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Exact vs. High-Energy symmetries in String Scattering Amplitudes

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with **Chuan-Tsung Chan** and **Dan Tomino**

High-energy scatterings in string theory

String theory scattering amplitudes (bosonic open 4-pt amplitudes)

$$\mathcal{A}_G = \int_{\Sigma_G} (\text{ghost}) \langle V_1(k_1,x_1) V_2(k_2,x_2) V_3(k_3,x_3) V_4(k_4,x_4) \rangle$$

$$V(k,x) = V^{\text{pol}}(\partial X, \partial^2 X, \cdots) e^{ik\cdot X}$$: vertex operators

Fixed-angle High-energy limit: $\alpha' s \to \infty \quad t/s = \text{fixed}$

→ Can be evaluated by the saddle point method [Gross-Mende, Gross-Manes, ...]

$$\mathcal{A}_G \sim \mathcal{A}_G^{\text{tachyon}} \cdot \prod_i \underline{V_i^{\text{pol}}(\{k_j\})} + \cdots$$

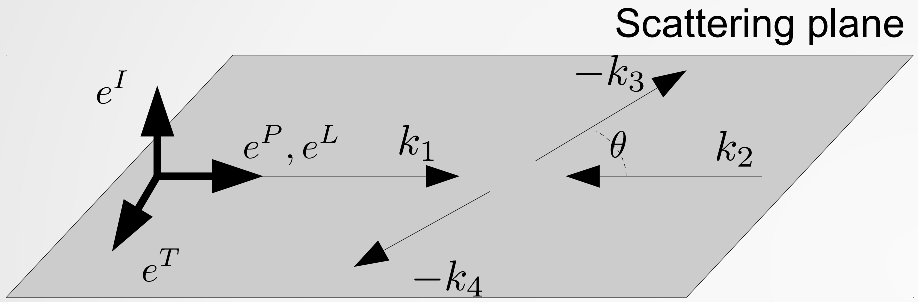
Polynomials in momenta

"Veneziano" part including $e^{-\frac{\alpha'}{G+1}(s\ln s + t\ln t + u\ln u)}$

Linear relations and high-energy symmetry?

Simple relations among amplitudes [Gross]

Helicity basis in the CM frame



$$\mathcal{T}^{\mu\nu} = \int \langle (- : \partial X^\mu \partial X^\nu) e^{ik_1\cdot X} :: e^{ik_2\cdot X} :: e^{ik_3\cdot X} :: e^{ik_4\cdot X} : \rangle$$

$$s = -(k_1 + k_2)^2$$
$$t = -(k_1 + k_3)^2$$

$$4\mathcal{T}^{LL} = \mathcal{T}^{TT}\left(1 + \mathcal{O}(s^{-1})\right)$$: linear relation

High-energy symmetry:

·Infinitely many linear relations
·New identity due to enhancement of symmetry?

cf) Decoupling of “high-energy zero-norm states”

[Lee, Chan, Yi, Ho, Teraguchi, Lin, Ko, Mitsuka, ...]

Plan

1. Introduction
2. Deformation of vertex operators and relation among amplitudes

   [Moore ('93)]
3. High-energy expansion
4. Conclusion and Discussion

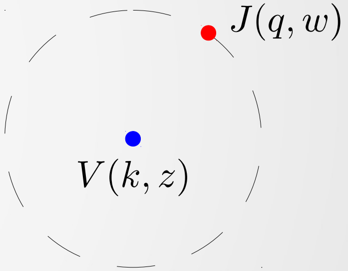
Bracket operation

$$\{\mathcal{J}(q), V(k,z)\} \equiv \oint_z \frac{dw}{2\pi i} J(q,w)V(k,z) = V^{\mathrm{br}}(\tilde{k}, z)$$

$(\tilde{k} \equiv k + q)$

Example: $J_{(1)}(q,w) = i\,\zeta_q \cdot \partial X e^{iq\cdot X}(w)$: “deformer” operator

$V_{(0)}(k,z) =: e^{ik\cdot X} : (z)$: “seed” operator



$$\oint_z \frac{dw}{2\pi i} J_{(1)}(q,w)V_{(0)}(k,z)$$

$$= \oint_z \frac{dw}{2\pi i} (w-z)^{q\cdot k} : \left[\frac{\zeta_q \cdot k}{w-z} + i\zeta_q \cdot \partial X(w)\right] e^{iq\cdot X(w) + ik\cdot X(z)} :$$

