Towards S-matrix theory of unstable particles

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KA PRD107 (2023) 4, 044045, KA and Yu-tin Huang, in prep.

The S-matrix programme

- **Challenges in the 60s and 70s.**

  How do we understand/compute strong interactions?

  The Standard Model was NOT a standard model yet...
  Hard (or impossible at that time?) to compute strongly-coupled systems...

- **The S-matrix bootstrap.**

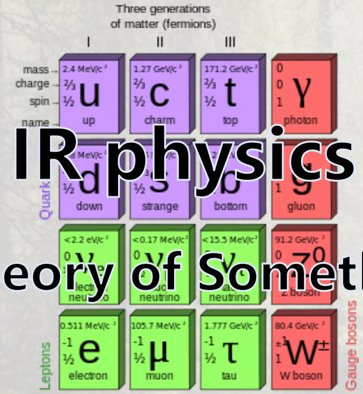
  Let's directly compute ("bootstrap") S-matrix from fundamental properties!
  The main ingredients are **Unitarity** & **Causality** ($\simeq$ analyticity).
  $\rightarrow$ led to Regge theory, string, CFT bootstrap, Cosmological bootstrap...

- **Why now?**

  How do we understand/compute quantum gravity (QG)?

  The string theory or any other candidates are NOT a standard model yet...
  Hard to compute low-energy predictions from QG candidates...

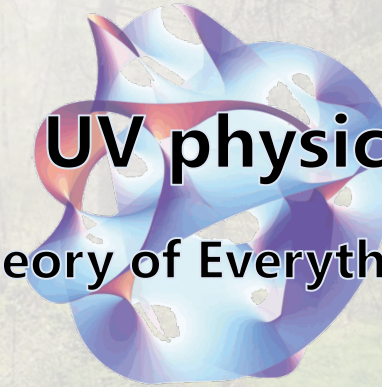
Consistency between IR and UV



IR physics

Theory of Something

↔



UV physics

Theory of Everything

- ❑ The underlying idea of Effective Field Theory (EFT):
  IR physics must be insensitive to UV, **but, not totally independent!**
- ❑ There are general consistency relations b/w IR and UV.
- ❑ The consistency relations may be used to
  - ✓ make predictions on IR physics,
  - ✓ or, extract information about the UV physics.

Cf. Swampland programme
Vafa, 2005.

Consistency between IR and UV

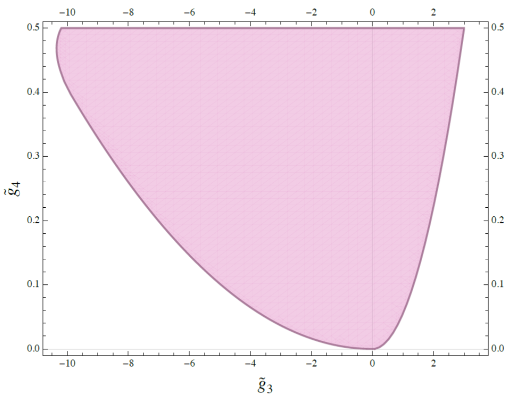
- **Scattering amplitudes do a nice job. Positivity bounds!**
  EFT constraints coming from unitarity and causality. A. Adams et al. 2006 and many.

- The bounds are well established and almost optimized
  for 2-to-2 scattering of the lightest state in a gapped system.
  B. Bellazzini+ 2020; A. J. Tolley+ 2020; S. Caron-Huot+ 2020; A. Sinha+ 2020; N. Arkani-Hamed+ 2020; L.-Y. Chiang+ 2021; ...

$$\mathcal{L}_{\rm EFT} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 + g_2(\partial\phi)^4 + g_3(\partial\phi)^2(\partial^2\phi)^2 + g_4(\partial^2\phi)^4 + \cdots$$

$$\Rightarrow 0 < g_2\,, \quad -\mathcal{O}(1) < \tilde{g}_3 \simeq g_3\frac{\Lambda_{\rm cutoff}^2}{g_2} < \mathcal{O}(1)\,, \cdots$$

There are upper bound on g_i themselves L.-Y. Chiang+ 2022.



From S. Caron-Hout&V. Van Duong, 2020.

1) Superluminal propagation is prohibited.
e.g. A. Adams et al. 2006.

2) The dimensional analysis is a theorem.
e.g. B. Bellazzini+ 2020; A. J. Tolley+ 2020; S. Caron-Huot+ 2020; A. Sinha+ 2020; N. Arkani-Hamed+ 2020; L.-Y. Chiang+ 2021.

3) Massive higher spin cannot be isolated.
e.g. B. Bellazzini+ 2023.

