Gravitational instability in AdS and thermalization of dual gauge theories

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Based on arXiv:

1304.4166 (with L.Lehner, S.Liebling)

1403.6471 (with V.Balasubramanian, S.Green, L.Lehner, S.Liebling);

1410.5381 (with S.Green, L.Lehner, S.Liebling);

1412.4761 (with S.Green, L.Lehner, S.Liebling);

1502.01574 (with L.Lehner); 1509.07780

1509.00774 (with M.Buchel); 1510.08415

YITP, November 10, 2015

There are two separate motivations for my work:

$\Longrightarrow$ First,

- A ground state solution to Einstein vacuum equations is Minkowski space-time: $R^{3,1}$
  - A fundamental question is whether this solution is stable? *i.e.*, , do small perturbations of it at $t = 0$ remain small for all future times (where small is defined in terms of an appropriate norm)?

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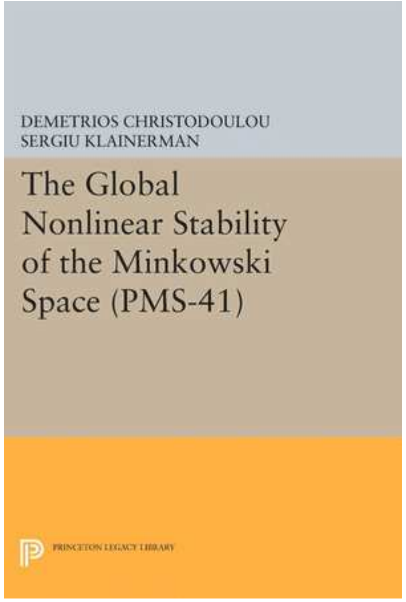
  - A fundamental question is whether this solution is stable? *i.e.*, , do small perturbations of it at $t = 0$ remain small for all future times (where small is defined in terms of an appropriate norm)?
- The answer (CK) (700+ citations, 432pages):



- CK proved that sufficiently small perturbations not only remain small but decay to zero with time in any compact region. The physical mechanism responsible for the asymptotic stability of Minkowski space is the dissipation by dispersion, that is the radiation of energy of perturbations to infinity — **”stuff” escapes to asymptotic infinity**

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- It is much more difficult to make similar statements for space-times with negative cosmological constant, the anti-de-Sitter space times:

  ■ This is exactly what P.Bizon and A.Rostworowski (BR) tried to tackle in their ground breaking paper arXiv:1104.3702!

  ■ Their main conjecture was:

$$\textit{The } AdS_{d+1} \textit{ space } (for\ d \geq 3) \textit{ is unstable against the formation of a black hole for a large class of arbitrarily small perturbations}$$

  ■ Moreover, they presented a technical and physical mechanism for the instability

- The basic question we wanted to address:
  - are the BR conjecture and the instability mechanism correct?

$\Longrightarrow$ Second,

- Recall,

$$\textit{The } AdS_{d+1} \textit{ space } (\textit{for } d \geq 3) \textit{ is unstable against the formation}$$

$$\textit{of a } \underline{\textbf{\textit{black hole}}} \textit{ for a large class of arbitrarily small perturbations}$$

▪ The black hole formation in AdS is a holographic representation to the thermalization of a dual strongly coupled gauge theory

▪ Thus, studying AdS (in-)stability we learn about the nonequilibrium dynamics of gauge theories

Outline of the talk:

- Review of basic AdS/CFT correspondence
- Review of Bizon and Rostworowski (BR) work
  - BR mechanism for weakly-nonlinear instability
- Boson stars in AdS (and motivation)
  - Stationary configurations and their properties (mass, charge)
  - Linearized fluctuations around boson stars (spectrum)
- Numerical simulation of boson star and their cousins
  - Surprises of fake boson stars
  - Surprises of original BR simulations
- *Re*: BR mechanism for weakly-nonlinear instability
  - two-time framework (TTF) for the AdS gravitational collapse
  - TTF$\Longrightarrow$ FPU (Fermi-Pasta-Ulam paradox)
  - Role of hidden conservation laws in the dual turbulent cascade
- What all of this have to do with thermalization of dual gauge theories?
  - comments, conclusion and future directions

Basic AdS/CFT correspondence:

gauge theory		string theory
$\mathcal{N}=4$ $SU(N)$ SYM	$\Longleftrightarrow$	N-units of 5-form flux in type IIB string theory
g^2_{YM}	$\Longleftrightarrow$	g_s

$\Longrightarrow$ Each of the duality frames are valid in complimentary regimes. In the 't Hooft limit (planar limit), $N \to \infty$, $g^2_{YM} \to 0$ with $N g^2_{YM}$ kept fixed:

- for $g^2_{YM} N \ll 1$ we can use a standard perturbation theory
- for $g^2_{YM} N \gg 1$ we can use effective supergravity description of type IIB string theory on $AdS_5 \times S^5$

$\Longrightarrow$ In the above regime we can incorporate corrections:

$$\frac{1}{N}\text{-corrections} \Longleftrightarrow g_s\text{-corrections}$$

$$\frac{1}{N g^2_{YM}}\text{-corrections} \Longleftrightarrow \alpha'\text{-corrections}$$

$\Longrightarrow$ We consider the planar ('t Hooft) limit:

$$N \to \infty\,, \qquad g_{YM}^2 \to 0\,, \qquad \text{with} \qquad \lambda \equiv N g_{YM}^2 = \text{const}$$

with

$$\lambda \gg 1$$

$\Longrightarrow$ In this limit, type IIB string theory is well approximated by type IIB supergravity. For now, we focus on static/dynamic phenomena in $\mathcal{N} = 4$ SYM with unbroken $SO(6)$ R-symmetry. KK reduction on the S^5 leads to the following effective action

$$S_5 = \frac{1}{16\pi G_5} \int_{\mathcal{M}_5} d\xi^5 \sqrt{-g} \left[R + \frac{12}{L^2} + \mathcal{L}_{matter} \right]$$

with

$$L^4 = g_{YM}^2 N \ell_s^4 = 4\pi g_s N \ell_s^4\,, \qquad G_5 \propto N^2$$

$\mathcal{L}_{matter}$ includes gravitational modes that are excited in dynamics. For example, one can prepare initial state specifying expectation value of $\mathcal{O}_4 = \langle \mathrm{Tr} F^2 \rangle$. In this case

$$\mathcal{L}_{matter} = -\frac{1}{2}(\partial\phi)^2$$

where ϕ is a dilaton.

$\Longrightarrow$ During evolution, operators of different dimensions can get excited. To be completely consistent, we should use consistent supergravity truncations in $\mathcal{L}_{matter}$.

$\Longrightarrow$ Consider SYM on S^3 of radius ℓ.

What are the candidates for the SYM $SO(6)$-invariant equilibrium states in the gravitational dual?

$\Longrightarrow$ To answer, we search for static solutions of the above gravitational action.

- The ground state is AdS_5; it has a nonzero energy to be identified with Casimir energy of the $\mathcal{N} = 4$ SYM on S^3:

$$E_{vacuum} = \frac{3(N^2-1)}{16\ell}$$

- All the states with

$$E = E_{vacuum}(1+\delta)\,, \qquad \delta > 0$$

  are AdS-Schwarzschild black hole:
  ▪ they exist for arbitrarily small δ;
  ▪ they are 'thermal' in that once can naturally associate to them the thermodynamic properties (entropy, temperature...)

$$S(\epsilon) = \frac{\pi N^2}{2^{3/2}}\left(\sqrt{1+\epsilon}-1\right)^{3/2}\,, \qquad (T\ell)^2 = \frac{1}{2\pi^2}\,\frac{1+\epsilon}{\sqrt{1+\epsilon}-1}$$

$\Longrightarrow$ The message:

$$\textit{Equilibrium states of SYM} \qquad \Longleftrightarrow \qquad \textit{Black holes in } AdS_5$$

thus,

$$\textit{Equilibration in SYM} \qquad \Longleftrightarrow \qquad \textit{Black holes formation in } AdS_5$$

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$\Longrightarrow$ Fits nicely with BR conjecture: from stat-mech we expect strongly interactive systems to equilibrate.

Moreover,

- No-gap* in the spectrum of equilibrium states suggests that thermalization would occur no matter how small the initial perturbation of the *AdS*

* (this innocent fact has important consequences — more later if time permits)

BR work

$\Longrightarrow$ In a groundbreaking paper, BR studied gravitational collapse of a real scalar in global AdS_4. (To avoid repeating myself, I will discuss generalization of BR with a complex scalar field — the BR analysis correspond to setting $\phi_2 = 0$)

The effective four-dimensional action is given by (we set the radius of AdS to one)

$$S_4 = \frac{1}{16\pi G_4} \int_{\mathcal{M}_4} d^4\xi \sqrt{-g} \left(R_4 + 6 - 2\partial_\mu \phi \partial^\mu \phi^*\right) ,$$

where $\phi \equiv \phi_1 + i\ \phi_2$ is a complex scalar field and

$$\mathcal{M}_4 = \partial\mathcal{M}_3 \times \mathcal{I}, \qquad \partial\mathcal{M}_3 = R_t \times S^2 , \qquad \mathcal{I} = \{x \in [0, \frac{\pi}{2}]\} .$$

The line element is

$$ds^2 = \frac{1}{\cos^2 x} \left(-Ae^{-2\delta} dt^2 + \frac{dx^2}{A} + \sin^2 x \, d\Omega_2^2 \right)$$

$d\Omega_2^2$ is the metric of unit radius S^2, and $A(x,t)$ and $\delta(x,t)$ are scalar functions describing the metric.