Mutually local: $q \cdot k \in \mathbf{Z}$

$(\alpha' = 1/2)$

Bracket operators

$$\{\mathcal{J}_{(1)}(q), V_{(0)}(k,z)\} \equiv \begin{cases} 0 & q\cdot k \geq 1 \\ \zeta_q \cdot k \, :e^{i\tilde{k}\cdot X}: & q\cdot k = 0 \\ i\zeta_{(1)}\cdot \partial X \, e^{i\tilde{k}\cdot X} & q\cdot k = -1 \\ :\big(-\zeta_{(2)\mu\nu}\partial X^{\mu}\partial X^{\nu} + i\zeta_{(2)\mu}\partial^2 X^{\mu}\big)\, e^{i\tilde{k}\cdot X}: & \\ \quad \vdots & q\cdot k = -2 \end{cases}$$

$$\zeta_{(2)\mu\nu} = \zeta_{q(\mu}q_{\nu)} + (\zeta_q\cdot k)q_\mu q_\nu \qquad (\tilde{k} = k+q)$$

$$\zeta_{(1)\mu} = \zeta_{q\mu} + (\zeta_q\cdot k)q_\mu$$

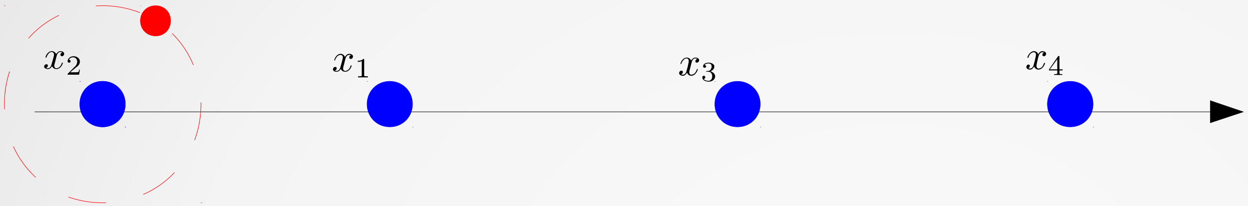
$$\zeta_{(2)\mu} = \zeta_{q\mu} + \frac{(\zeta_q\cdot k)}{2}q_\mu$$

Observation:

·Deformation = Specific form of the polarization tensor
·The resultant operator level is determined by q. k
·There are infinitely many choices to give an operator at a level

Moore's exact identity: Sketch

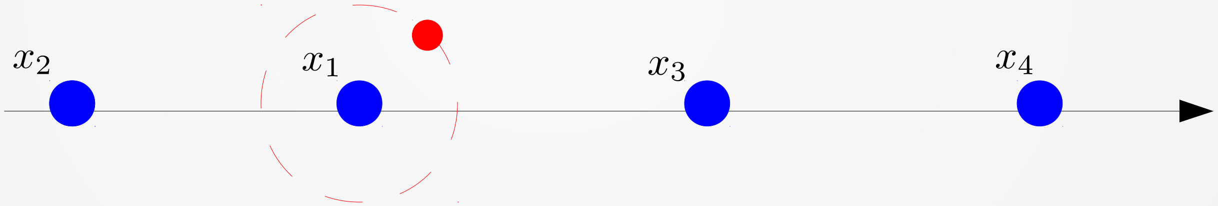
$$\left\langle V_1(k_1,x)V_2^{\rm br}(\tilde{k}_2,0)V_3(k_3,1)V_4(k_4,\infty)\right\rangle \qquad (0<x<1)$$



$V_2^{\rm br}(\tilde{k}_2,x_2)\equiv\{\mathcal{J}(q),V_2(k_2,x_2)\}$

Contour deformation

$=$



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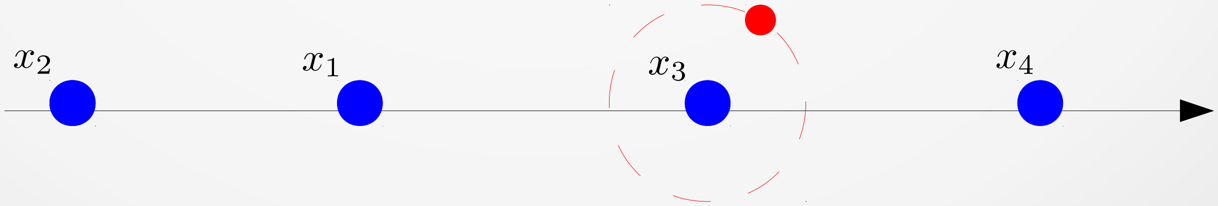

