漸近的安全性による重力の量子論

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1 Introduction – why do wee need quantum gravity? –

In the gravitational theory (Eintein gravity) there are mysterious objects called black holes.

Another mystery is that our universe seems to be expanding with acceleration even though gravity is attractive.

These objects/fact put us several puzzles.

- Black holes ··· have singularites at the center:
  The curvature diverges, and the Einstein equation does not make sense
- Big Bang Cosmology (our universe is expanding) ⇒ initial singularity
  Here again the Einstein equation is not valid
- Other problems associated Hawking radiation

Quantum Mechanics ⇒ Uncertainity principle forbids point-like state
  ··· QM could be rescue to remove singularities and resolve the puzzles

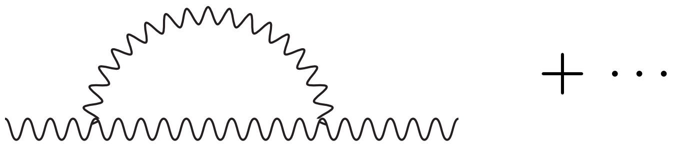
Gravitational wave ··· Graviton (de Brogli)

How? ··· "Quantum Gravity"

The Einstein theory is of little help; it is a non-renormalizable theory.

What "non-renormalizable" means?

According to the modern technology to incorporate quantum effects in field theory (Einstein gravity is one of such theory based on "fields"), we have to take into account the "loop effects."



The loops contains infinite number of states which contribute, and these cause divergences $\Rightarrow$ must be absorbed into infinite number of couplings $\Rightarrow$ unpredictability

These divergences are typical in quantum field theory, but they can be absorbed into finite number of couplings and mass etc. in renormalizable theories.

The renormalizable theories include

Quantum Electrodynamics: theory of electromagnetism

Quantum Chromodynamics: theory of strong interaction acting on quarks

Electroweak (Weinberg-Salam) theory: theory of weak interactions which

causes decay of elementary particles like beta-decay

It is known that there are only four interactions at the fundamental level of elementary particles. These three are among these.

There remains only gravitational interaction which turns out non-renormalizable because its coupling constant has negative mass dimension.

How to make sense of quantum theory of gravity $\cdots$ Quantum Gravity

At the moment, it is one of the most important unsolved problems in theoretical physics:

Only with such a theory, we believe the problems mentioned before, in black holes and Big Bang cosmology, may be understood.

Many people think superstring theory is the leading candidate for such theory. However

Problems

- **String theory is now understood only perturbatively.**
- **As a consequence, it cannot describe phenomena on curved spacetime, whereas the significant problems such as singularities appear at situation in early universe and/or black holes**

Want to consider other formulation that can deal with such phenomena.

⇒ Quantum gravity within the framework of local field theory.

I do not mean to exclude string theory. Rather it may be compatible because the scale invariance would be realized at high energy.

Some remarks.

1. **The first attempt to quantize gravity and calculate loop effects was made by Feynman at one loop.**

   R. P. Feynman, "Quantum theory of gravitation," Acta Phys. Polon. 24 (1963), 697

   He found the necessity of ghosts for quantization of gravity and *Yang-Mills*!

2. **Quantization is made by DeWitt 1967.**
3. **Understanding Faddeev-Popov ghost is made in 1967.**
4. **One-loop calculation is made by 't Hooft-Veltman 1974, but they found counterterms in pure gravity are quadratic in curvatures**

$$\Delta\mathcal{L}_{grav} = \frac{\sqrt{g}}{\varepsilon}\left(\frac{1}{120}R^2 + \frac{7}{20}R_{\mu\nu}^2\right)$$

which vanishes upon using field equations, after use of Gauss-Bonnet theorem. This means that they may be removed by field redefinition in 4-dimension. (Terms proportional to field equations can be removed by field redefinition.)
They also showed that gravity coupled scalar is nonrenormalizable.

5. **Nonrenormalizability of pure gravity is shown at two-loop level by Goroff and Sagnotti in 1985! (Counterterm of the form $R_{\mu\nu\rho\sigma}R^{\rho\sigma\alpha\beta}R_{\alpha\beta}{}^{\mu\nu}$) (nonrenormalizability with matter were shown before.)**
6. **Quadratic theory (in curvature) is shown to be renormalizable (Stelle, 1977), but most probably suffer from the trouble in unitarity (but see Donoghue).**

2 Asymptotic safety

How to make sense of quantum gravity?

Possible if we can restrict theory such that all the UV divergences are under control.

The object we deal with is the effective average action Γ_k obtained by integrating out all fluctuations of the fields with momenta larger than k.

$$e^{W_k(J)} = \int [D\phi] e^{-(S[\phi]+\Delta S_k[\phi])+\int J\phi} \quad \textbf{where } \Delta S_k[\phi] = \frac{1}{2}\int d^d q \phi(-q) R_k(q^2)\phi(q)$$

$R_A(q)$: a cutoff which gives suppression of IR modes
Its role is to remove the IR mode from the action,
so that the path integral is carried out over UV modes
$\Rightarrow$ Standard Legendre transf. $\Rightarrow$ $\Gamma_k[\phi]$

When evaluated at tree level, this correctly describes gravitational phenomena with quantum effects included. An important consequence of the quantum effects is that the Newton coupling depends on the energy scale k. $\Rightarrow$ $\Gamma_{k=0}[\phi]$: effective action (divergent)

This means that near the singularity, Newton coupling may vanish so that the singularity is resolved. (We will see that this indeed may happen in balck hole.)

- **The effective average action itself is divergent, because all high energy modes – infinite number of them – are integrated.**

Important fact

We look at the dependence of the effective average action on k, which gives the RG flow, free from any divergence and can be used to define quantum theory.

$$k\partial_k\Gamma_k(\Phi) = \frac{1}{2}\mathbf{tr}\left[\left(\frac{\partial^2\Gamma_k}{\partial\Phi^A\partial\Phi^B}+R_k\right)^{-1}k\partial_k R_k\right] \quad \Leftarrow \quad \text{there is no divergence!}$$

because $k\partial_k R_k$ has contribution from modes only around $\sim k$

Functional renormalization group equation (FRGE)!

R_k: the cutoff function.

FRGE gives flow of the effective action in the theory (coupling) space defined by suitable bases $\mathcal{O}_i$.

$$\Gamma_k = \sum_i g_i(k)\mathcal{O}_i \quad \Rightarrow \quad \frac{d\Gamma_k}{dt} = \sum_i \beta_i \mathcal{O}_i, \quad \beta_i = \frac{dg_i}{dt}$$

$$t \equiv \ln k$$

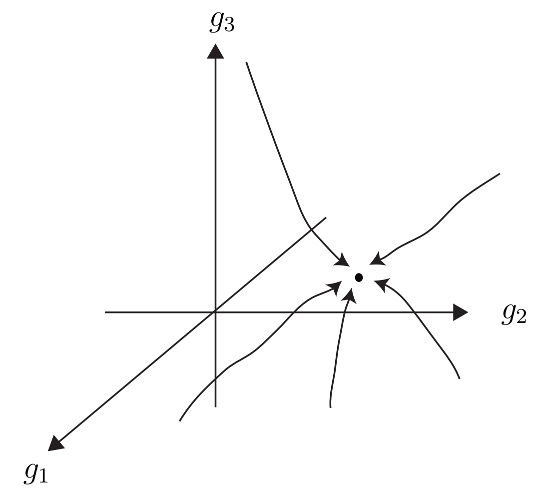


図 1: RG flow

We set initial conditions at some point and then flow to $k \to \infty$.