For numerical simulations is it convenient to rescale the matter fields as

$$\hat{\phi}_i \equiv \frac{\phi_i}{\cos^2 x}$$

$$\hat{\Pi}_i \equiv \frac{e^\delta}{A} \frac{\partial_t \phi_i}{\cos^2 x}$$

$$\hat{\Phi}_i \equiv \frac{\partial_x \phi_i}{\cos x}$$

From effective action we find the following equations of motion (we drop the caret from here forward)

$$\dot{\phi}_i = A e^{-\delta} \Pi_i$$

$$\dot{\Phi}_i = \frac{1}{\cos x} \left(\cos^2 x A e^{-\delta} \Pi_i\right)_{,x}$$

$$\dot{\Pi}_i = \frac{1}{\sin^2 x} \left(\frac{\sin^2 x}{\cos x} A e^{-\delta} \Phi_i\right)_{,x}$$

$$A_{,x} = \frac{1 + 2\sin^2 x}{\sin x \cos x} (1 - A) - \sin x \cos^5 x A \left(\frac{\Phi_i^2}{\cos^2 x} + \Pi_i^2\right)$$

$$\delta_{,x} = -\sin x \cos^5 x \left(\frac{\Phi_i^2}{\cos^2 x} + \Pi_i^2\right)$$

There is one constraint equation

$$A_{,t} + 2\sin x \cos^4 A^2 e^{-\delta} \left(\Phi_i \Pi_i\right) = 0$$

where a sum over $i = \{1, 2\}$ is implied.

We are interested in studying the solution to above subject to the boundary conditions:

- Regularity at the origin implies these quantities behave as

$$\phi_i(t,x) = \phi_0^{(i)}(t) + \mathcal{O}(x^2)$$
$$A(t,x) = 1 + \mathcal{O}(x^2)$$
$$\delta(t,x) = \delta_0(t) + \mathcal{O}(x^2)$$

- at the outer boundary $x = \pi/2$ we introduce $\rho \equiv \pi/2 - x$ so that we have

$$\phi_i(t,\rho) = \phi_3^{(i)}(t)\rho + \mathcal{O}(\rho^3)$$
$$A(t,\rho) = 1 - M\frac{\sin^3\rho}{\cos\rho} + \mathcal{O}(\rho^6)$$
$$\delta(t,\rho) = 0 + \mathcal{O}(\rho^6)$$

The asymptotic behaviour determines the boundary CFT observables: the expectation values of the stress-energy tensor T_{kl}, and the operators $\mathcal{O}_3^{(i)}$, dual to ϕ_i,

$$8\pi G_4 \langle T_{tt} \rangle = M\,, \qquad \langle T_{\alpha\beta} \rangle = \frac{g_{\alpha\beta}}{2}\, \langle T_{tt} \rangle$$

$$16\pi G_{d+1} \langle \mathcal{O}_3^{(i)} \rangle = 12\, \phi_3^{(i)}(t)$$

where $g_{\alpha\beta}$ is a metric on a round S^2. Additionally note that the conserved $U(1)$ charge is given by

$$Q = 8\pi \int_0^{\pi/2} dx\; \sin^2 x\; \cos^2 x\; \left(\Pi_2(0,x)\phi_1(0,x) - \Pi_1(0,x)\phi_2(0,x)\right)$$

and that since $\partial_t Q = 0$, above integral can be evaluated at $t = 0$.

$\Longrightarrow$ The gravitational momentum constraint ensures that

$$\partial_t\, \langle T_{tt} \rangle = 0\,,$$

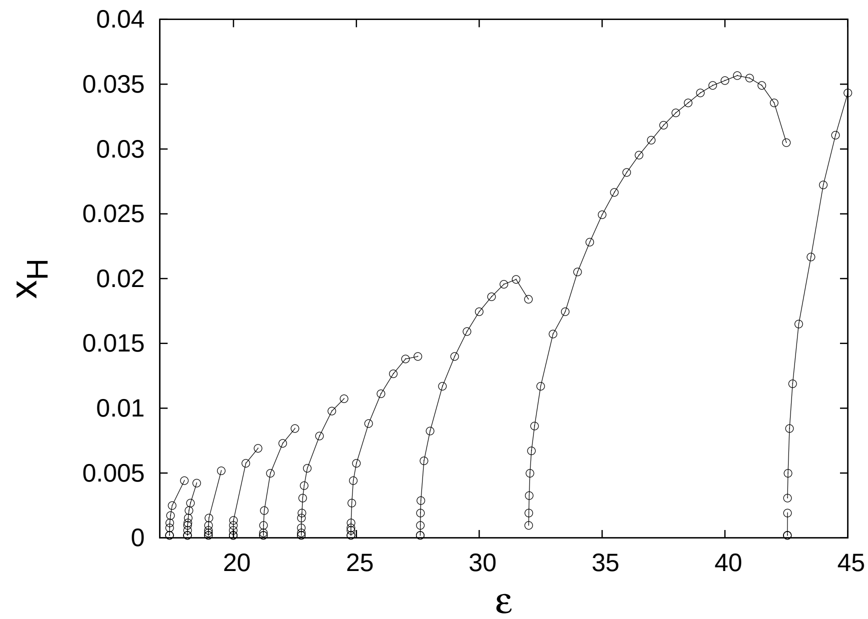
which in turn implies that M is time-independent.

$\Longrightarrow$ BR considered the following initial data

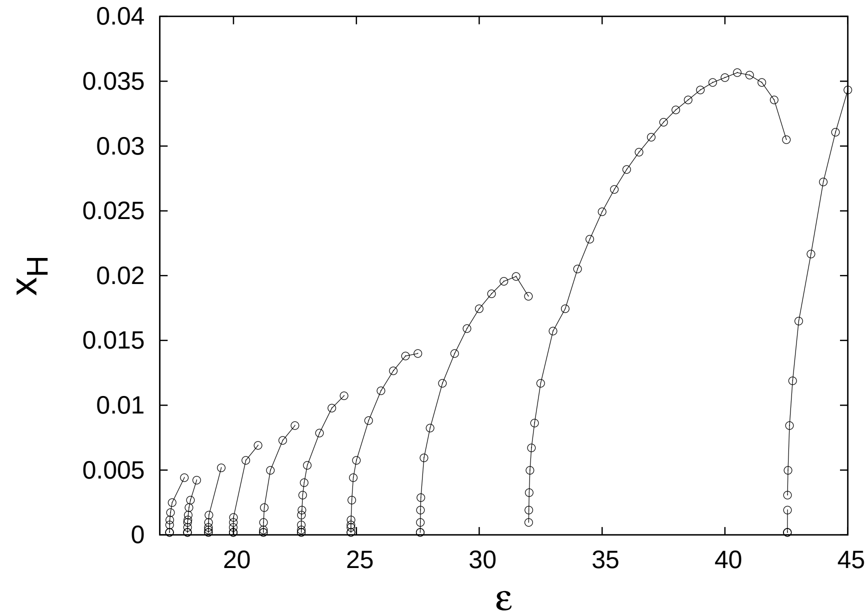
$$\Phi(0,x) = 0\,, \qquad \Pi(0,x) = \frac{2\epsilon}{\pi}\;\exp\left(-\frac{4\tan^2 x}{\pi^2\sigma^2}\right)\;\frac{1}{\cos^2 x}\,, \qquad \sigma = \frac{1}{16}$$

and changing ϵ

$\cdots$ they found:



(Figure from BR) FIG.1: Horizon radius vs amplitude for initial data (9). The number of reflections off the AdS boundary before collapse varies from zero to nine (from right to left).



(Figure from BR) FIG.1: Horizon radius vs amplitude for initial data (9). The number of reflections off the AdS boundary before collapse varies from zero to nine (from right to left).

$\Longrightarrow$ Matter bounces in the gravitational cavity (produced by AdS), sharpening all the time under the influence of gravity $\rightarrow$ formation of trapped surface

What is the mechanism leading to horizon formation?

Consider the solution of gravitational EOMs, perturbative in the bulk scalar amplitudes ϵ:

$$\phi_i = \sum_{j=0}^{\infty} \epsilon^{2j+1}\ \phi_{i,2j+1}\,, \qquad A = 1 - \sum_{j=1}^{\infty} \epsilon^{2j}\ A_{2j}\,, \qquad \delta = \sum_{j=1}^{\infty} \epsilon^{2j}\ \delta_{2j}\,,$$

where $\phi_{i,2j+1}\,, A_{2j}\,, \delta_{2j}$ are functions of (t, x).

$\Longrightarrow$ It is convenient to decompose these functions in terms of a complete basis. A natural basis is provided by the AdS_{d+1} massless scalar eigenvalues and eigenfunctions (which we refer to from now on as *oscillons*)

$$\omega_j = d + 2j\,, \quad e_j(x) = d_j\ \cos^d x\ {}_2F_1\left(-j, d+j, \frac{d}{2}, \sin^2 x\right)\,, \quad j = 0, 1, \cdots\,,$$

where d_j are normalization constants such that

$$\int_0^{\pi/2} dx\ e_i(x) e_j(x) \tan^{d-1} x = \delta_{ij}\,.$$

A remarkable observation of BR was that initial conditions which represent at a linearized level (at order $\mathcal{O}(\epsilon)$) a superposition of several oscillons with different index j appear to be unstable at time scales $t_{\text{instability}} \sim \mathcal{O}(\epsilon^{-2})$; on the other hand, nonlinear effects of a single oscillon do not lead to destabilization. Specifically, the instabilities occur whenever oscillons with indices[a]
$\{j_1, j_2, j_3\}$ are present at order $\mathcal{O}(\epsilon)$, while the oscillon with index j_r, such that

$$\omega_{j_r} = \omega_{j_1} + \omega_{j_2} - \omega_{j_3} \,,$$

is not excited at this order.

[a]The indices could be repeated.