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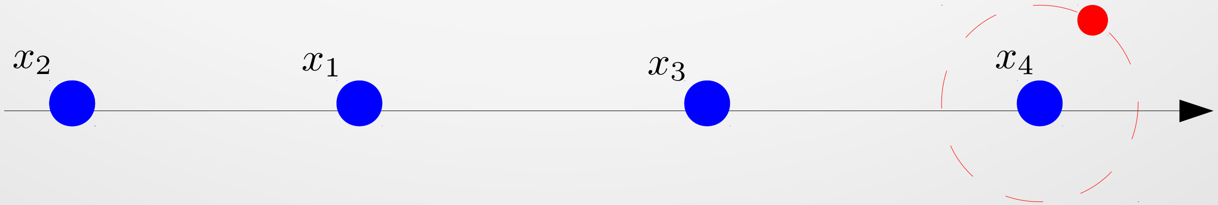

Moore's exact identity: 4-pt amplitudes

With $\int_0^1 dx(\text{ghost})\times$ this becomes a relation among amplitudes

$$\begin{aligned}
0 =& \mathcal{A}[\mathcal{V}_1(k_1)\mathcal{V}_2^{\text{br}}(\widetilde{k}_2)\mathcal{V}_3(k_3)\mathcal{V}_4(k_4)] \\
&+ (-1)^{q\cdot k_1}\mathcal{A}[\mathcal{V}_1^{\text{br}}(\widetilde{k}_1)\mathcal{V}_2(k_2)\mathcal{V}_3(k_3)\mathcal{V}_4(k_4)] \\
&+ (-1)^{q\cdot(k_1+k_3)}\mathcal{A}[\mathcal{V}_1(k_1)\mathcal{V}_2(k_2)\mathcal{V}_3^{\text{br}}(\widetilde{k}_3)\mathcal{V}_4(k_4)] \\
&+ (-1)^{q\cdot(k_1+k_3+k_4)}\mathcal{A}[\mathcal{V}_1(k_1)\mathcal{V}_2(k_2)\mathcal{V}_3(k_3)\mathcal{V}_4^{\text{br}}(\widetilde{k}_4)] \, .
\end{aligned}$$

In general, $$0 = \sum_{m=1}^{M}(-1)^{q\cdot(k_1+\cdots k_m)}\mathcal{A}\left[\mathcal{V}_1(k_1)\cdots\tilde{\mathcal{V}}_m^{\text{br}}(\tilde{k}_m)\cdots\mathcal{V}_M(k_M)\right]$$

$$\left(q+\sum_i k_i = 0,\ \sum_i n_i = m_q^2\right)$$

Example: from exact relation to H.E. relations

Deformer: $J_{(1)}(q) = i\zeta_q \cdot \partial X e^{iq\cdot X}$

Seed:
$$V_{(1)} = i\zeta_1 \cdot \partial X e^{ik_1\cdot X}$$
$$V_{(0)} =: e^{ik_2\cdot X} :$$

$$n_i \equiv q \cdot k_i \qquad n_1 = n_2 = -1, \quad n_3 = n_4 = 1$$

$$\mathcal{A}[\mathcal{V}^{\mathrm{br}}_{(2)}(\widetilde{k}_1)\mathcal{V}_{(0)}(k_2)\mathcal{V}_{(0)}(k_3)\mathcal{V}_{(0)}(k_4)] = \mathcal{A}[\mathcal{V}_{(1)}(k_1)\mathcal{V}^{\mathrm{br}}_{(1)}(\widetilde{k}_2)\mathcal{V}_{(0)}(k_3)\mathcal{V}_{(0)}(k_4)]$$