The flow stops at fixed point (FP) where $\beta = 0$.

- **If all couplings go to finite FPs at UV, physical quantities are well defined, giving the UV finite theory $\Rightarrow$ Asymptotic safety**

Note:

The couplings with positive mass dimensions go to infinity as dictated by their dimensions.

Those operators whose couplings go to FPs in the infinite energy are called relevant operators, and repell irrelevant operators and others marginal.

We are interested in only those relevant operators! $\Rightarrow$ well-defined $\Gamma_{k=0}$

Important question: How can we determine which operator is relevant?

Look at the behavior close to a fixed point at g_*, and define the difference of the couplings and the FP:

$$y_i = g_i - g_{i*}$$

We consider the linearized equation near the FP:

$$\frac{dy_i}{dt} = M_{ij} y_j, \quad M_{ij} \equiv \left. \frac{\partial \beta_i}{\partial g_j} \right|_{g_*}$$

The eigenvalues λ_i of this stability matrix is important.

If λ_i is negative, the coupling goes into the FP in UV: relevant operator

If λ_i is positive, it goes away from the FP in UV: irrelevant operator

If λ_i is zero, it stays: marginal operator

Asymptotic safety

All the operators in our theory are relevant
$\Rightarrow$ all the couplings are well defined in UV, and the effective action $\Gamma_{k=0}$ describes gravitational phenomena without divergences. UV complete.

Suppose that we have an operator $\mathcal{O}_i$ of dimension d_i, its coupling has dimension

$$d_{g_i} = 4 - d_i.$$

The beta functions for dimensionless couplings $\tilde{g}_i \equiv k^{-d_{g_i}} g_i$ are then given as

$$\beta_i(\tilde{g}_j) = -d_{g_i}\tilde{g}_j + O(\tilde{g}_j^2) = -(4 - d_i)\tilde{g}_j + O(\tilde{g}_j^2)$$

and the operators with $d_i > 4\,(d_{g_i} < 0)$ is irrelevant near origin (i.e. $\tilde{g}_j \approx 0$). (The FPs at the origin are called Gaussian FPs, and are always FPs for any couplings, defining perturbation theory.)

These are precisely non-renormalizable interactions in perturbation theory! $\Rightarrow$ Finite number of operators expected

However, when the coupling is not very small, we have to check really if the stability matrix has negative eigenvalues at the fixed points.

Note that the larger the d_i is, the more irrelevant it tends to.

In the Asymptotic Safety, we mainly consider nontrivial (nonzero) FPs.

The trajectories with the same FP make a surface, called critical surface of dimension given by the number of relevant operators (in the space of the couplings). See Fig. 1.

The Important problem

How many relevant operators we need?

What is the dimension of critical surface?

$\cdots$ "Nonperturbatively Renormalizable theory" or Predictability

Ideally we should consider the whole "theory space" of infinite number of possible operators and check how many relevant operators we have. But in practice it is impossible.

What we do is to use approximation:

1. **Truncation: restrict the theory space to finite number of operators**
   $\Rightarrow$ At the next stage, we examine if there are additional relevant operators and if the previous fixed point is affected.
   $\Rightarrow$ If yes, we go to next stage; if not, we can stop there.
2. **Local potential approximation**
3. **Derivative expansion**

Some technical details: **(This part may be just lightly touched.)**

How to calculate the rhs of the FRGE:

What we need to calculate is

$$\frac{1}{2}\mathrm{Tr}\left[\frac{\dot{R}_k}{\partial^2\Gamma_k/\partial\Phi_A\partial\Phi_B + R_k}\right] = \frac{1}{2}\mathrm{Tr}\left[\frac{\dot{R}_k}{\Delta_{AB}+R_k}\right]$$

More generally, using Laplace transform $W(z)=\int_0^\infty e^{-zs}\tilde{W}(s)$**, one evaluates**

$$\mathrm{Tr}[W(\Delta)] = \sum_n W[\lambda_n] = \sum_n \int_0^\infty dse^{-\lambda_n s}\tilde{W}(s) = \int_0^\infty ds\tilde{W}(s)\mathrm{Tr}[e^{-s\Delta}]$$

and use heat kernel expansion

$$\mathrm{Tr}[e^{-s\Delta}] = \frac{1}{(4\pi s)^{d/2}}\sum_{n=0}^\infty B_{2n}(\Delta)s^n$$

to get

$$\mathrm{Tr}[W(\Delta)] = \frac{1}{(4\pi s)^{d/2}}\big(B_0(\Delta)Q_{d/2}(W) + B_2(\Delta)Q_{d/2-1}(W) + B_4(\Delta)Q_{d/2-2}(W) + \cdots\big)$$

where

$$Q_n(W) = \frac{1}{\Gamma(n)}\int_0^\infty dzW(z)z^{n-1};$$

$$B_0 = \int d^dx\sqrt{g}\mathrm{tr}(\mathbf{1}),\quad B_2 = \int d^dx\sqrt{g}\left(\frac{1}{6}\mathrm{tr}\mathbf{1} - \mathrm{tr}\mathbf{E}^2\right),\quad B_4 = \int d^dx\sqrt{g}\left(\frac{1}{180}(R^2_{\mu\nu\alpha\beta} - R^2_{\mu\nu} + \frac{5}{2}R^2)\mathrm{tr}\mathbf{1} + \cdots\right)$$

For $d=4$**, we need only these 3 terms.**

The lhs of the FRGE is

$$\Gamma_k = \int d^dx\sqrt{g}\frac{1}{16\pi G_N}[2\Lambda - R + \cdots] \quad\Rightarrow\quad \dot{\Gamma}_k = \int d^dx\sqrt{g}\left[\left(\frac{-\Lambda}{8\pi G_N^2}\beta_{G_N} + \frac{1}{8\pi G_N}\beta_\Lambda\right) + \frac{1}{16\pi G_N^2}\beta_{G_N}R + \cdots\right]$$

We compare the lhs and rhs, and obtain

$$-\frac{\Lambda}{8\pi G_N^2}\beta_{G_N}+\frac{1}{8\pi G_N}\beta_\Lambda=\textbf{rhs constant term},\qquad \frac{1}{16\pi G_N^2}\beta_{G_N}=\textbf{Coefficient of } R \textbf{ on the rhs}$$