Let's consider a single oscillon excited at linear level:

$$\phi_2(t,x) \equiv \phi_1(t,x)\,, \qquad \phi_1(0,x) = \epsilon\; e_0(x) + \mathcal{O}(\epsilon^3)\,, \qquad \partial_t \phi_1(0,x) = 0\,.$$

developing expansion to $\mathcal{O}(\epsilon^3)$ we find

$$\phi_1 = \epsilon\,\left[e_0(x)\;\cos\left(\left(3 - \frac{135}{4\pi}\epsilon^2\right)t\right)\right] + \epsilon^3\,\left[F_{3,3}(x)\;\cos(3t)\right.$$
$$\left. + F_{3,9}(x)\;\cos(9t)\right] + \mathcal{O}(\epsilon^5)$$

with

$$F_{3,3} = \frac{3\sqrt{2}\cos^3 x}{\pi^{3/2}}\Big(12\cos^8 x - 88\cos^6 x + 108\cos^4 x - 63\cos^2 x + 63\pi^2$$
$$-252x^2 - 252x\cot x(2 - \cos^2 x)\Big)$$

$$F_{3,9} = \frac{4\sqrt{2}}{\pi^{3/2}}\cos^9 x(9\cos^2 x - 4)$$

Notice that in above we absorbed a term linearly growing in time

$$\propto \epsilon^3 t \sin(\omega_0 t)$$

into $\mathcal{O}(\epsilon^2)$ shift of the leading-order oscillon frequency ω_0:

$$\omega_0 \to w_0 - \frac{135}{4\pi}\epsilon^2$$

Obviously, we could do so because an oscillon with such a frequency has already been present in the initial condition. For this initial configuration the instability condition is satisfied only for $j_1 = j_2 = j_3 = j_r = 0$.

Consider now a slightly more general initial condition

$$\phi_2(t,x) \equiv \phi_1(t,x)\,, \qquad \phi_1(0,x) = \epsilon\,(e_0(x)+e_1(x))+\mathcal{O}(\epsilon^3)\,, \qquad \partial_t\phi_1(0,x) = 0$$

Here,

$$\phi_1 = \epsilon\left[e_0(x)\ \cos\left(\left(3 - \frac{335}{2\pi}\epsilon^2\right)t\right) + e_1(x)\ \cos\left(\left(5 - \frac{1519}{6\pi}\epsilon^2\right)t\right)\right]$$

$$+\epsilon^3\left[\sum_{k=1}^{8} F_{3,2k-1}(x)\ \cos((2k-1)t) + \frac{\sqrt{6}\pi}{105}e_2(x)\ t\ \sin(7t)\right] + \mathcal{O}(\epsilon^5)$$

where $F_{3,2j+1}(x)$ are some analytically determined functions.

Here, we have three different terms at order $\mathcal{O}(\epsilon^3)$, which grow linearly with time

$$\propto \epsilon^3 t \times \left\{\sin(\omega_0 t)\,,\ \sin(\omega_1 t)\,,\ \sin(\omega_2 t)\right\}$$

The last secular term comes from the resonance condition:

$$\omega_1 + \omega_1 = \omega_0 + \omega_2 \qquad w_2\Big|_{resonance} = \omega_1 + \omega_1 - \omega_0$$

The presence of $j = \{0, 1\}$ oscillons in order $\mathcal{O}(\epsilon)$ initial conditions allows us to absorb the first two terms into the shifts of the leading-order oscillon frequencies

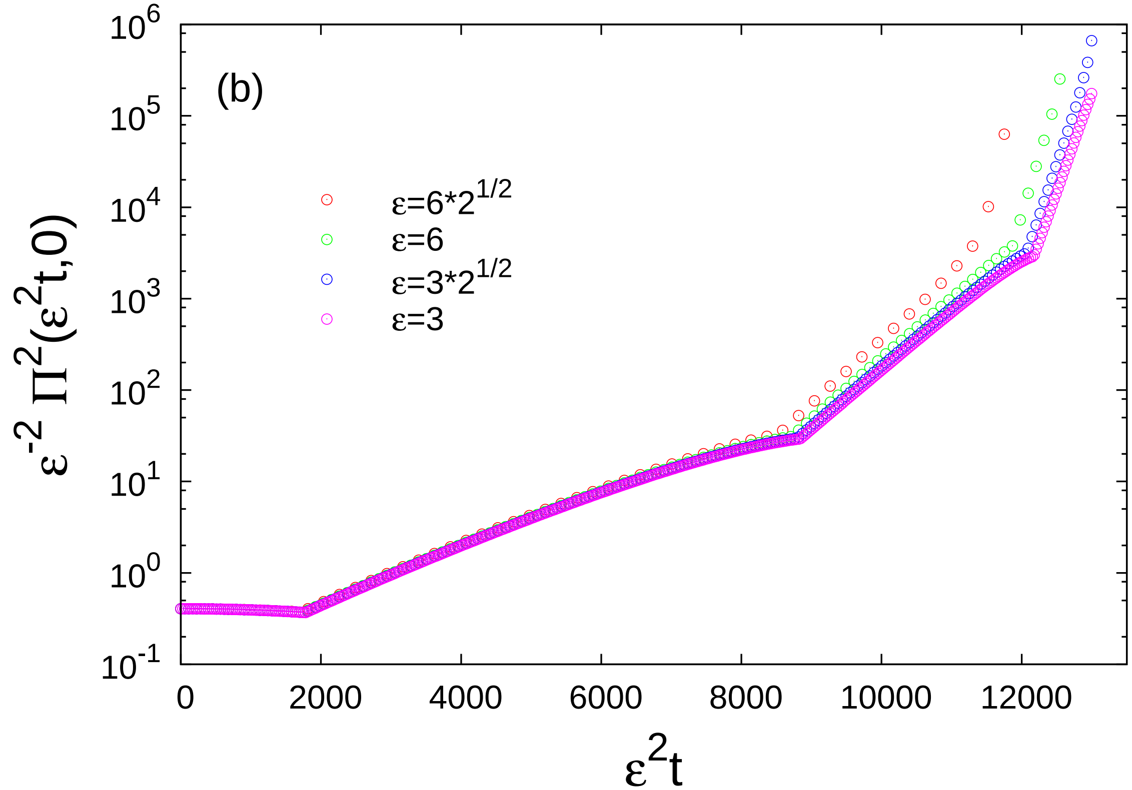
$$\omega_0 \to w_0 - \frac{335}{3\pi}\epsilon^2\,, \qquad \omega_1 \to w_1 - \frac{1519}{6\pi}\epsilon^2$$

$\Longrightarrow$ We cannot do the same with the remaining term in — for this to happen $\phi_1(0, x)$ must contain a term $\propto \epsilon\ e_2(x)$.

$\Longrightarrow$ Of course, the presence of $e_2(x)$ at order $\mathcal{O}(\epsilon)$ in the initial conditions, while eliminating $\epsilon^3 t \times \sin(\omega_2 t)$ term, would generate new resonances at $j > 2$.

This is the basically the backbone of BR arguments that 'weakly-nonlinear instability' is universal (generic) :

- Lower frequency modes excite higher frequency on a (slow) time-scale $\tau = \epsilon^2 t$
- Eigenmodes of the scalar profile at higher frequencies have a large backreaction at the origin, eventually leading to the formation of the trapped surface
- The latter is illustrated in the upper envelope of the Ricci scalar at the origin:



From BR arXiv:1104.3702v5: scaling of $\Pi(t,0)^2$ at the origin; this is proportional to (upper envelope of the) Ricci scalar at the origin; growth signals formation of the BH.

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$\Longrightarrow$ But, the end point need not be static!

Stationary configurations in AdS — Boson stars

$\Longrightarrow$ A complex scalar field in AdS can support some interesting stationary, but not static configurations:

- assuming

$$\phi_1(x,t) + i\phi_2(x,t) = \frac{\phi(x)}{\cos^2 x} e^{i\omega t}\,, \qquad A(t,x) = a(x)\,, \qquad \delta(t,x) = d(x)$$

- we find ODEs:

$$0 = \phi'' + \left(\frac{2}{\cos x \sin x} + \frac{a'}{a} - d'\right)\ \phi' + \omega^2 e^{2d} a^{-2}\ \phi$$

$$0 = d' + \sin x \cos x\ a^{-2}\left((\phi')^2 a^2 + \phi^2\omega^2 e^{2d}\right)$$

$$0 = a' + \frac{2\cos^2 x - 3}{\cos x \sin x}(1-a) + \sin x \cos x\ a^{-1}\left((\phi')^2 a^2 + \phi^2\omega^2 e^{2d}\right)$$

The charge and the mass determined by these solutions are given by:

$$Q = 8\pi \int_0^{\pi/2} dx \, \frac{\omega \sin^2 x \phi(x)^2 e^{d(x)}}{a(x) \cos^2 x}$$

$$M = \int_0^{\pi/2} dx \, \frac{\sin^2 x}{a(x) \cos^2 x} \left(a(x)^2 (\phi'(x))^2 + e^{2d(x)} \omega^2 \phi(x)^2 \right)$$

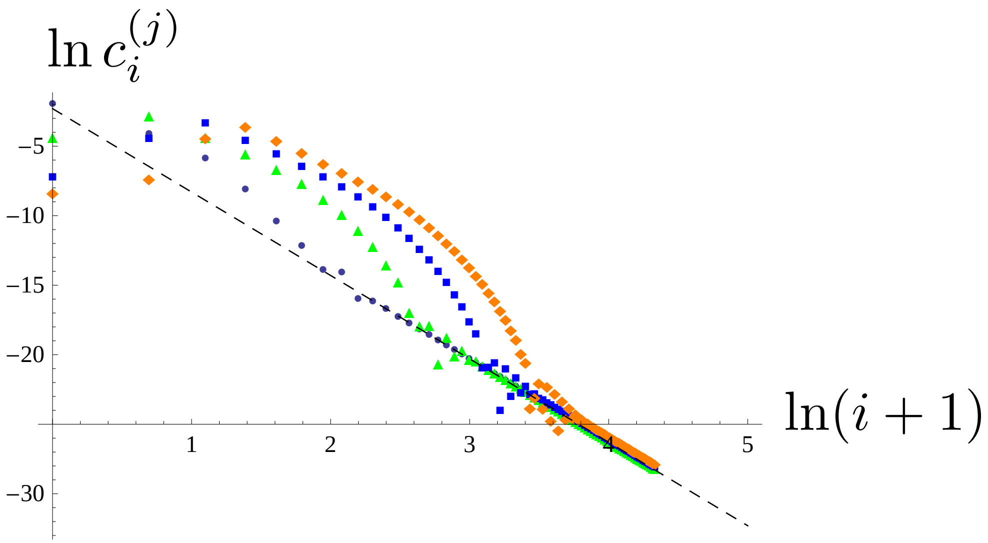
$\Longrightarrow$ Physical solutions are characterized by a discrete integer $j = 0, 1, \cdots$, denoting the number of nodes of the complex scalar radial wave-function, and a continuous value of the global charge Q (or equivalently the amplitude of the complex scalar modulus):

$$M = \epsilon^2 \, \frac{\pi (3+2j)^2}{8(j+1)(j+2)} + \mathcal{O}(\epsilon^4) \,, \qquad Q = \epsilon^2 \, \frac{\pi^2 (3+2j)}{2(j+1)(j+2)} + \mathcal{O}(\epsilon^4)$$

$$M = \frac{3+2j}{4\pi} \, Q + \mathcal{O}(Q^2) = \frac{\omega_0^{(j)}}{4\pi} \, Q + \mathcal{O}(Q^2)$$

where $\omega_0^{(j)}$ is the level-j oscillon frequency.