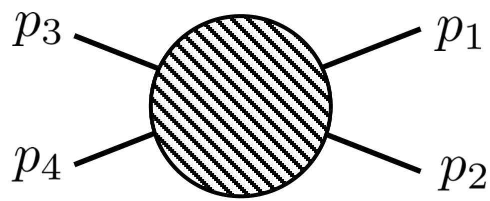
4) Possible proof of weak gravity conjecture.
e.g. C. Cheung+ 2014; S. Andriolo+ 2018; Y. Hamada+ 2019; B. Bellazzini+ 2019; KA+ 2021.

...

What's next?

- ❑ The bounds are well established
  for 2-to-2 scattering of the lightest state in a gapped system.

$$\mathcal{M}(s,t) =$$



$$s = -(p_1 + p_2)^2$$
$$t = -(p_1 - p_3)^2$$
$$u = -(p_1 - p_4)^2$$

Nice properties (analytic structure, high energy behaviour) are known.
Unitarity gives a simple positivity constraint $\mathrm{Im}\mathcal{M}|_{t=0} > 0$.

- ❑ **However, our world is more complicated!**

✓ There are massless particles (photon, graviton)
→ Graviton may give non-trivial constraints a la swampland.

See Sota's next talk!

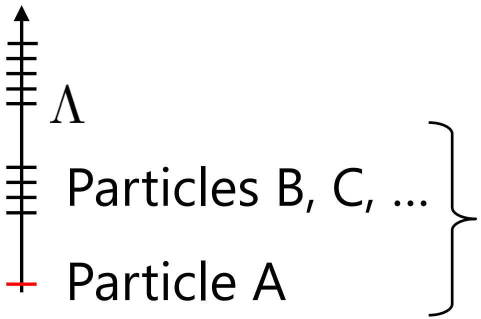
✓ There are many massive particles.

✓ We are living in curved spacetime. …

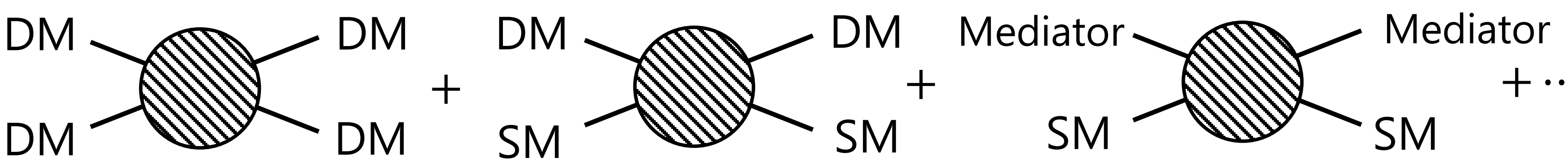
Towards "global" S-matrix bootstrap

- The bounds are well established for 2-to-2 scattering of ~~the lightest state~~ in the gapped system.

→ **What are the bounds arising from all possible 2-to-2 scatterings?**



Below the cutoff, we can study
$AA \to AA, AB \to AB, BB \to BB, BC \to BC, \ldots$



Of course, this is not a new idea but must be crucial,
e.g. Higgs boson was predicted by the "global bootstrap"!

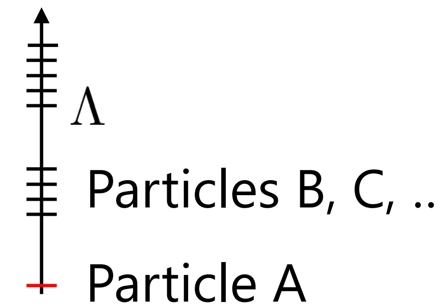
Four-fermi interaction → W boson scattering→ Higgs is required.

Why care about unstable particles?

- ❑ Typically, particles B, C, … are unstable.

  Stable or unstable does not matter if long-lived?
  There are no distinctions at tree level.

  → However, non-trivial constraints arise at loop level!



- ❑ Global S-matrix bootstrap for quantum gravity?

  (Perturbative) UV completion of gravity requires higher-spin particles.

$$\mathcal{M} = \frac{P(s,t)}{stu} \to -P(s,t)\frac{\Gamma(-\alpha' s/4)\Gamma(-\alpha' t/4)\Gamma(-\alpha' u/4)}{\Gamma(1+\alpha' s/4)\Gamma(1+\alpha' t/4)\Gamma(1+\alpha' u/4)}$$

$P \propto (s^2u^2 + t^2u^2 + s^2t^2)$ for scalar scattering

Type-II superstring amplitude

There are an infinite # of possible "UV complete" amplitudes at tree level.

e.g. Y.-t. Huang and G. N. Remmen, 2022.

Infinite # of quantum gravity or Infinite # of hidden constraints?

S-matrix theory of unstable particles

- **We need a general theory of unstable particles!**

  There are many possible applications.
  In any case, most of the particles in nature are unstable!!

  It seems general knowledge of unstable particles is completely missing…
  Why difficult? Unstable particles do not appear in the asymptotic states.

  Veltman 1963.