We write the beta functions for the dimensionless couplings by writing

$$G_N=k^{-(d-2)}\tilde{G}_N,\qquad \Lambda=k^2\tilde{\Lambda}$$

Example: General Relativity

Action

$$S(g)=Z_N\int d^dx\sqrt{g}(2\Lambda-g^{\mu\nu}R_{\mu\nu}(g))$$

Hessian

$$S^{(2)}=\frac{Z_N}{2}\int d^dx\,\sqrt{\bar{g}}\bigg\{\frac{1}{2}h_{\mu\nu}(-\bar{\nabla}^2)h^{\mu\nu}+h_{\mu\nu}\bar{\nabla}^\mu\bar{\nabla}^\rho h_\rho{}^\nu-h\bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu}+\frac{1}{2}h\bar{\nabla}^2h$$
$$-h^{\mu\nu}\left[\bar{R}_{\mu\alpha\nu\beta}-\frac{1}{2}\left(\bar{g}_{\mu\nu}\bar{R}_{\alpha\beta}+\bar{g}_{\alpha\beta}\bar{R}_{\mu\nu}\right)+\frac{1}{2}\left(\bar{g}_{\mu\alpha}\bar{R}_{\nu\beta}+\bar{g}_{\nu\beta}\bar{R}_{\mu\alpha}\right)+\frac{\bar{R}-2\Lambda}{4}(\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}-2\,\delta_{\mu\nu,\alpha\beta})\right]h^{\alpha\beta}\bigg\}$$

Guge fixing and ghost

$$\mathcal{L}_{GF+FP}/\sqrt{\bar{g}}=\frac{Z_N}{2a}(F^\mu)^2+i\bar{c}_\mu\Big[\delta^\mu_\nu\bar{\nabla}^2+\Big(1-2\frac{1+b}{d}\Big)\bar{\nabla}^\mu\bar{\nabla}_\nu+\bar{R}^\mu{}_\nu\Big]c^\nu,\quad F_\mu\equiv\bar{\nabla}_\rho h^\rho_\mu-\frac{1+b}{d}\bar{\nabla}_\mu h\ ,\quad (h\equiv h^\mu_\mu)$$

We separate $h_{\mu\nu}=h^T_{\mu\nu}+\frac{\bar{g}_{\mu\nu}}{d}h$ (traceless and trace parts)
Introduce regularization function (type Ia)

$$\boldsymbol{R}^T=\frac{Z_N}{4}R_k,\quad \boldsymbol{R}=\frac{(d-2)Z_N}{8d}R_k\quad \boldsymbol{R}_{gh}=R_k,\quad R_k\equiv P_k(-\bar{\nabla}^2)+\bar{\nabla}^2$$

Then we get

$$\dot{\Gamma}_k = \frac{1}{2}\mathrm{Tr}\left(\frac{\dot{\boldsymbol{R}}^T}{\Delta^T + \boldsymbol{R}^T_{\mu\nu,\alpha\beta}}\right) + \frac{1}{2}\mathrm{Tr}\left(\frac{\dot{\boldsymbol{R}}}{\Delta + \boldsymbol{R}}\right) - \mathrm{Tr}\left(\frac{\dot{\boldsymbol{R}}_{gh}}{\Delta_{gh} + \boldsymbol{R}_{gh}}\right)$$

$$= \frac{1}{2}\mathrm{Tr}\left[(\mathbf{1}-\mathbf{P})\frac{\dot{R}_k + \eta R_k}{P_k - 2\Lambda + \frac{d^2-3d+4}{d(d-1)}\bar{R}}\right] + \frac{1}{2}\mathrm{Tr}\left[\mathbf{P}\frac{\dot{R}_k + \eta R_k}{P_k - 2\Lambda + \frac{d-4}{d}\bar{R}}\right] - \mathrm{Tr}\left[\frac{\dot{R}_k}{P_k - \frac{1}{d}\bar{R}}\right]$$

$\eta = \frac{1}{Z_N}\frac{dZ_N}{dt}\cdots$ **the anomalous dimension**
$\mathbf{P} \equiv P_{\mu\nu}{}^{\alpha\beta} = \frac{1}{d}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\cdots$ **projector to trace part**

Simplest choice: Litim cutoff $R_k(-\bar{\nabla}^2) = (k^2+\bar{\nabla}^2)\theta(k^2+\bar{\nabla}^2) \Rightarrow P_k(-\bar{\nabla}^2) = -\bar{\nabla}^2 + R(-\bar{\nabla}^2) \sim k^2\theta(k^2+\bar{\nabla}^2)$

$$\dot{\Gamma}_k = \frac{1}{(4\pi)^{d/2}}\int\sqrt{\bar{g}}\,\left[\frac{1}{2}\left(\frac{d(d+1)}{2}-1\right)\left[b_0(\Delta)Q^T_{d/2}(W^T) + b_2(\Delta)Q^T_{d/2-1}(W^T)\right]\right.$$
$$\left. +\frac{1}{2}\left[b_0(\Delta)Q_{d/2}(W) + b_2(\Delta)Q_{d/2-1}(W)\right] - d\left[b_0(\Delta)Q_{d/2}(W_{gh}) + b_2(\Delta)Q_{d/2-1}(W_{gh})\right]\right]$$

$$Q^T_n(W^T) = \frac{1}{\Gamma(n)}\int_0^{k^2}\frac{\dot{P}_k + \eta(P_k - z)}{P_k - 2\Lambda + \frac{d^2-3d+4}{d(d-1)}\bar{R}}z^{n-1}dz,$$

$$Q_n(W) = \frac{1}{\Gamma(n)}\int_0^{k^2}\frac{\dot{P}_k + \eta(P_k - z)}{P_k - 2\Lambda + \frac{d-4}{d}\bar{R}}z^{n-1}dz, \quad Q_n(W_{gh}) = \frac{1}{\Gamma(n)}\int_0^{k^2}\frac{\dot{P}_k}{P_k - \frac{1}{d}\bar{R}}z^{n-1}dz$$

Cmparing the constant part and R **part, we get**

$$\frac{d}{dt}\left(\frac{2\Lambda}{16\pi G}\right) = \frac{k^d}{16\pi}(A_1 + A_2\eta), \quad -\frac{d}{dt}\left(\frac{1}{16\pi G}\right) = \frac{k^{d-2}}{16\pi}(B_1 + B_2\eta) \quad \Rightarrow \beta \text{ functions}$$

See my book or related papers for further details.

Known facts

- **In all studies made so far, nontrivial fixed points are found and asymptotic safety seems to be realized.**
- **We need higher-derivative (curvature) terms with dimensionless couplings.**
- **In 4D, quadratic (higher derivative) theory is renormalizable!**

  [K. S. Stelle, Phys. Rev. D16 (1977) 953.]
- **But it is non-unitary! (on flat backgrounds)**

- **We need all terms with dimensionless couplings at least.**

So let's study the theory

HDG

$$S_{HDG} = \int d^4x\sqrt{-g}\left[\mathcal{V} - Z_N R + \frac{1}{2\lambda}{C_{\mu\nu\rho\lambda}}^2 + \frac{1}{\xi}R^2 - \frac{1}{\rho}E\right],$$

$${C_{\mu\nu\rho\lambda}}^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\mu\nu} + \frac{1}{3}R^2, \quad E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2, \quad Z_N = \frac{1}{16\pi G}, \quad \mathcal{V} = 2\Lambda Z_N.$$

3 Known results on beta functions

To formulate the theory, we need truncation (keep finite no. of operators). (We cannot deal with infinite no. of couplings.)