$\Longrightarrow$ We can construct (numerically) boson stars at different excitation level and for wide range of Q



Spectral decomposition of level $j = \{0, 1, 2, 3\}$ ({purple,green,blue,orange}) boson stars in oscillon basis:

$$c_i^{(j)} \equiv \left| \int_0^{\pi/2} dx \; \phi^{(j)}(x) e_i(x) \tan^2 x \right|$$

Note that the maxima of $c_i^{(j)}$ are achieved for $i = j$, much like in the small-Q limit. For all levels considered $c_i^{(j)}$ approach a universal fall-off:

$$c_i^{(j)} \propto (1+i)^{-6}\,, \qquad i \gg j\,,$$

represented by a dashed black curve.

$\Longrightarrow$ Boson stars are examples of infinite sets of 'oscillons' that are stable at linearized level.

Consider perturbations of stationary boson stars to leading order in λ:

$$\phi_1(x,t) + i\phi_2(x,t) = \cos^{-2} x \Big(\phi(x) + \lambda(f_1(t,x) - i\phi(x) g_1(t,x)) \Big) e^{i\omega t}$$

$$A(t,x) = a(x) + \lambda \; a_1(t,x)$$

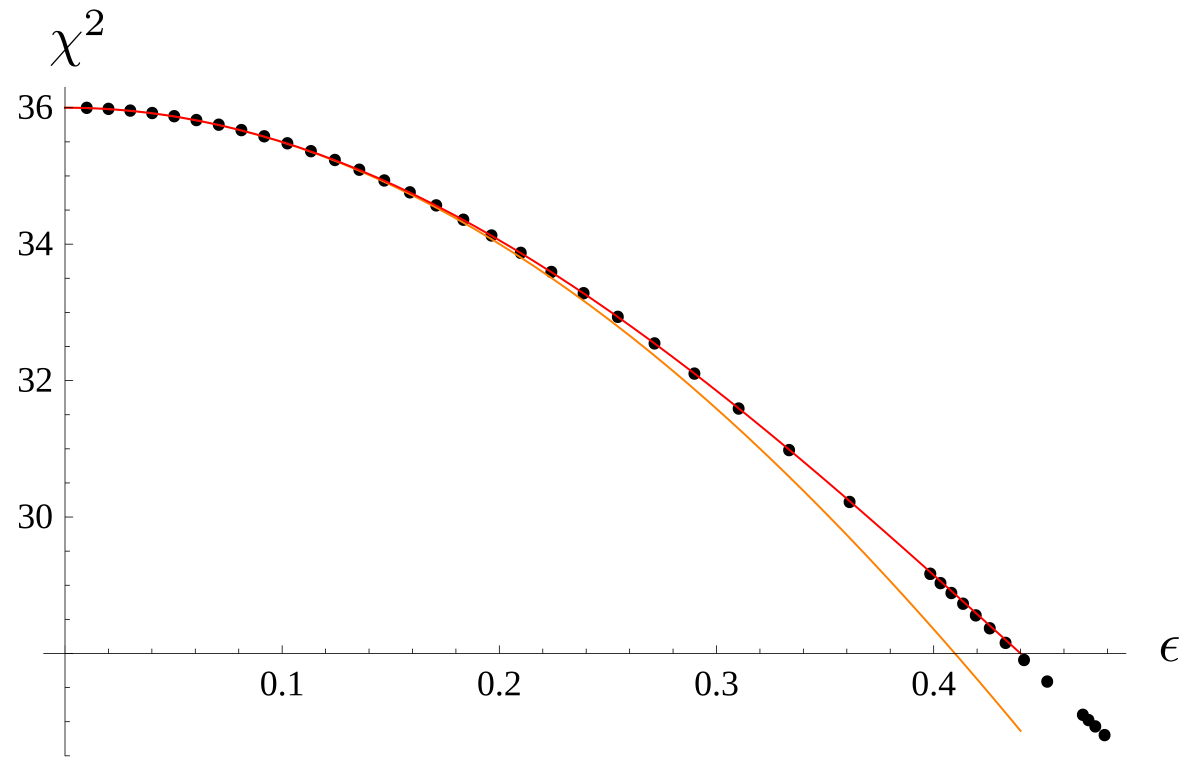
$$\delta(t,x) = d(x) + \lambda \; \delta_1(t,x)$$

Further introducing

$$f_1(t,x) = F_1(x) \cos(\chi t)\,, \qquad g_1(t,x) = -G_1(x) \; \sin(\chi t)$$

the equations for $a_1(t,x)$ and $\delta_1(t,x)$ can be solved explicitly, and $F_1(x)$ and $G_1(x)$ satisfy a (complicated=long) coupled system of ODEs

$\Longrightarrow$ Numerically, we compute $\chi(\epsilon)$ for different excitation levels of a boson star. I show results for the ground state only.



Spectrum of linearized fluctuations about $j = 0$ boson stars as a function of ϵ (black dots). The solid orange/red curves are successive approximations to $\chi^2 = (\chi(\epsilon))^2$ in ϵ^2:

$$\chi = 6 - \frac{135}{32}\ \epsilon^2 + \left(\frac{1215}{128}\pi^2 - \frac{113892831}{1254400}\right)\ \epsilon^4 + \mathcal{O}(\epsilon^6)$$

Summary of numerical simulations

We performed different simulations:

- Perturbed, Genuine Boson Stars:

$$\Phi_i = \left[\frac{\phi'}{\cos^2 x} + G'(x)\right]\delta_i^1\,, \qquad \Pi_i = \left[\omega\phi\frac{e^d}{a} + G'(x)\right]\delta_i^2$$

$$G(x) = \epsilon e^{-(r-R_0)^2/\Delta^2}$$

- Fake Boson Stars:

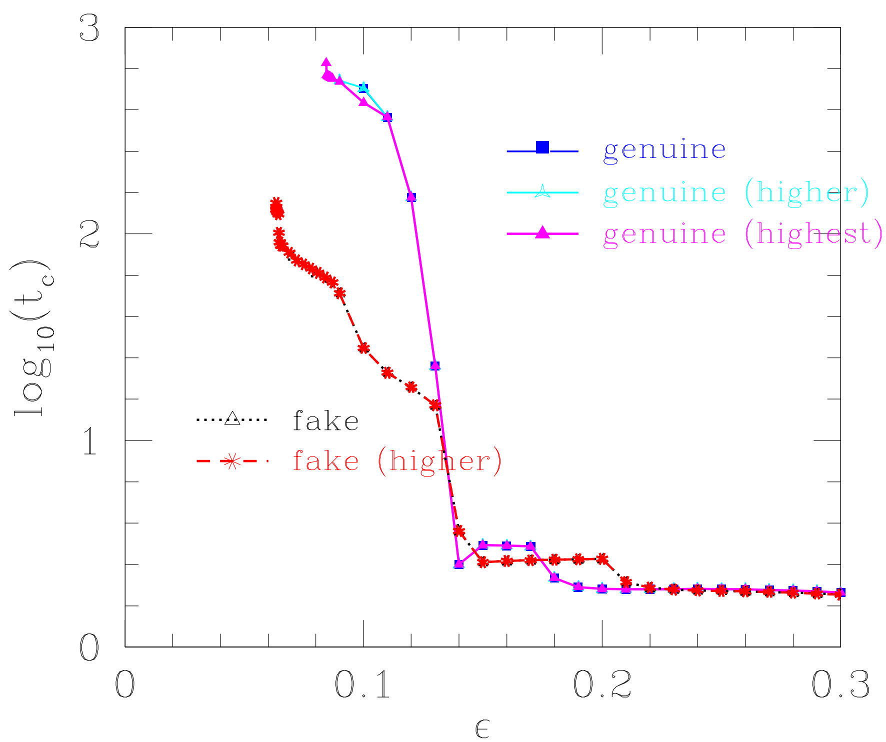
$$\phi_1^{fake} = \phi_1^{BS}\,, \qquad \Pi_1^{fake} = \Pi_2^{BS}\,, \qquad \phi_2^{fake} = \Pi_2^{fake} = 0$$

- Large σ:

