bar{v}(t, y),$$
$$d\bar{x} = \frac{\partial \bar{x}}{\partial t}dt + \frac{\partial \bar{x}}{\partial y}dy,\ d\bar{v} = \frac{\partial \bar{v}}{\partial t}dt + \frac{\partial \bar{v}}{\partial y}dy. \quad (21)$$

This map expresses a 2D *surface* in the 4D space (Fig.7).

Sec 3. Burridge-Knopoff Model i. Map figure

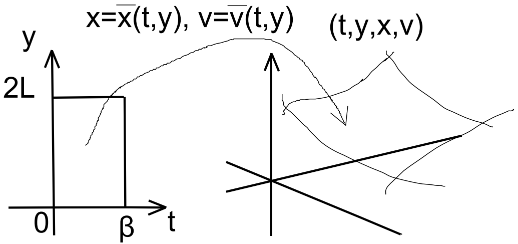

Figure: *The two dimensional surface, (21), in 4D space (t,y,x,v).*

Sec 3. Burridge-Knopoff Model j.Geometry

On the surface, the line element (20) reduces to

$$\widetilde{ds}^2 \quad - \text{on surface} \rightarrow \ e_1 g_{ij}(X) dX^i dX^j, \quad g_{00} =$$
$$\frac{a^2}{e_1}\left\{ -H(\bar{x},\bar{v}) + ma^2 (\frac{\partial \bar{v}}{\partial t})^2 - \frac{\partial G}{\partial t}(\frac{\partial \bar{x}}{\partial t})^2 + Ka^2(\frac{\partial \bar{v}}{\partial y})^2 \right\},$$
$$g_{01} = g_{10} = \frac{a^2\sqrt{m}}{{e_1}^{3/2}}\left\{ ma^2 \frac{\partial \bar{v}}{\partial t}\frac{\partial \bar{v}}{\partial y} - \frac{\partial G}{\partial t}\frac{\partial \bar{x}}{\partial t}\frac{\partial \bar{x}}{\partial y} \right\},$$
$$g_{11} = \frac{a^2}{e_1}\left\{ H(\bar{x},\bar{v}) + ma^2(\frac{\partial \bar{v}}{\partial y})^2 - \frac{\partial G}{\partial t}(\frac{\partial \bar{x}}{\partial y})^2 \right\},$$
$$H(\bar{x},\bar{v}) \equiv -2\bar{x}F(\bar{v}) + m\bar{v}^2 - k(\bar{x}-Vt)^2, \qquad (22)$$

where $\frac{\partial G}{\partial t} = \frac{dG(\bar{v})}{d\bar{v}}\frac{\partial \bar{v}}{\partial t}$ and $i = 0, 1$.

Sec 3. Burridge-Knopoff Model k.Distribution

Using the (dimensionless) surface area A, the partition function $\mathrm{e}^{-\beta F}$ is given by

$$A[\bar{x}(t,y),\bar{v}(t,y)] = \frac{1}{d_0 d_1}\int_0^{\beta} dt \int_0^{2L} dy \sqrt{\det g_{ij}},$$
$$\mathrm{e}^{-\beta F} = \int \prod_{t,y} \mathcal{D}\bar{x}(t,y)\mathcal{D}\bar{v}(t,y)\mathrm{e}^{-\frac{1}{\alpha}A}, \qquad (23)$$

where α is a dimensionless model parameter.

Sec 3. Burridge-Knopoff Model I.Minimal Area Surface

The *minimum area surface*, which gives the main contribution to the above quantity, is given by the following equation.

$$\frac{\partial A}{\partial \bar{x}(t,y)} = 0 \ , \quad \frac{\partial A}{\partial \bar{v}(t,y)} = 0. \tag{24}$$

Sec 4. Conclusion a.What has been done

Two friction (earthquake) models: the spring-block model and Burridge-Knopoff model.

How to evaluate the statistical fluctuation effect.

Based on the geometry appearing in the system dynamics.

Sec 4. Conclusion b.Multiple Scales

Multiple scales exist in both models.
SB model: 1. the natural length of the string $\bar{\ell}$
2. the external velocity $\bar{V}$.
BK model; 1. the external velocity V
2. the spring constant K
3. the block spacing a.
The use of dimensionless quantities clarifies the description.
The multiple scales indicate the existence of the fruitful phases in the present statistical systems.

Sec 4. Conclusion c.Minimal Principle

- The dissipative systems are treated by using the *minimal principle*.
- The difficulty of the *hysteresis* effect (non-Markovian effect) [3]is avoided in the present approach. These are the advantage of the discrete Morse flow method. We do not use the ordinary time t, instead, exploit the step number n ($t_n = nh$).
- Several theoretical proposals for the statistical ensembles appearing in the friction phenomena.
- Necessary to *numerically* evaluate the models with the proposed ensembles and compare the result with the real data appearing both in the natural phenomena and in the laboratory experiment.

Sec 5. References

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