2.1 Gauss-Bonnet term

Due to its topological nature, it is known that the beta function of ρ must have the form

$$\beta_\rho = -\frac{1}{16\pi^2} a \rho^2 \quad \cdots \quad a \textbf{ does not contain } \rho \textbf{ itself}$$

The UV behavior of ρ

$$\begin{cases} a = 0 \Rightarrow \textbf{any value in the UV} \\ a > 0 \Rightarrow \textbf{run logarithmically to zero from above} \\ a < 0 \Rightarrow \textbf{run logarithmically to zero from below} \end{cases}$$

Holds *independently of the truncation*.

2.3 One-loop beta functions for G and Λ:

J. Julve, M. Tonin, Nuovo Cim. 46B, 137 (1978).

E.S. Fradkin, A.A. Tseytlin, Phys. Lett. 104 B, 377 (1981).

I.G. Avramidi, A.O. Barvinski, Phys. Lett. 159 B, 269 (1985).

A. Codello and R. Percacci, Phys. Rev. Lett. 97 (2006) 22.

M. Niedermaier, Nucl. Phys. B 833 (2010) 226.

N. O. and R. Percacci, Class. Quant. Grav. 31 (2014) 015024 [arXiv:1308.3398]

Beta functions for dimensionless couplings:

$$\beta_\lambda = -\frac{1}{(4\pi)^2}\frac{133}{10}\lambda^2$$

$$\beta_\omega = -\frac{1}{(4\pi)^2}\frac{25+1098\omega+200\omega^2}{60}\lambda$$

$$\beta_\theta = -\frac{1}{(4\pi)^2}\frac{7(56-171\theta)}{90}\lambda$$

where $\omega = -\frac{3\lambda}{\xi}, \quad \theta = \frac{\lambda}{\rho}$**.**

These are closed by themselves: do not depend on $\tilde{G}$ and $\tilde{\Lambda}$.

Gaussian FPs ($\lambda = 0, \xi = 0$) are observed for dimensionless couplings.

The ratios have nontrivial fixed points: $(\omega_*, \theta_*) = (-5.47, 0.33), (-0.023, 0.33)$**.**

Beta functions for Newton and cosmological couplings:

$$\beta_{\tilde{\Lambda}} = -2\tilde{\Lambda} + p(\lambda,\omega)\tilde{\Lambda} - q(\omega)\tilde{G}\tilde{\Lambda} + r(\omega)\tilde{G} + s(\lambda,\omega) + \frac{t(\lambda,\omega)}{\tilde{G}},$$
$$\beta_{\tilde{G}} = 2\tilde{G} - u(\lambda,\omega)\tilde{G} - q(\omega)\tilde{G}^2$$

$$p(\omega) = \frac{1}{(4\pi)^2}\frac{1+86\omega+40\omega^2}{12\omega}\lambda, \quad q(\omega) = \frac{171+298\omega+152\omega^2+16\omega^3}{36\pi(1+\omega)}, \quad s(\lambda,\omega) = -\frac{1}{(4\pi)^2}\frac{1+10\omega}{4\omega}\lambda,$$
$$r(\omega) = \frac{283+664\omega+204\omega^2-128\omega^3-32\omega^4}{144\pi(1+\omega)^2}, \quad t(\lambda,\omega) = \frac{1}{(4\pi)^2}\frac{1+20\omega^2}{256\omega^2}\lambda^2, \quad u(\lambda,\omega) = \frac{1}{(4\pi)^2}\frac{3+26\omega-40\omega^2}{12\omega}\lambda$$

Flow in $\tilde{\Lambda}$–$\tilde{G}$ plane for $(\omega_*, \theta_*) = (-0.023, 0.33)$.
(for the other FP, $\tilde{G}_* < 0$)

$$\beta_{\tilde{\Lambda}} = -2\tilde{\Lambda} + \frac{2\tilde{G}}{\pi} - q_*\tilde{G}\tilde{\Lambda}$$
$$\beta_{\tilde{G}} = 2\tilde{G} - q_*\tilde{G}^2$$

where $q_* = q(\omega_*) \approx 1.440$ **(type III)**

$$\tilde{\Lambda}_* = \frac{r_*}{2q_*} \approx 0.209\ , \qquad \tilde{G}_* = \frac{2}{q_*} \approx 1.346\ .$$

Both are relevant couplings!

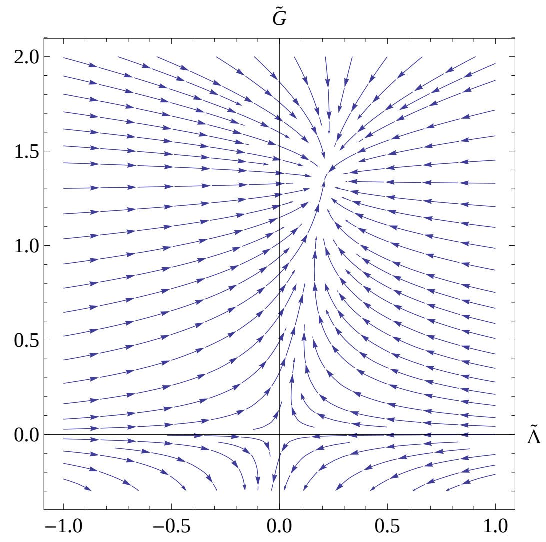


図 2: The flow in the $(\tilde{\Lambda}, \tilde{G})$–plane for $(\tilde{\Lambda}_*, \tilde{G}_*) \approx (0.209, 1.346)$.

What about quadratic curvature terms with dimensionless couplings?

Earlier result found that the dimension of the critical surface is 3 (Λ, R, R^2) within the power series of scalar curvature (up to R^{70} order!), with non-trivial couplings of dimension 0.

K. Falls, D. Litim, K. Nikolakopulos and C. Rahmede, "A bootstrap towards asymptotic safety," Phys. Rev. D 93 (2016) 104022 [arXiv:1410.4815 [hep-th]].

But this result consider only scalar curvature but no other tensors.

Perturbative renormalizability suggests that $R_{\mu\nu}^2$ is also needed.

Namely the number of relevant directions is 4.

$$\Lambda, R, R^2, R_{\mu\nu}^2 \cdots \text{ on dimensional grounds.}$$

A surprise:

D. Benedetti, P. F. Machado and F. Saueressig, Nucl. Phys. B 824 (2010) 168 [arXiv:0902.4630 [hep-th]]

studied the problem on Einstein space, keeping R^2 and ${R_{\mu\nu\rho\lambda}}^2$.

$$R_{\mu\nu} = \Lambda g_{\mu\nu}$$

$\Rightarrow$ Claims that there are only 3 relevant operators, the same as $f(R)$ theory, in contrast to perturbation theory which requires $R_{\mu\nu}^2$.