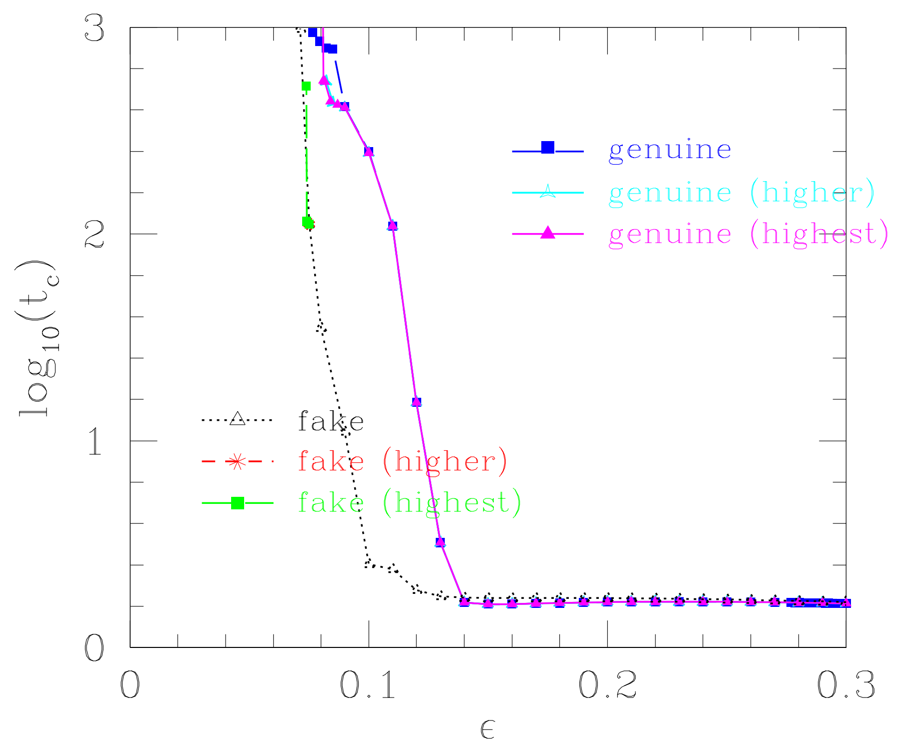
$$\Phi_i(0,x) = 0\,, \qquad \Pi_i(0,x) = \frac{2\epsilon}{\pi} e^{-\frac{4\tan^2 x}{\pi^2\sigma^2}}\ \cos^{1-d}\ \delta_i^1$$

$\Longrightarrow$ *Perturbed, Genuine & Fake Boson Stars*:

- We would like to verify nonlinear stability of boson stars
- Understand whether a global charge plays any role in the stability — note that *Fake Boson Stars* do not carry any charge.



Collapse times for Gaussian perturbations of a ground state boson star ($\phi_1(0,0) = 0.253$) and its corresponding fake star. Increasing resolutions are shown. For short collapse times, resolutions agree. However, for the longest evolutions, higher resolutions are needed. Even with very high resolutions, small ϵ evolutions show no sign of collapse.



Collapse times for Gaussian perturbations of a first excited state boson star ($\phi_1(0,0) = -0.272$) and its corresponding fake star. As before, higher resolutions are also shown with differences among the resolution appearing only at very late times.

$\Longrightarrow$ Fake solutions are not stationary and have no charge, two seemingly essential features of genuine boson stars, and so their apparent immunity to this weakly turbulent instability is surprising.

$\Longrightarrow$ This "stability" is apparently not tied to special features (e.g. charge or stationarity) but instead suggests that the dynamics undergoes something akin to a *frustrated resonance* in which amplitudes increase at times but then disperse.

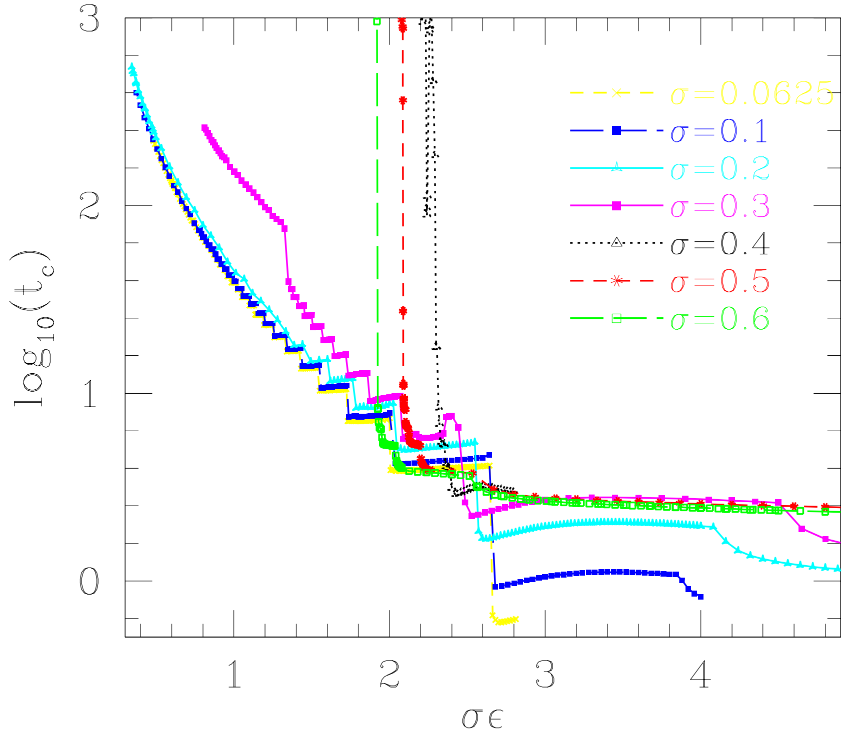
$\Longrightarrow$ One essential aspect common to both genuine and fake boson stars appears to be their non-compact, long-wavelength nature. Because they have energy distributed throughout the domain, modes no longer propagate coherently. Instead there is a continuing competition between dispersion and gravitational contraction; collapse to a black hole or not is then determined by the outcome of this competition.

⟹ Back to BR simulations with *Large* σ

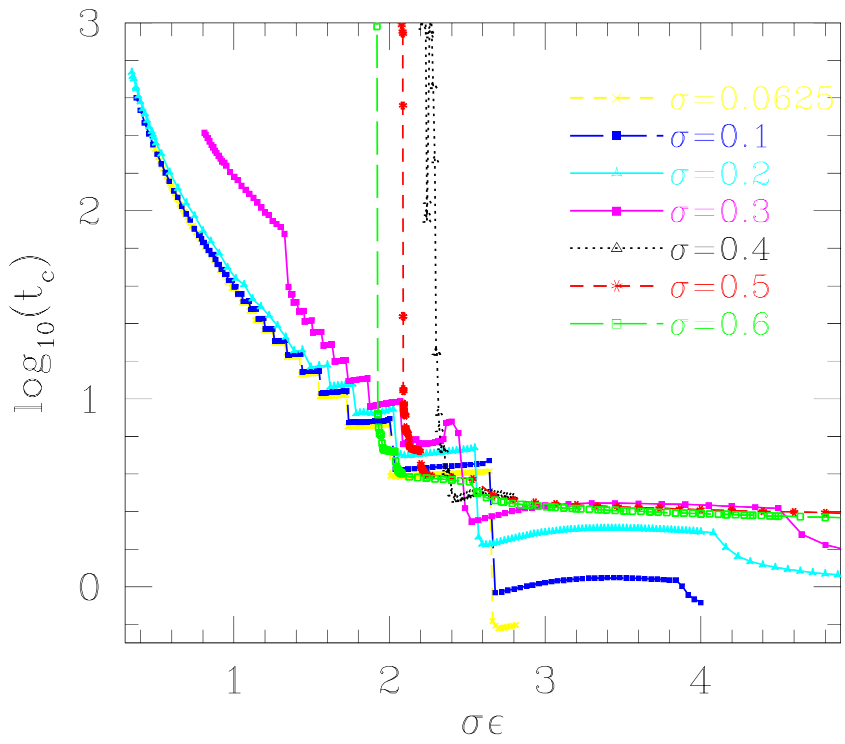
- Admittedly, above argument is far from rigorous. But if it holds, then it would imply many other forms of stable initial data.
- In particular, perhaps other forms of initial data may be immune to this weakly-turbulent instability when its extent is large.

⟹ To explore this conjecture, we adopt the same form of data considered in many previous studies of this instability (as BR)

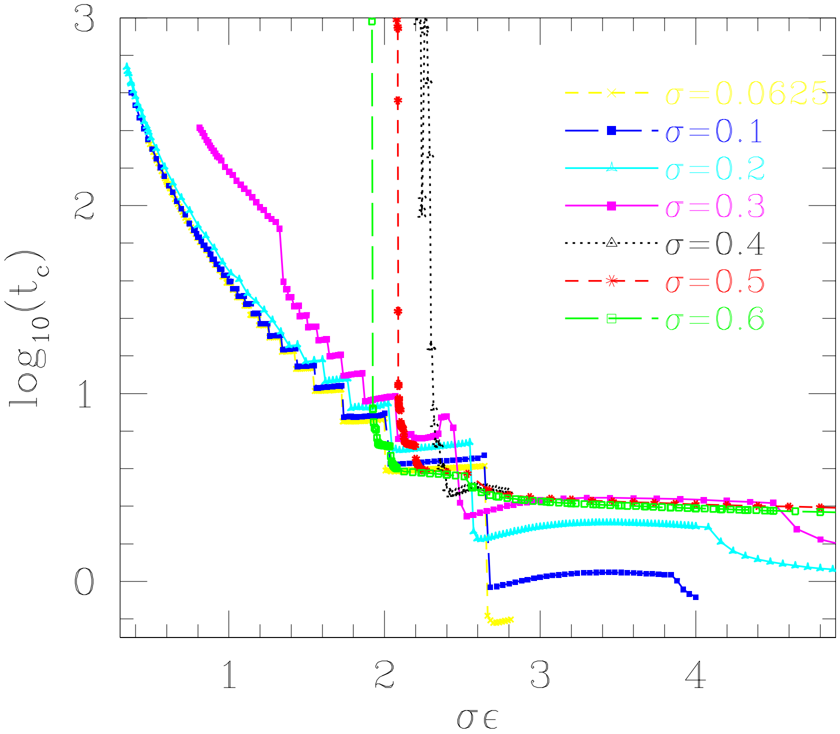
- Note: BR themselves originally did the simulations with large σ (private communication)



Collapse times for initial data of the BR form with varying width values, σ. Because changes to σ affect the amount of mass, the natural parameter against which to plot is $\sigma\epsilon$.