$$V^{\mathrm{br}}_{(2)}(\tilde{k}_1, z) = :\big(-\zeta^{(2)}_{\mu\nu}\partial X^\mu \partial X^\nu + i\zeta^{(2)} \cdot \partial^2 X\big)e^{i\tilde{k}_1\cdot X} : (z)$$
$$\begin{aligned}\zeta^{(2)}_{\mu\nu}(\zeta_1, \zeta_q) =&(\zeta_q \cdot k_1)q_{(\mu}\zeta_{1\nu)} - (\zeta_1 \cdot q)q_{(\mu}\zeta_{q\nu)} + \zeta_{q(\mu}\zeta_{1\nu)} \\ &- \frac{1}{2}\big(-(\zeta_q \cdot \zeta_1) + (\zeta_q \cdot k_1)(\zeta_1 \cdot q)\big)q_\mu q_\nu\end{aligned}$$
$$\zeta^{(2)}_{\mu}(\zeta_1, \zeta_q) = -(\zeta_1 \cdot q)\zeta_{q\mu} - \frac{1}{2}\big(-(\zeta_q \cdot \zeta_1) + (\zeta_q \cdot k_1)(\zeta_1 \cdot q)\big)q_\mu$$

$$V^{\mathrm{br}}_{(1)}(\tilde{k}_2, z) = i\zeta_R \cdot \partial X e^{i\tilde{k}_2\cdot X}(z)$$
$$\zeta_{R\mu}(\zeta_q) = (\zeta_q \cdot k_2)q_\mu + \zeta_{q\mu}$$

Deformation of 3rd and 4th operators trivially vanish.

Explicit forms of the exact relation

$$\mathcal{T}^{\mu\nu}_{[2000]} = \int \langle (- : \partial X^\mu \partial X^\nu) e^{ik_1\cdot X} :: e^{ik_2\cdot X} :: e^{ik_3\cdot X} :: e^{ik_4\cdot X} : \rangle$$

Using

$$\mathcal{T}^{\mu}_{[2000]} = \int \langle : i\partial^2 X^\mu e^{ik_1\cdot X} :: e^{ik_2\cdot X} :: e^{ik_3\cdot X} :: e^{ik_4\cdot X} : \rangle$$

$$\mathcal{T}^{\mu|\nu}_{[1100]} = \int \langle i\partial X^\mu e^{ik_1\cdot X}\, i\partial X^\nu e^{ik_2\cdot X} \; : e^{ik_3\cdot X} :: e^{ik_4\cdot X} : \rangle$$

$$\begin{aligned} &\Big[(\zeta_q\cdot k_1) q_{(\mu}\zeta_{1\nu)} - (\zeta_1\cdot q) q_{(\mu}\zeta_{q\nu)} + \zeta_{q(\mu}\zeta_{1\nu)}\Big]\mathcal{T}^{\mu\nu}_{[2000]} \\ &+\frac{1}{2}\big((\zeta_q\cdot\zeta_1) - (\zeta_q\cdot k_1)(\zeta_1\cdot q)\big)\Big[q_\mu q_\nu \mathcal{T}^{\mu\nu}_{[2000]} + q_\mu \mathcal{T}^{\mu}_{[2000]}\Big] \quad = \quad \zeta_{1\mu}\big[(\zeta_q\cdot k_2) q_\nu + \zeta_{q\nu}\big]\mathcal{T}^{\mu|\nu}_{[1100]} \end{aligned}$$

This holds for arbitrary ζ_1, ζ_q

Want to translate them to asymptotic high-energy relations.

High-energy limit and set of “Ward identities”

$$\mathcal{A}[\mathcal{V}_1^{\rm br}(k_1+q)\mathcal{V}_2(k_2)\mathcal{V}_3(k_3)\mathcal{V}_4(k_4)] = \mathcal{A}[\mathcal{V}_1(k_1)\mathcal{V}_2^{\rm br}(k_2+q)\mathcal{V}_3(k_3)\mathcal{V}_4(k_4)]$$

We may want

- ·Different set of vertex operators
- ·Equal set of momenta
- ·The same basis for polarizations (the scattering planes are tilted)

→ Deformation of momentum: $\tilde{k}_1 = k_1 + q$

Mass shell conditions: $-k_i^2 = m_i^2$

→ High-energy limit $\alpha' s \to \infty$

$q \cdot k = -1 \,\text{or}\, 1$ → In CM frame, $q \sim \mathcal{O}(1)$

$$\mathcal{A}[\mathcal{V}_1^{\rm br}(k_1)\mathcal{V}_2(k_2)\mathcal{V}_3(k_3)\mathcal{V}_4(k_4)] \overset{\text{leading}}{\simeq} \mathcal{A}[\mathcal{V}_1(k_1)\mathcal{V}_2^{\rm br}(k_2)\mathcal{V}_3(k_3)\mathcal{V}_4(k_4)]$$

A convenient basis for physical amplitudes

Standard helicity basis: $e^P,\ e^L,\ e^T,\ e^I,\ e^{J_i}$
(for 1st state)