However, Einstein background is not enough.

The reason is that $R^2, R_{\mu\nu}^2, R_{\mu\nu\rho\lambda}$ are not independent on the Einstein space, and we cannot distinguish these.
$\Rightarrow$ We have to study this problem on general backgrounds.

The question: Does the above result (only 3 relevant operators) persist on general backgrounds?

A point to notice:

The earlier results give only Gaussian fixed points for dimensionless couplings. $\cdots$ asymptotic safety (except $f(R)$ approach.)

To find nontrivial results, we have to keep higher orders in $1/G_N$.

We have studied this problem with quadratic curvatures on general backgrounds.

K. Falls, N. Ohta and R. Percacci, "Towards the determination of the dimension of the critical surface in asymptotically safe gravity," Phys. Lett. B 810 (2020) 135773 [arXiv:2004.04126 [hep-th]].
The technical details of the calculations is given in the arXiv page as ancillary file, and the mathematica file containing full beta functions is https://doi.org/10.5281/zenodo.4017671.

Result

Keeping linear order in $Z_N \equiv 1/G_N$ as well, we have found there are nontrivial fixed points for dimensionless couplings.

3.1 Beta functions without cosmological term

$$
\begin{aligned}
\beta_\lambda &= -\frac{133}{160\pi^2}\lambda^2 + \tilde{Z}_N \lambda^3 \frac{251\xi - 58\lambda}{120\pi^2\xi} \\
\beta_\xi &= -\frac{5(72\lambda^2 - 36\lambda\xi + \xi^2)}{576\pi^2} + \tilde{Z}_N \frac{9720\lambda^3 - 1980\lambda^2\xi + 489\lambda\xi^2 - 14\xi^3}{6480\pi^2} \\
\beta_\rho &= -\frac{49}{180\pi^2}\rho^2 + \tilde{Z}_N \lambda\rho^2 \frac{233\xi - 58\lambda}{240\pi^2\xi} \\
\beta_{\tilde{Z}_N} &= \left(-2 + \frac{(30\lambda - \xi)(4\lambda + \xi)}{192\pi^2\xi}\right)\tilde{Z}_N + \frac{-3168\lambda^2 + 654\lambda\xi + 35\xi^2}{1152\pi^2\xi(6\lambda + \xi)} \\
&\quad - \frac{72\lambda^2 - 84\lambda\xi + 65\xi^2}{192\pi^2(6\lambda + \xi)^2} \log\left(\frac{2}{3} - \frac{2\lambda}{\xi}\right) .
\end{aligned}
$$

Fixed points without cosmological term $(\Lambda = 0)$

	λ_*	ξ_*	ρ_*	ω_*	$\tilde{Z}_{N*}$	$\tilde{G}_*$
FP_1	0	0	0	-0.02286	0.00833	2.388
FP_2	29.26	-220.2	0	0.4040	0.01318	1.509
FP_3	52.61	1672	0	-0.0944	0.00761	2.614

3.2 Fixed points with cosmological term

Beta functions are too complicated to be given here, but FP can be found.

	λ_*	ξ_*	ω_*	$\tilde{Z}_{N*}$	$\tilde{\mathcal{V}}_*$	$\tilde{G}_*$	$\tilde{\Lambda}_*$
FP_1	0	0	-0.02286	0.00833	0.00649	2.388	0.3894
FP_2	24.91	-287	0.2603	0.01635	0.00457	1.217	0.1399
FP_3	28.24	175	-0.4825	0.01499	0.00693	1.327	0.2310
FP_4	0	-312	0	0.009222	0.00609	2.157	0.3303

Scaling exponents

	1	2	3	4	5
FP_1	4	2	0	0	0
FP_2	$2.352 + 1.677i$	$2.352 - 1.677i$	1.767	0	-3.200
FP_3	$2.327 + 1.521i$	$2.327 - 1.521i$	1.237	0	-5.277

The complex eigenvalues mean that the trajectory in the coupling space makes spiral before converging into FP.

We expect that the independent operator ${R_{\mu\nu}}^2$ would play a role in this approach!

The above result is strong evidence that the dimension of critical surface is probably 3.

One possible problem in our results is that we kept only terms linear order in $Z_N = 1/G_N$.

This means that we cannot integrate the FRGE down to $k = 0$ where we expect that the Newton couplings would go to zero.

So we tried to include terms to all order in Z_N, but it was difficult to finish the calculation by hand.

With the assistance of Benjamin Knorr who gave us a program which calculate the beta functions without expanding in Z_N, we tried to find fixed points.

We failed to find nontrivial fixed points after scrutinizing large range of parameter spaces when we use higher-derivative gauge fixing.

However we have found that there are nontrivial fixed points for dimensionless couplings when we use lower(two)-derivative gauge fixing.

We think that this probably means that the fixed points found are fake, so we have not published these results (but thought we may publish them in a forthcoming "Handbook of Quantum Gravity" to be published early next year from Springer).

In the meantime, results with lower-derivative gauge fixing are presented in

S. Sen, C. Wetterich and M. Yamada, "Asymptotic freedom and safety in quantum gravity," JHEP 03 (2022) 130 [arXiv:2111.04696 [hep-th]].

Higher-derivative gauge fixing was not studied in this. (Our lower-derivative gauge fixings and the results are different $\cdots$ Bad sign)

We suspect that the Gaussian points may be the only correct fixed points, but then ... $\Rightarrow$ This needs further study!

Moreover, to really conclude that the dimension of critical surface is 3, we have to look at further higher curvature terms, whether there are really no further relevant terms of higher dimensions.

Y. Kluth and D. F. Litim, "Fixed Points of Quantum Gravity and the Dimensionality of the UV Critical Surface," [arXiv:2008.09181 [hep-th]].

$$\lambda_0 + \lambda_1 R + \lambda_2 R^2_{\mu\nu\rho\sigma} + \lambda_3 R(R^2_{\mu\nu\rho\sigma}) + \lambda_4 (R^2_{\mu\nu\rho\sigma})^2 + \cdots$$

was studied on the sphere, and it is suggested that the cubic term is also relevant.

However we cannot trust this result, because different results may be obtained if we restrict our analysis to sphere!

$$aR^2_{\mu\nu\rho\sigma} + bR^2 \sim (a+6c)R^2_{\mu\nu\rho\sigma} + (b-c)R^2 \cdots \textbf{(ambiguity on the sphere)}$$

$$a = b = 0 \quad \textbf{or} \quad a + 6c = b - c = 0$$

It is also plausible that higher curvature terms with higher mass dimensions are unlikely to be relevant!

It is very difficult to go higher curvatures on general backgrounds.

One of the reasons:
There are various contractions of Riemann tensors and Ricci tensors!
We have to sort out independent combinations.