$\Longrightarrow$ For $\sigma \lesssim 0.3$ the standard behaviour is observed where collapse eventually occurs for any ϵ.



$\Longrightarrow$ For $\sigma \gtrsim 0.3$, there appears to exist a threshold ϵ^* below which collapse does not occur. For initial data above the transition, $\sigma > 0.3$, evolutions with smaller ϵ than shown reached at least $t \approx 2000$ with no signs of eventually collapse.

Recall:

$\Longrightarrow$ This "stability" is apparently not tied to special features (e.g. charge or stationarity) but instead suggests that the dynamics undergoes something akin to a *frustrated resonance* in which amplitudes increase at times but then disperse.

$\Longrightarrow$ I am going to present the refined analysis of the BR collapse

Re: BR mechanism for weakly-nonlinear instability

$\Longrightarrow$ To understand the physics, let's solve equations perturbatively in ϵ:

- introducing a *slow time* $\tau = \epsilon^2 t$ in addition to *fast time* t,

$$\phi = \epsilon\phi_{(1)}(t,\tau,x) + \epsilon^3\phi_{(3)}(t,\tau,x) + O(\epsilon^5)$$

$$A = 1 + \epsilon^2 A_{(2)}(t,\tau,x) + O(\epsilon^4)$$

$$\delta = \epsilon^2\delta_{(2)}(t,\tau,x) + O(\epsilon^4)$$

- at $\mathcal{O}(\epsilon)$:

$$\partial_t^2\phi_{(1)} = \phi''_{(1)} + \frac{2}{\sin x\cos x}\phi'_{(1)} \equiv -L\phi_{(1)}.$$

The operator L has eigenvalues $\omega_j^2 = (2j+3)^2$ $(j = 0,1,2,\ldots)$ and eigenvectors $e_j(x)$ ("oscillons"); up to normalization constant d_j,

$$e_j(x) = d_j\cos^3 x\ {}_2F_1\left(-j, 3+j; \frac{3}{2}; \sin^2 x\right),$$

$\Longrightarrow$

$$\phi_{(1)}(t,\tau,x) = \sum_{j=0}^{\infty}\left(A_j(\tau)e^{-i\omega_j t} + \bar{A}_j(\tau)e^{i\omega_j t}\right)e_j(x)$$

So far, the slow time dependence in decomposition $A_j(\tau)$ is not fixed.

- at $\mathcal{O}(\epsilon^2)$:

$$A_{(2)}(x) = -\frac{\cos^3 x}{\sin x} \int_0^x \left(|\Phi_{(1)}(y)|^2 + |\Pi_{(1)}(y)|^2\right) \tan^2 y \, \mathrm{d}y$$

$$\delta_{(2)}(x) = -\int_0^x \left(|\Phi_{(1)}(y)|^2 + |\Pi_{(1)}(y)|^2\right) \sin y \cos y \, \mathrm{d}y$$

- finally, at $\mathcal{O}(\epsilon^3)$:

$$\partial_t^2 \phi_{(3)} + L\phi_{(3)} + 2\partial_t \partial_\tau \phi_{(1)} = S_{(3)}(t, \tau, x)$$

with the source term

$$S_{(3)} = \partial_t (A_{(2)} - \delta_{(2)}) \partial_t \phi_{(1)} - 2(A_{(2)} - \delta_{(2)}) L\phi_{(1)} + (A'_{(2)} - \delta'_2)\phi'_{(1)}$$

In general, the source term $S_{(3)}$ contains resonant terms — proportional to $e^{\pm i\omega_j t}$). Such resonances occur for all triads (j_1, j_2, j_3), with

$$\omega_j = \omega_{j_1} + \omega_{j_2} - \omega_{j_3}$$

In ordinary perturbation theory these resonances lead to secular growths in $\phi_{(3)}$, and is the origin of the early 'linear-in-slow-time' growth of $\Pi(t, 0)^2$.

$\Longrightarrow$ A standard trick of the multiscale dynamics is to remove resonance terms in the source via slow-time dynamics of $A_j(\tau)$:

- first, project $\mathcal{O}(\epsilon^3)$ equations onto oscillon modes e_j,

$$\left(e_j, \partial_t^2 \phi_{(3)} + \omega_j^2 \phi_{(3)}\right) - 2i\omega_j \left(\partial_\tau A_j e^{-i\omega_j t} - \partial_\tau \bar{A}_j e^{i\omega_j t}\right) = \left(e_j, S_{(3)}\right)$$

- By exploiting the presence of terms proportional to $e^{\pm i\omega_j t}$ on the left hand side of the equation, we may cancel off the resonant terms on the right hand side. Denoting by $f[\omega_j]$ the part of f proportional to $e^{i\omega_j t}$, we set

$$-2i\omega_j \partial_\tau A_j = (e_j, S(t,\tau,x))[-\omega_i] = \sum_{klm} \mathcal{S}^{(j)}_{klm} \bar{A}_k A_l A_m$$

where $\mathcal{S}^{(j)}_{klm}$ are real constants representing different resonance channel contributions.

No resonances $\Longleftrightarrow$ *no secular growth in perturbation theory*

$\Longrightarrow$ An equivalent framework, a standard renormalization group analysis (resummation of $\mathcal{O}(\epsilon^3)$ terms), was developed by Ben Craps, Oleg Evnin and Joris Vanhoof in arXiv:1407.6273.

So far, for arbitrarily high but finite truncation in the number of modes, :

- small-ϵ dynamics can be resumed to $\mathcal{O}(\epsilon^3)$ using TTF (two-time framework) or renormalization group
- there is no unbounded growth of $\Pi(t,0)^2$ in TTF

$\Longrightarrow$ no BH formation

$\Longrightarrow$ no equilibration!

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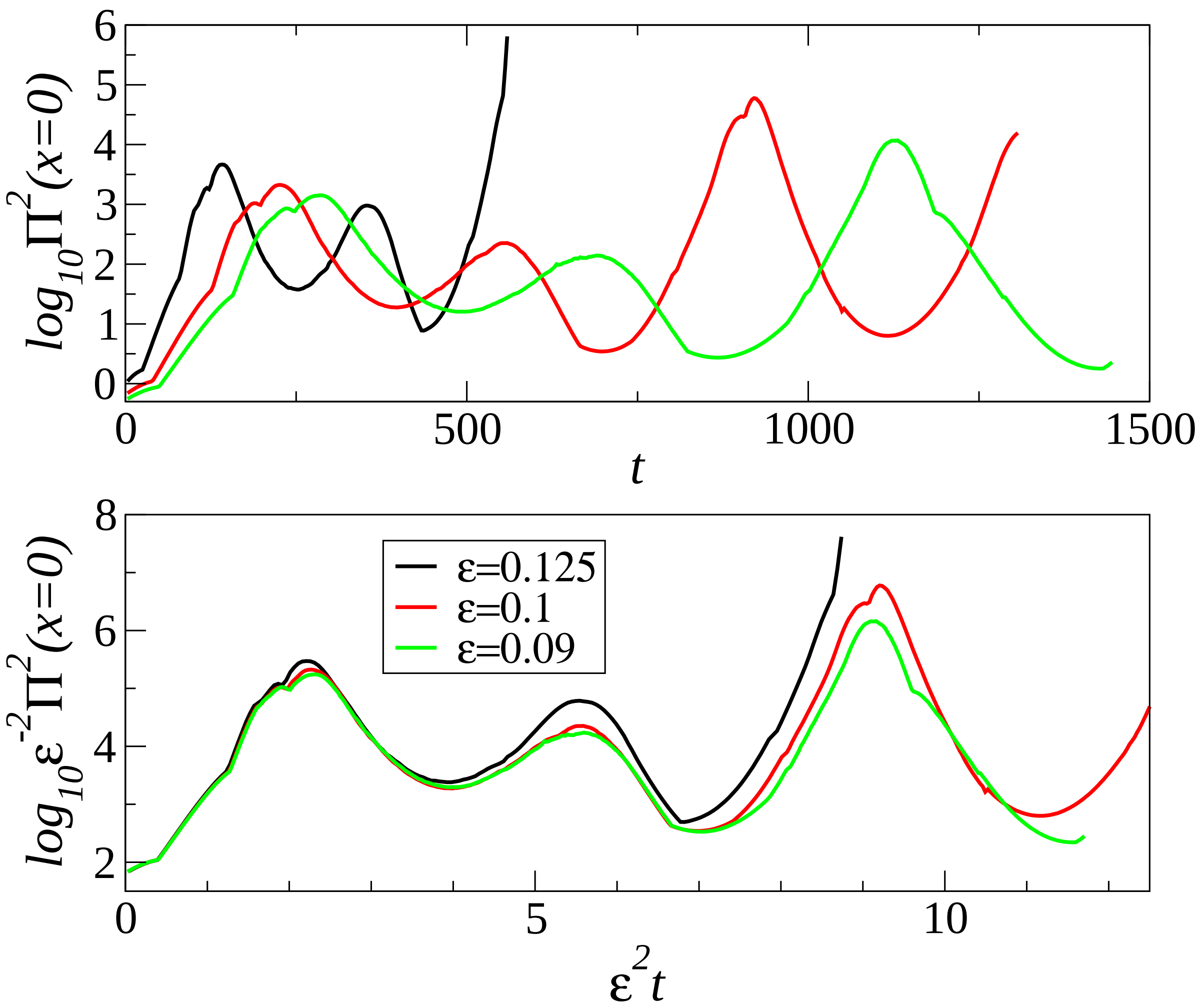
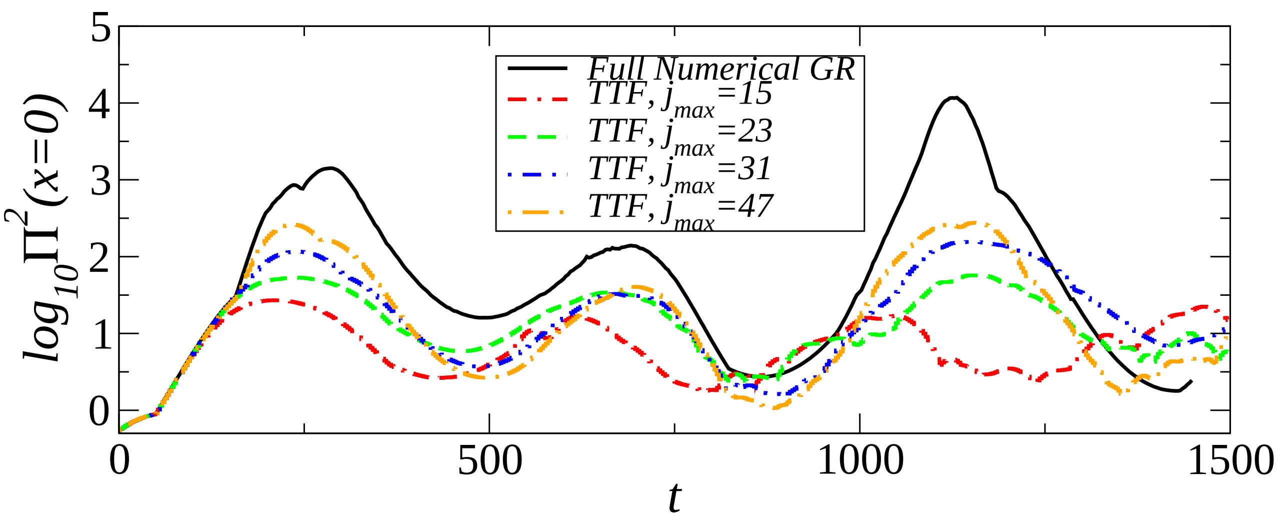
BUT: is TTF a good approximation to full numerics?