Rearrange → $e^{T_q},\ e^{I_q},\ e^Q$

Helicity basis w.r.t. the deformation momentum q

The physical bracket operator: $\zeta_1 = e^A,\ \zeta_q = e^B,\ e^{A,B} = e^{T_q},\ e^{I_q},\ e^{J_i}$

$$V^{\mathrm{br}}_{(2)}(\tilde{k}_1, z) = :\big(-\zeta^{(2)}_{\mu\nu}\partial X^\mu \partial X^\nu + i\zeta^{(2)}\cdot\partial^2 X\big)e^{i\tilde{k}_1\cdot X}:(z)$$

$$\zeta^{(2)}_{\mu\nu} = e^A_{(\mu}e^B_{\nu)} + \frac{\delta^{AB}}{2}q_\mu q_\nu \qquad \zeta^{(2)}_\mu = \frac{\delta^{AB}}{2}q_\mu$$

$$\boxed{\begin{gathered} e^{A'} = \sum_{a'=L,T,I,J_i} C^{A'}{}_{a'}e^{a'} \\ A' = Q, T_q, I_q, J_i \end{gathered}}$$

Corresponding state

$$\left[\alpha^{AB}_{-1} + \frac{\delta^{AB}}{2}\left(\alpha^{qq}_{-1} + \alpha^q_{-2}\right)\right]\left|0;\tilde{k}_1\right\rangle$$

Original basis

$$\left[(G^{AB}_{TT} + G^{AB})\alpha^T_{-1}\alpha^T_{-1} + 2G^{AB}_{LT}\alpha^L_{-1}\alpha^T_{-1} + (G^{AB}_{LL} + G)\alpha^L_{-1}\alpha^L_{-1} + \cdots\right]|0;\tilde{k}_1\rangle$$

$(\alpha^{\mu\nu}_{-1} \equiv \alpha^\mu_{-1}\alpha^\nu_{-1})$

Asymptotic expansion of the exact relations

Moore's relation in terms of "familiar amplitudes"

$$(G^{AB}_{TT}+G^{AB})\mathcal{T}^{TT}_{[2000]}+2G^{AB}_{LT}\mathcal{T}^{LT}_{[2000]}+(G^{AB}_{LL}+G)\mathcal{T}^{LL}_{[2000]}$$
$$=C^{A}{}_{T_R}\tilde{G}^{B}_{T_R}\mathcal{T}^{T_R|T_R}_{[1100]}+C^{A}{}_{I_R}\tilde{G}^{B}_{I_R}\mathcal{T}^{I_R|I_R}_{[1100]}+\sum_i C^{A}{}_{J_i}\tilde{G}^{B}_{J_i}\mathcal{T}^{J_i|J_i}_{[1100]}$$

Fixed angle expansion: $s\to\infty$ $\quad\hat{t}=t/s=\text{fixed}$

Expand the amplitudes and the coefficients: Coefficients are functions of $\hat{t}$

$$\mathcal{T}^{TT}_{[2000]}=\mathcal{T}^{TT}_{[2000](3)}s^{3}+\mathcal{T}^{TT}_{[2000](2)}s^{2}+\cdots \qquad G^{T_qT_q}_{TT}=G^{T_qT_q}_{TT(0)}+G^{T_qT_q}_{TT(-1)}s^{-1}+\cdots$$

$G^{AB}_{ab},C^{A}{}_{a}$: Known from the kinematics $\quad\mathcal{T}^{\mu\nu}_{[2000]}$: unknowns to be determined

From this expansion, we find constraints on the leading order amplitudes.

Asymptotic expansion of the exact relations

$$(A,B)=(T_q,T_q):$$

$$\mathcal{O}(s^3): \qquad 0=\frac{19}{20}\mathcal{T}^{TT}_{[2000](3)}+\frac{1}{5}\mathcal{T}^{LL}_{[2000](3)}-\mathcal{T}^{T_R|T_R}_{[1100](3)}$$

$$\mathcal{O}(s^2): \qquad 0=\frac{19}{20}\mathcal{T}^{TT}_{[2000](2)}-2\mathcal{T}^{LL}_{[2000](3)}+\frac{1}{5}\mathcal{T}^{LL}_{[2000](2)}-\mathcal{T}^{T_R|T_R}_{[1100](2)}$$