Perhaps with the aid of computer, we may be able to do so
$\Rightarrow$ Benjamin Knorr

4 Summary and discussions 1

Asymptotic safety may define nonperturbatively renormalizable theory.

Study on general backgrounds shows

- **It is found and confirmed that there are fixed points in all cases studied so far, indicating that asymptotic safety may be realized.**
- **There are 3 relevant operators up to curvature squares.**

  Such theory possibly gives nonperturbatively renormalizable theory.

  – should be confirmed further including higher curvature terms.

Problems to be understood

- **Can we flow from the UV fixed point to the perturbative gravity regime in the low energy?**

  – should be important for phenomenology.
- **What is the dimension of critical surface really?**

  Analysis of higher order terms may be necessary

  $\cdots$ very complicated on general backgrounds

- **Can the unitarity problem be resolved?**

When the curvature square terms exist, ghost states with negative metric appear. We have to get rid of these.

This is an important problem to be understood. Probably the unitarity cannot be recovered in perturbation, and this is why the AS looks for nontrivial FPs. Still they have to be explicitly resolved.

There are several suggestions like:

1) Donoghue: unitarity is OK but micro-causality is slightly violated,

J. F. Donoghue and G. Menezes, "Unitarity, stability and loops of unstable ghosts," Phys. Rev. D 100 (2019) 105006 [arXiv:1908.02416 [hep-th]].

The ghost is unstable and never appear as asymptotic state.

2) Holdom: gravity becomes strong coupling so that the ghosts are confined,

B. Holdom and J. Ren, "QCD analogy for quantum gravity," Phys. Rev. D 93 (2016) 124030 [arXiv:1512.05305 [hep-th]].

3) the threshold that ghost appears recedes in the higher energy and the ghost never appears in the asymptotic states.

- **Issue of essential coupling vs inessential coupling**

The coefficients of the terms proportional to field equation can be changed!

In general, given an action S, we have a field equation

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0$$

Typically this gives Einstein equation proportional to Ricci tensor $R_{\mu\nu}$.

If we get additional terms generated by quantum effects

$$S[g_{\mu\nu}] + \frac{\delta S}{\delta g_{\mu\nu}} \Psi^{\mu\nu} \quad \Rightarrow \quad S[g_{\mu\nu} + \Psi_{\mu\nu}]$$

Namely such term proportional to the field equation may be absorbed into the field redefinition.

This may mean that R^2 and $R_{\mu\nu}^2$ terms may not be necessary for the formulation of AS? $\Rightarrow$ should be studied further.

If this is true, we should look at not only $R_{\mu\nu}^2$ and R^2 but terms like $R_{\mu\nu\rho\lambda} R^{\rho\lambda\alpha\beta} R_{\alpha\beta}{}^{\mu\nu}$ etc. $\cdots$ Goroff-Sagnotti terms.

One caveat:
Eliminating R^2 and $R_{\mu\nu}^2$ by field redefinition changes the spectrum in the theory completely.

The theories with quadratic terms may belong to different universality class?

- **How much do the results depend on matter?**

  There are many works.

- **Some discussions on implications to particle physics, e.g. Higgs mass, fermion mass, but universal results have not been obtained due to gauge ambiguity and so on.**

  Example:
  There is a claim by Robinson and Wilczek

  S. P. Robinson and F. Wilczek, "Gravitational correction to running of gauge couplings," Phys. Rev. Lett. 96 (2006) 231601 [arXiv:hep-th/0509050 [hep-th]].

  that gravitational corrections render all gauge couplings asymptotically free!

  However this comes in as quadratic divergences, which are known to be gauge dependent and not universal. See for example

  A. R. Pietrykowski, "Gauge dependence of gravitational correction to running of gauge couplings," Phys. Rev. Lett. 98 (2007), 061801 [arXiv:hep-th/0606208 [hep-th]].

  Most probably we cannot make definite statements on the gravitational corrections to beta funactions of gauge coupling.

  Debate still continues.

- **Problems with gauge dependence, cutoff scheme dependence and field parametrization dependence:**

eg. parametrization dependence:

It is possible to define the quantum fluctuation as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, but this is not the only way to do so.

We can define other quantum fields as $g_{\mu\nu} = \bar{g}_{\mu\rho} \left(e^h\right)^\rho{}_\nu$ or $g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu}$.

What would be the difference?

At the one-loop level, we need only up to quadratic order of the fluctuation, which we write as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g^{(1)}_{\mu\nu} (\equiv h_{\mu\nu}) + \delta g^{(2)}_{\mu\nu}$$

The contribution to the Hessian (quadratic terms) is

$$\boxed{\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \delta g^{(2)}_{\mu\nu}} + \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial g_{\mu\nu} \partial g_{\rho\sigma}} \delta g^{(1)}_{\mu\nu} \delta g^{(1)}_{\rho\sigma}$$

Both terms do contribute! The second term is common to all parametrization. It is the first term which makes the difference.

However, it is proportional to the field equation, and vanishes on shell!

Time for lunch?

5 Application to black holes

Renormalization group equation for Newton coupling (with $\tilde{G}(k) = k^2 G(k), \Lambda = 0$):

$$k\partial_k \tilde{G}(k) = 2\tilde{G}(k)\frac{1+\omega'\tilde{G}(k)}{1+(\omega+\omega')\tilde{G}(k)}, \quad \Rightarrow \quad \tilde{G}(k)[1+\omega'\tilde{G}(k)]^{\omega/\omega'} = \tilde{G}(k_0)[1+\omega'\tilde{G}(k_0)]^{\omega/\omega'}(k/k_0)^2$$

$$\omega = \tfrac{4}{\pi}(1-\tfrac{\pi^2}{144}) \approx 1.2, \quad \omega' = \tfrac{2}{3\pi}(-7+\tfrac{\pi^2}{24}) \approx -1.4, \quad \omega'/\omega \approx -1.18$$

For simplicity, we set $\omega' \approx -\omega$ to get analytic solution

$$G(k) = \frac{G(k_0)}{1+\omega G(k_0)k^2}$$

k_0 **is a small energy scale, which may be taken $k_0 = 0$.**

This means that the Newton constant tend to vanish at high energy!