Note that initially, the growth in $\Pi(t,0)^2$ is the same as in BR, but for sufficiently small ϵ, the forward energy cascade is followed with the reverse one. It appears the number of forward/backward sequences can continue forever, as $\epsilon \to 0$.



Full numerical and TTF results for 2-mode equal-energy initial data with $\epsilon = 0.09$. As $j_{\max}$ is increased, the TTF solutions achieve better agreement with the full numerics. Recurrence behavior observed in the full numerical solution is reasonably well captured by TTF.

$\Longrightarrow$ I presented a strong numerical evidence that there are initial configurations that do no equilibrate

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■ no equilibration is not surprising, as slow-time EOMs have the same structure as FPU β-model (an infinite set of nonlinearly coupled oscillators), which *paradoxically* does not equilibrate.

$\Longrightarrow$ Let me borrow couple slides from David K. Campbell presentation from "First Symposium of the Institute for Basic Science February 21, 2014"

“In the beginning…” was FPU

Los Alamos, Summers 1953-4 Enrico Fermi, John Pasta, and Stan Ulam decided to use the world's then most powerful computer, the

MANIAC-1

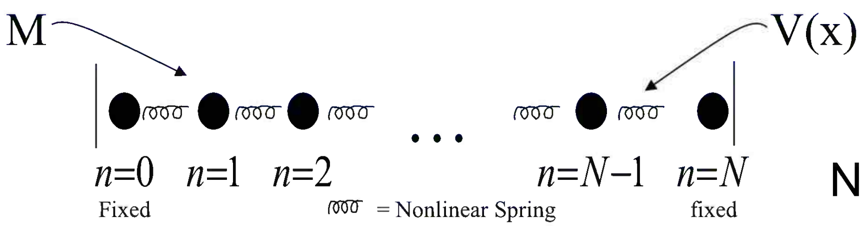
(Mathematical Analyzer Numerical Integrator And Computer)

to study the equipartition of energy expected from statistical mechanics in the simplest classical model of a solid: a ID chain of equal mass particles coupled by *nonlinear** springs. Fermi expected “these were to be studied preliminary to setting up ultimate models …where “mixing” and “turbulence” could be observed. The motivation then was to observe the *rates* of the mixing and thermalization with the hope that the calculational results would provide hints for a future theory.” [S. Ulam].

*They knew linear springs could not produce equipartition

Aside: Birth of computational physics (“experimental mathematics”)

“In the beginning…” was FPU



N=32, 64

$$V(x) = \frac{1}{2} kx^2 + \alpha/3\ x^3 + \beta/4\ x^4$$

“The results of the calculations (performed on the old MANIAC machine) were interesting and quite surprising to Fermi. He expressed to me the opinion that they really constituted a little discovery in providing limitations that the prevalent beliefs in the universality of “mixing and thermalization in *non-linear* systems may not always be justified.”

[S. Ulam]

Role of hidden conservation laws in the dual turbulent cascade

$\Longrightarrow$ Can we gain an analytical understanding for the sequence of forward/reverse energy cascades?

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- Recall, to order $\mathcal{O}(\epsilon^3)$ the energy transfer between different modes in

$$\phi_{(1)}(t,\tau,x) = \sum_{j=0}^{\infty} \left(A_j(\tau) e^{-i\omega_j t} + \bar{A}_j(\tau) e^{i\omega_j t} \right) e_j(x),$$

  is governed by TTF equations:

$$-2i\omega_j \frac{dA_j}{d\tau} = \sum_{klm} \mathcal{S}^{(j)}_{klm} \bar{A}_k A_l A_m$$

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- TTF equations has a trivial conservation law (the energy conservation to $\mathcal{O}(\epsilon^2)$)

$$E \equiv \sum_j E_i = \sum_j 4\omega_j^2 |A_j(\tau)|^2 \,, \qquad \frac{d}{d\tau} E = 0$$

- Turns out TTF equations have an **additional/accidental** conservation quantity ("the particle number"):

$$N \equiv \sum_j 4\omega_j |A_j|^2 , \qquad \frac{d}{d\tau} N = 0$$

- Thus, we can rewrite conservation laws as

$$E = \sum_j E_j , \qquad N = \sum_j (2j+3)^{-1} E_j$$

- It is clear that "forward-only" energy cascade is not possible —- this would lead to violation of the particle number

- It is also easy to see that the TTF does no allow equipartitioning of the energy (the equilibration generically):

$$N_{\rm final} = \sum_{j=0}^{j_{\rm max}} \frac{E_j}{\omega_j} = \sum_{j=0}^{j_{\rm max}} \frac{E}{\omega_j (j_{\rm max}+1)} = \frac{H_{j_{\rm max}+\frac{3}{2}} - 2 + \log 4}{2(j_{\rm max}+1)} E$$

where H_n is the nth harmonic number. Unless finely-tunes,

$$N_{\rm final} \neq N_{initial}$$

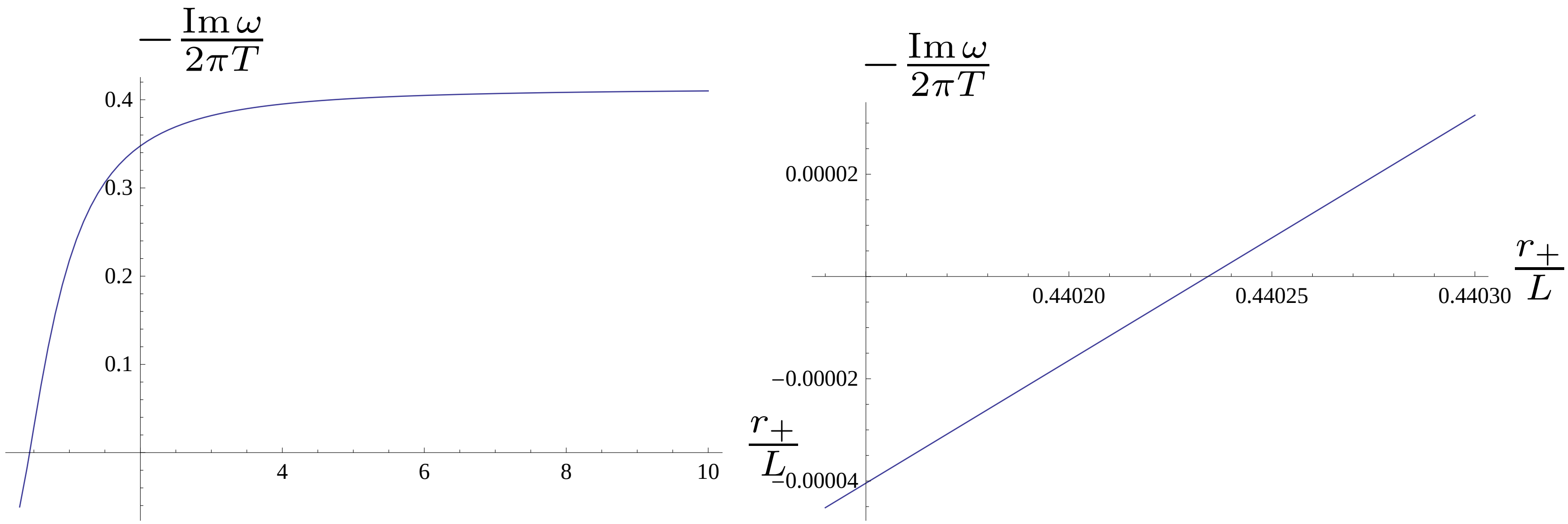
What all of this have to do with thermalization of dual gauge theories?

What all of this have to do with thermalization of dual gauge theories?

$\Longrightarrow$ Sadly, not much:

- So far, I discussed the spectrum of small BHs (spectrum of dual gauge theory equilibrium states) in effective 5d gravitational description. The full holography is in 10d. Thus, we focused only on the states that preserve the symmetry of the compact manifold in the holography — S^5 [$SO(6)$ symmetry] for the $\mathcal{N} = 4$ SYM.