$$(A,B)=(I_q,I_q):$$

$$\mathcal{O}(s^3): \qquad 0=\mathcal{T}^{TT}_{[2000](3)}-4\mathcal{T}^{LL}_{[2000](3)}$$

$$\mathcal{O}(s^2): \qquad 0=-\frac{1}{20}\mathcal{T}^{TT}_{[2000](2)}+6\mathcal{T}^{LL}_{[2000](3)}+\frac{1}{5}\mathcal{T}^{LL}_{[2000](2)}+\mathcal{T}^{I_R|I_R}_{[1100](2)}$$

$$(A,B)=(J,J):$$

$$\mathcal{O}(s^3): \qquad 0=\mathcal{T}^{TT}_{[2000](3)}-4\mathcal{T}^{LL}_{[2000](3)}$$

$$\mathcal{O}(s^2): \qquad 0=-\frac{1}{20}\mathcal{T}^{TT}_{[2000](2)}-2\mathcal{T}^{LL}_{[2000](3)}+\frac{1}{5}\mathcal{T}^{LL}_{[2000](2)}-\mathcal{T}^{J|J}_{[1100](2)}$$

$$(A,B)=(T_q,I_q),(I_q,T_q):$$

$$\mathcal{O}(s^2): \qquad 0=(2\hat{t}+1)\mathcal{T}^{TT}_{[2000](3)}+\sqrt{-2\hat{t}(1+\hat{t})}\mathcal{T}^{TL}_{[2000](5/2)}$$

$$0=-2(4\hat{t}^2+6\hat{t}+1)\mathcal{T}^{TT}_{[2000](3)}+(2\hat{t}^2+3\hat{t}+1)\mathcal{T}^{TT}_{[2000](2)}+\sqrt{-2\hat{t}(1+\hat{t})}(1+\hat{t})\mathcal{T}^{TL}_{[2000](3/2)}$$

Asymptotic expansion of the exact relations

For leading order part, we can find some linear relations:

$$\underbrace{\mathcal{T}^{TT}_{[2000](3)} = 4\mathcal{T}^{LL}_{[2000](3)}\,,}_{\text{Known linear relation}} \qquad \underbrace{\mathcal{T}^{TT}_{[2000](3)} = \mathcal{T}^{T_R|T_R}_{[1100](3)}}_{\text{An inter-level relation}}$$

Subleading relations: Rotational symmetry: $\mathcal{T}^{I_R|I_R}_{[1100]} = \mathcal{T}^{J|J}_{[1100]}$

$$4\mathcal{T}^{LL}_{[2000](3)} + \mathcal{T}^{I_R|I_R}_{[1100](2)} = 0\,, \quad \mathcal{T}^{TT}_{[2000](2)} - \mathcal{T}^{T_R|T_R}_{[1100](2)} + \mathcal{T}^{I_R|I_R}_{[1100](2)} = 0$$

In this way, we can extract lots of nontrivial relations among amplitudes.

Another example considered

We have also calculated a bit more involved example:

Massive deformer and a level 3 state appears

→ Derive various (known) linear relations,
but not all of them

$$\mathcal{T}^{TTT}_{[3000](9/2)} = 8\mathcal{T}^{TLL}_{[3000](9/2)} = -8\mathcal{T}^{[L;T]}_{[3000](9/2)}$$

→ Amplitudes are related to one another in a complicated manner.

There are infinitely many ways to construct a given level vertex operator.

→ Through many other amplitudes, they would be related.

Conclusion (or observation)

We have understood:

High-energy expansion of the relations from bracket deformation leads to high-energy relations systematically.

“Change of frame” coefficients from the deformation momentum q

(q indeed connects asymptotic amplitudes)

High-energy symmetry in String Theory? Hint?

$\mathcal{T}^{TT} = 4\mathcal{T}^{LL}$: Leading energy part with respect to the scattering plane

$\longrightarrow$ Reduction of degrees of freedom? [Gross-Manes]

DDF operators in closed string theory $\longrightarrow$ Kac-Moody algebra

[West-Gaberdiel]

$\longrightarrow$ Some algebra from Bracket deformation?
So far, not promising.

Special choice of q: Referring to other states

[West][Moore]

$\longrightarrow$ Troidal compactification

Future directions...

We want to understand ...

Multi-point amplitudes and higher genus

Another limit, such as Regge limit [NCTU group]

Deformation of vertex operators and world-sheet symmetries

....

What is the (high-energy) stringy symmetry?

Thank you for your attention!