The scale identification

How the energy scale is related to the distance scale? — dimensional grounds

$$k = \frac{\xi}{d(P)} \cdots$$ d **is a distance, ξ is a dimensionless number of order 1.**

5.1 Schwarzschild black holes

There are several possibilities:

- **Choose geodesic distance:** **(A. Bonnano and M. Reuter, PRD 62 (2000) 043008.)**

The Schwarzschild black hole

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

with

$$f(r) = 1 - \frac{2G_0 M}{r}$$

The behaviors of the geodesic distance are

$$\begin{aligned} d(r) &\to r, \qquad r \to \infty. \\ d(r) &\to r^{3/2}, \qquad r \to 0 \end{aligned}$$

If we replace G_0 by $G(r)$, then the singularity at the origin $r = 0$ disappears!

$$G(r) \to r^{3/2}, \qquad r \to 0$$

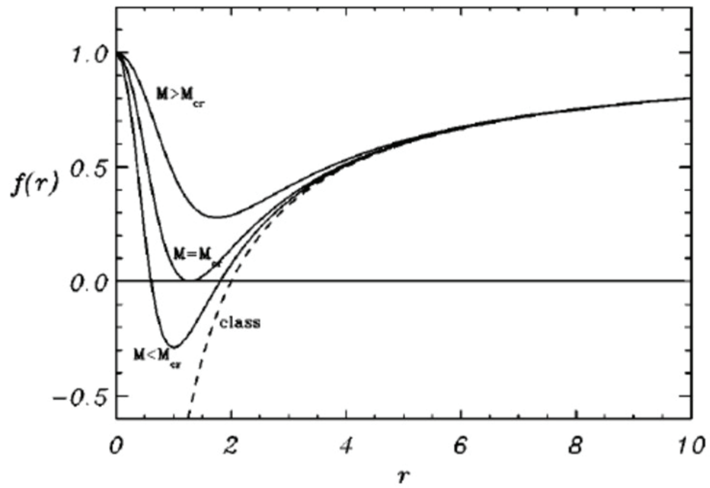
The price is that the structure of "black hole" changes by parameters, and this identification depends on coordinate system.

Bonnano-Reuter chose the interpolating function

$$d(r) = \left(\frac{r^3}{r + \gamma G_0 M} \right)^{1/2}$$

$$\Rightarrow G(r) = \frac{G_0 r^3}{r^3 + \tilde{\omega} G_0 (r + \gamma G_0 M)}$$

$$f(r) \Rightarrow 1 - \frac{2 G_0 M r^2}{r^3 + \tilde{\omega} G_0 (r + \gamma G_0 M)}$$



The dotted line is classical Schwarzschild. If M is large, quantum effects are small. For small M, there are two horizons, which may degenerate for critical mass.

- **Another choice which does not depend on the coordinates:**

(J. M. Pawlowski and D. Stock, PRD 98 (2018) 106008)

$$k = (R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma})^{1/4} \cdots \textbf{ Kretschmann invariant}$$

Short distance behavior is even better

$$G(r) \to r^2 \quad \textbf{for} \quad r \to 0.$$

The singularity at the origin may be resolved, since the gravity becomes weaker there or even zero.

The same problem has been studied for Reisnner-Nordström solution: with Kretschmann identification and found similar conclusion (singlarity may be resolved).

A. Ishibashi, N. Ohta and D. Yamaguchi, "Quantum improved charged black holes," Phys. Rev. D 104 (2021) 066016

So far so good.

For Kerr solution,
both identifications give results depending on angle, and then the horizon position depends on the angle?

This would be strange! ⇒ Then should we keep arbitrary angle?

This leads to several difficulties:

- **The surface gravity is not constant on the horizon ··· black hole may not be thermal equilibrium!**
- **The entropy does not satisfy the 1st law of thermodynamics!**
- **In general, there appears singularity on the horizon.**

None of these may be accepted for physical situation.

Need to specify what is the correct identification of k with distance r.

The problem

There has not been any physical principle to determine the scale identification.
What is physical principle to specify the identification?

We have studied this problem with Chiang-Mei Chen, Yi Chen (NCU student), Ishibashi and Yamaguchi (Kindai student)

C. M. Chen, Y. Chen, A. Ishibashi, N. Ohta and D. Yamaguchi, "Running Newton coupling, scale identification, and black hole thermodynamics," Phys. Rev. D 105 (2022) 106026 [arXiv:2204.09892 [hep-th]].

We propose that we must impose the consistency of the first law of thermodynamics as a physical guiding principle, at least near horizon.

The first law of black hole:

$$dM = TdS + \Omega dJ,$$

Choose the horizon radius r_+ and angular momentum parameter $a(=$

J/M) **as independent variable.**

Mass is determined by the vanishing condition of lapse at the horizon:

$$M(r_+, a) = \frac{r_+^2 + a^2}{2Gr_+}$$

Writing everything as

$$dS = \partial_+ S dr_+ + \partial_a S da, \qquad (\partial_+ \equiv \partial_{r_+})$$

we find

$$\partial_+ S = \frac{\partial_+ M - \Omega \partial_+ J}{T}, \quad \partial_a S = \frac{\partial_a M - \Omega \partial_a J}{T}$$

The consistency condition

$\partial_a(\partial_+ S) = \partial_+(\partial_a S) \quad \Rightarrow \quad$ **This gives nontrivial constraint!**

5.2 Kerr black holes

If we take the Newton coupling depends only on r_+, we get trivial solution $G =$ const.

Adding the dependence on a as well to G, we calculate the temperature from the surface gravity defined by

$$\kappa = \sqrt{-\frac{(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)}{2}}$$

$\chi = \partial_t + \Omega_H \partial_\varphi$ **is chosen to be null ($\chi_\mu^2 = 0$) on the horizon.**

$$T = \frac{\kappa}{2\pi} = \frac{(r_+^2 - a^2)G(r_+, a) - r_+(r_+^2 + a^2)\partial_+ G(r_+, a)}{4\pi r_+(r_+^2 + a^2)G(r_+, a)}$$

κ **is the surface gravity.**

Then the derivatives of S from the first law are

$$\partial_+ S = \frac{2\pi r_+}{G(r_+, a)}, \qquad \partial_a S = \frac{2\pi \left[r_+^2(r_+^2 + a^2)\partial_a G(r_+, a) - a(r_+^2 - a^2)G(r_+, a)\right]}{G(r_+, a)\left[r_+(r_+^2 + a^2)\partial_+ G(r_+, a) - (r_+^2 - a^2)G(r_+, a)\right]}.$$

The consistency condition gives rather complicated equation

$$\partial_+ \left[\frac{r_+^2 (r_+ \partial_a G - a \partial_+ G)}{r_+ (r_+^2 + a^2) \partial_+ G - (r_+^2 - a^2) G} \right] = 0,$$

(This may look simple, but what you really get is much more messy equation, with the derivative acted on the expression.)

This can be integrated once, but we have to integrate it more (difficult)! However it is easy to find simple solution.

That is the Newton constant should be a function of the horizon area $A = 4\pi(r_+^2 + a^2)$!

Substituting this condition back into the derivatives of S

$$\partial_+ S = \frac{2\pi r_+}{G(A)}, \qquad \partial_a S = \frac{2\pi a}{G(A)}.$$

This gives us the quantum entropy

$$\boxed{S = \int \frac{dA}{4G(A)}}$$

Evidence: this gives Bekenstein-Hawking formula for constant G.

We have thus found that this leads to a solution that the scale identification should be made through surface area at fixed radius.

Very interestingly, we find that this is a universal formula valid not only for Kerr solutions but also for various higher-dimensional solutions, not mention Schwarzschild and Reissner-Nordstrom black holes.