- However, global symmetries can be broken dynamically (Gregory-Laflamme instability in the gravity dual) at low-energies:



The dependence of the $g = -\mathrm{Im}\,(\omega)$ as a function of $\rho_+ = \frac{r_+}{L}$ for $\ell = 1$ fluctuations of $SO(6)$ symmetric black holes in $AdS_5 \times S^5$. Black holes with $g < 0$ are unstable with respect to condensation of these fluctuations.

- So, we would like to relax $SO(6)$ symmetry
- Study gravitational collapse which allows for the symmetry breaking at low-energies (work in progress)

Conclusions:

- I argued that low-energy dynamics in AdS (and equilibration of dual gauge theories) is a fascinated subject
- There are initial configurations in AdS that collapse to black hole in the limit $\epsilon \to 0$
- There are also initial configurations that do not lead to equilibration
- TTF provides a nice framework to understand why some configurations do not collapse; it also provides an understanding why some initial configurations (like BR original profiles) do collapse ("Islands of stability and recurrence times in AdS" by Stephen Green et.al)
- **I did not talk**: why can't we further extend TTF to $\mathcal{O}(\epsilon^5)$? is AdS stable or not? does the CFT have R-symmetric states at low-energies or R-symmetry is *always* spontaneously broken?

Thanks you!

$\Longrightarrow$ Consider a phenomenological model of AdS/CFT correspondence with the action

$$S = \frac{1}{2\ell_p^3}\int_{\mathcal{M}_5} d^5\xi\sqrt{-g}\biggl(\frac{12}{L^2}+R+\mathcal{L}_{\rm matter}$$

$$+\frac{\lambda_{\rm GB}}{2}L^2\left(R^2-4R_{\mu\nu}R^{\mu\nu}+R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\biggr)$$

Once again, for the $\mathcal{L}_{matter}$ we take the action of the massless scalar (dual to a marginal operator). The role of the higher-derivative term with $\lambda_{\rm GB}$ coupling is to generalize the conformal anomaly of the dual boundary CFT:

$$\langle T^\mu{}_\mu\rangle_{\rm CFT} = \frac{c}{16\pi^2}I_4 - \frac{a}{16\pi^2}E_4$$

$$E_4 = r_{\mu\nu\rho\lambda}r^{\mu\nu\rho\lambda}-4r_{\mu\nu}r^{\mu\nu}+r^2\,,\qquad I_4 = r_{\mu\nu\rho\lambda}r^{\mu\nu\rho\lambda}-2r_{\mu\nu}r^{\mu\nu}+\frac{1}{3}r^2$$

where E_4 and I_4 correspond to the four-dimensional Euler density and the square of the Weyl curvature of $\mathcal{M}_4=\partial\mathcal{M}_5$

$\Longrightarrow$

$$c = \frac{\pi^2\tilde{L}^3}{\ell_p^3}\left(1-2\frac{\lambda_{\rm GB}}{\beta^2}\right)\,,\qquad a = \frac{\pi^2\tilde{L}^3}{\ell_p^3}\left(1-6\frac{\lambda_{\rm GB}}{\beta^2}\right)$$

$$\tilde{L}\equiv\beta L\,,\qquad \beta^2\equiv 1/2+1/2\sqrt{1-4\lambda_{\rm GB}}$$

$\Longrightarrow$ Let's begin with the equilibrium states of the theory (within gravity approximation):

- First, we have a vacuum:

$$ds_5^2 = \frac{L^2\beta^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x\, d\Omega_3^2\right)$$

which is AdS, with a modified radius $L \to \tilde{L} = \beta L$. Requiring that β^2 is real, $i.e$, we have AdS asymptotic, constraints

$$\lambda_{GB} \leq \frac{1}{4}$$

Using the machinery of the holographic renormalization we can compute the vacuum (Casimir) energy:

$$\mathcal{E}_{vacuum} = \frac{3c}{4\tilde{L}}\ \frac{a}{c}$$

- We also have BHs:

$$ds^2 = \frac{L^2\beta^2}{\cos^2 x}\left(-A(x)dt^2 + \frac{dx^2}{A(x)} + \sin^2 x\, d\Omega_3^2\right)$$

$$A = 1 - \frac{1}{2\lambda_{\rm GB}}\Bigg((2\lambda_{\rm GB} - \beta^2)\sin^2 x + \Big(4\lambda_{\rm GB}(\beta^2 - 2\lambda_{\rm GB})M\cos^4 x$$
$$+ (2\lambda_{\rm GB} - \beta^2)^2\cos^4 x - \beta^4(1 - 4\lambda_{\rm GB})\cos(2x)\Big)^{1/2}\Bigg)$$

A free parameter $M > 0$ (why positive will be clear later) in the solution is related to the BH mass (boundary energy of equilibrium CFT states):

$$\mathcal{E}_{BH} = \frac{3c}{4\tilde{L}}\left(\frac{a}{c} + 4M\right)$$

It is straightforward to observe that BHs have regular horizons only if

$$M \geq \begin{cases} \frac{1-\beta^2}{2\beta^2-1}\,, & \text{if } \lambda_{\rm GB} > 0\,, \\ (\beta^2 - 1)(2\beta^2 - 1)\,, & \text{if } \lambda_{\rm GB} < 0\,. \end{cases}$$

Otherwise, the $\sin^2 x$ (the warp factor of S^3) vanishes **before** vanishing $A(x)$. The saturation occurs for the zero-size BHs.

$\Longrightarrow$ So, introducing

$$\delta E = \mathcal{E}_{BH} - \mathcal{E}_{vacuum} > 0$$

in phenomenological AdS/CFT dualities with $c \neq a$,

$$\frac{\delta E}{|\mathcal{E}_{vacuum}|} \;\geq\; \epsilon_{gap} = \frac{4(1-\beta^2)}{|6\beta^2-5|} \times \begin{cases} 1\,, & \lambda_{\rm GB} > 0\,, \\ -(2\beta^2-1)^2\,, & \lambda_{\rm GB} < 0 \end{cases}$$

Notice that ϵ_{gap} can become arbitrarily large: ϵ_{gap} is unbounded as $\lambda_{\rm GB} \to -\infty$ and $\lambda_{\rm GB} \to 5/36 < \frac{1}{4}$.

$$\Downarrow$$

In a dual CFT any state $|\xi\rangle$, if exist, with

$$\delta E/|\mathcal{E}_{vacuum}| = \frac{\mathcal{E}_\xi - \mathcal{E}_{vacuum}}{\mathcal{E}_{vacuum}} \;<\; \epsilon_{gap}$$

can not equilibrate!

I am going to show now that arbitrary low excitations are allowed in the GB-model of holography

$\Longrightarrow$ As before, we write the 5-dimensional metric describing an asymptotically AdS spacetime with $SO(4)$ symmetry in the form

$$ds^2 = \frac{L^2\beta^2}{\cos^2 x}\left(-Ae^{-2\delta}dt^2 + \frac{dx^2}{A} + \sin^2 x\, d\Omega_3^2\right)$$

where

$$A = A(x,t)\,, \qquad \delta = \delta(x,t)\,, \qquad \phi = \phi(t,x)$$

$\Longrightarrow$ EOMs: $\Box\phi = 0$,

$$A_{,x} = \frac{1}{\cos x(\beta^2\sin^2 x + 2\lambda_{\rm GB}(\cos^2 x - A))}\Big(2\sin x(\beta^2(1+\sin^2 x)(\beta^2 - A)$$

$$-\beta^2(\beta^2-1)\cos^2 x - 2\lambda_{\rm GB}A(\cos^2 x - A))\Big)$$

$$-\frac{\beta^2\sin^3 x\cos x}{A(\beta^2\sin^2 x + 2\lambda_{\rm GB}(\cos^2 x - A))}\left(e^{2\delta}(\partial_t\phi)^2 + A^2(\partial_x\phi)^2\right)$$

$$\delta_{,x} = -\frac{\beta^2\sin^3 x\cos x}{A^2(\beta^2\sin^2 x + 2\lambda_{\rm GB}(\cos^2 x - A))}\left(e^{2\delta}(\partial_t\phi)^2 + A^2(\partial_x\phi)^2\right)$$

$$A_{,t} + \frac{2\beta^2\sin^3 x\cos x A}{\beta^2\sin^2 x + 2\lambda_{\rm GB}(\cos^2 x - A)}\partial_t\phi\partial_x\phi = 0$$

Again, introduce the mass-aspect function $\mathcal{M}(t,x)$ as

$$A(t,x) = 1 - \frac{1}{2\lambda_{\rm GB}}\Bigg((2\lambda_{\rm GB}-\beta^2)\sin^2 x + \Bigg(4\lambda_{\rm GB}(\beta^2-2\lambda_{\rm GB})\mathcal{M}(t,x)\cos^4 x$$
$$+(2\lambda_{\rm GB}-\beta^2)^2\cos^4 x - \beta^4(1-4\lambda_{\rm GB})\cos(2x)\Bigg)^{1/2}\Bigg)$$

we can explicitly solve for $\mathcal{M}(t,x)$:

$$\mathcal{M}(t,x) = \frac{1}{2\beta^2-1}\int_0^x dz \frac{\tan^3 z}{A(t,z)}\left[e^{2\delta}(\partial_t\phi)^2 + A^2(\partial_x\phi)^2\right]$$

Furthermore, from the boundary stress-energy tensor,

$$M = \mathcal{M}(t,x)\Big|_{x=\frac{\pi}{2}} \qquad \propto \qquad (\mathcal{E}_\xi - \mathcal{E}_{vacuum})$$

Note:

- $M \geq 0$
- scaling down the amplitude of ϕ allows one to make M arbitrarily small

$\Longrightarrow$**Indeed, we can prepare arbitrary low-energy excitations in GB gravity.**