Simple dimensional analysis suggests the identification

$$k = \frac{\xi}{\sqrt{A}} = \frac{\tilde{\xi}}{\sqrt{r_+^2 + a^2}}, \qquad (\xi \text{ is a dimensionless constant})$$

The entropy is then

$$S = \frac{\pi(r_+^2 + a^2)}{G_0} + \pi\tilde{\omega}\ln(r_+^2 + a^2).$$

Typical logarithmic corrections to the Bekenstein-Hawking formula (first term)!

If this identification is true for the whole range of r, the singularities of

black holes may not be resolved.

However we may consider something like

$$k = \frac{\xi}{\sqrt{A}} \left(1 + \frac{G_0}{\sqrt{A}} \right)^6$$

which diverges faster in the limit of $A \to 0$, and such modification may resolve singularity. (This does not work for Kerr black hole because A never vanishes, but OK for Schwarzschild.)

We have determined that the identification should be made by horizon area (small progress), but do not have a criterion which form is correct.

Moreover, strictly speaking, the identification is valid only near the horizon. Away from the horizon, we may need different identification.

In particular, there is some evidence that our identification should be modified (high energy behavior).

Then there remains the important question of what is the physical principle for the identification away from the horizon.

5.3 Five-dimensional Myers-Perry black holes

Mertic

$$ds^2 = -dt^2 + \frac{Gmr^2}{\Pi F}(dt - a_1 \sin^2\theta' d\phi_1 - a_2 \cos^2\theta' d\phi_2)^2 + \frac{\Pi F}{\Pi - Gmr^2} dr^2$$
$$+ (r^2 + a_1^2)(\cos^2\theta' d\theta'^2 + \sin^2\theta' d\phi_1^2) + (r^2 + a_2^2)(\sin^2\theta' d\theta'^2 + \cos^2\theta' d\phi_2^2),$$

with

$$\Pi = (r^2 + a_1^2)(r^2 + a_2^2), \qquad F = 1 - \frac{a_1^2 \sin^2\theta'}{r^2 + a_1^2} - \frac{a_2^2 \cos^2\theta'}{r^2 + a_2^2}.$$

Horizon

$$\Pi(r_+) - Gmr_+^2 = 0$$

Calculate the temperature from the surface gravity

$$T = \frac{2(r_+^4 - a_1^2 a_2^2)G(r_+, a_1, a_2) - r_+(r_+^2 + a_1^2)(r_+^2 + a_2^2)\partial_+ G(r_+, a_1, a_2)}{4\pi r_+(r_+^2 + a_1^2)(r_+^2 + a_2^2)G(r_+, a_1, a_2)}.$$

First law gives

$$\partial_+ S = \frac{\pi^2(3r_+^4 + a_1^2 r_+^2 + a_2^2 r_+^2 - a_1^2 a_2^2)}{2r_+^2 G(r_+, a)},$$
$$\partial_{a_1} S = \frac{\pi^2(r_+^2 + a_2^2)\left[4a_1(r_+^4 - a_1^2 a_2^2)G(r_+, a_1, a_2) - (r_+^2 + a_1^2)(3r_+^4 + a_1^2 r_+^2 + a_2^2 r_+^2 - a_1^2 a_2^2)\partial_{a_1} G(r_+, a_1, a_2)\right]}{2r_+ G(r_+, a_1, a_2)\left[2(r_+^4 - a_1^2 a_2^2)G(r_+, a_1, a_2) - r_+(r_+^2 + a_1^2)(r_+^2 + a_2^2)\partial_+ G(r_+, a_1, a_2)\right]},$$
$$\partial_{a_2} S = \frac{\pi^2(r_+^2 + a_1^2)\left[4a_2(r_+^4 - a_1^2 a_2^2)G(r_+, a_1, a_2) - (r_+^2 + a_2^2)(3r_+^4 + a_1^2 r_+^2 + a_2^2 r_+^2 - a_1^2 a_2^2)\partial_{a_2} G(r_+, a_1, a_2)\right]}{2r_+ G(r_+, a_1, a_2)\left[2(r_+^4 - a_1^2 a_2^2)G(r_+, a_1, a_2) - r_+(r_+^2 + a_1^2)(r_+^2 + a_2^2)\partial_+ G(r_+, a_1, a_2)\right]}.$$

Consistency

$$\partial_{a_1}\partial_+ S = \partial_+\partial_{a_1} S, \qquad \textbf{etc. 3 conditions}$$

It turns out that if we take the Newton coupling to be a function of the horizon area

$$A \equiv 2\pi^2 \frac{(r_+^2 + a_1^2)(r_+^2 + a_2^2)}{r_+}.$$

all the consistencies are satisfied!

We find

$$\partial_{a_1} S = \frac{\pi^2 a_1 (r_+^2 + a_2^2)}{r_+ G(r_+, a_1, a_2)}, \qquad \partial_{a_2} S = \frac{\pi^2 a_2 (r_+^2 + a_1^2)}{r_+ G(r_+, a_1, a_2)},$$

This again leads to the universal formula

$$\boxed{S = \int \frac{dA}{4G(A)}}$$

5.4 Kaluza-Klein Black Strings

We have also studied Kaluza-Klein black strings in $d = n + 1$ dimensions

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n-2}^2 + dz^2,$$
$$f(r) = 1 - \frac{G\mu}{r^{n-3}},$$

Imposing the consistency of the first law, we again find precisely the universal formula!

$$\boxed{S = \int \frac{dA}{4G(A)}}\,, \qquad A \equiv \Omega_{n-2}\, L\, r_+^{n-2}$$

6 Summary and discussions 2

Asymptotic safety may define nonperturbatively renormalizable theory.

The approach gives scale-dependent Newton coupling, and may resolve singularity at the center of black holes.

- **The consistency of the thermodynamics gives a physical principle to determine the dependence**
- **The Newton coupling should be a function of the horizon area.**
- **We find a universal formula**

$$S = \int \frac{dA}{4G(A)}$$

This formula is valid not only for all the four-dimensional black hole solutions but also for higher-dimensional black holes!

- **The singularities of the Schwarzschild or Reissner-Nordström black holes may be resolved for suitable choice of the function, but not for Kerr black holes $\cdots$ problem to be understood.**

Problems to be understood

- **Black hole physics**

  The above identification is only near the horizon, and possibly away to infinity.

  But we do not have physical principle to determine what functional form we should take even on the horizon, not to mention away!

  Can we find a physical principle to make scale identification away from the horizon other than compatibility with thermodynamics 1st law?

- **The black hole singularities**

  Can they be resolved? $\cdots$ to some extent

- **Cosmology**

  There are several works on asymptotic safe cosmology. But it is not clear what role the energy cutoff k play, or to what we can identify it?

  There is suggestion that it may be identified to Hubble parameter, but what requires it and what functional form are not known.

- **What about other situation?**

  Can we understand some quantum gravity "results" in the context of AS?

- **What happens to Hawking radiation?**

- **Island idea in Hawking radiation**

- **Breakdown of global symmetry?**

- **Weak gravity conjecture?**