- **No S-matrix, no constraints?**

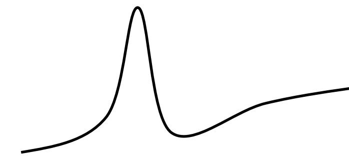
  Then, what is the S-matrix of unstable particles?
  If exists, what is the consequence of unitarity? …

  What does "the W scattering is unitary" mean?

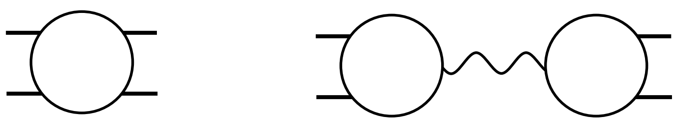
  Let's understand the general properties of unstable particles under (i) Lorentz inv. (ii) Unitarity, (iii) Analyticity ($\simeq$ Causality).

Unstable particle

- How do unstable particles appear?
  Its existence is only seen as resonance in physical process.
  → Complex pole! Unitarity ⇒ The residue is factorized.



$$\mathcal{A}_{\varphi\varphi\to\varphi\varphi} \sim \mathcal{A}_{\mathcal{A}\to\varphi\varphi}\frac{1}{s-M^2}\mathcal{A}_{\varphi\varphi\to\mathcal{A}}$$



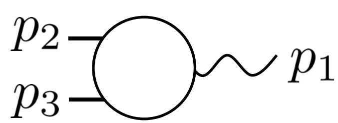
$$M^2 \in \mathbb{C}$$

$$M^2 = M_R^2 - iM_R\Gamma$$

(physical) mass   width

The amplitude for $\mathcal{A} \to \varphi\varphi$ can be defined by the residue.

See e.g. *The analytic S-matrix*, R. J. Eden et al, 1966

$\mathcal{A}_{\mathcal{A}\to\varphi\varphi} =$ 

Solid line: stable particle φ. Wavy line: unstable particle A.

The "on-shell" conditions are understood as

$$p_2^2 = p_3^2 = -\mu^2\,, \quad p_1^2 = -M^2 \in \mathbb{C}$$

p_1 is decaying mode.

Analyticity

$$\langle \text{out}|S|\text{in}\rangle \to \mathcal{A}^{(+)},\ \langle \text{out}|S^\dagger|\text{in}\rangle \to \mathcal{A}^{(-)}$$

The physical amplitude and its hermitian conjugate are real boundary values of the same analytic function whose singularities are inferred from unitarity. In particular, the unitarity equations are supposed to be the sums of discontinuities across individual thresholds.

$$\mathcal{A}^{(\pm)}_{n'n}(s_A) = \lim_{\varepsilon\to 0+} \mathcal{A}_{n'n}(s_A \pm i\varepsilon)$$

Analytic continuation of the physical amplitude

Analyticity

$$\langle \mathrm{out}|S|\mathrm{in}\rangle \to \mathcal{A}^{(+)},\ \ \langle \mathrm{out}|S^\dagger|\mathrm{in}\rangle \to \mathcal{A}^{(-)}$$

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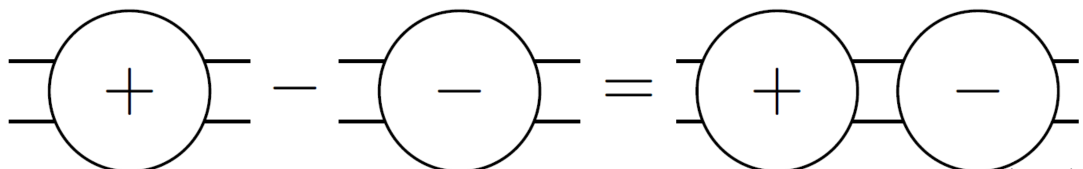
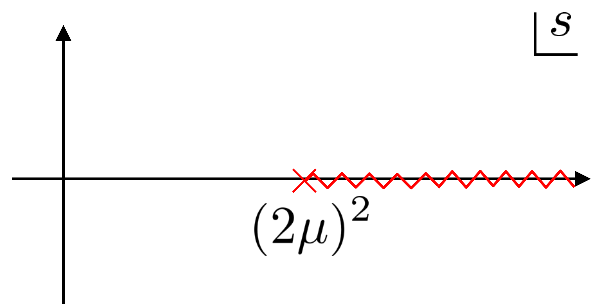
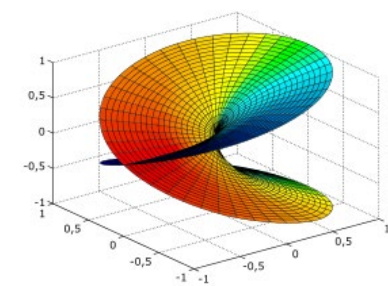
$SS^\dagger = 1 \Rightarrow$  $,$

$(2\mu)^2 < s < (3\mu)^2$

$$\mathrm{l.h.s.} = \mathcal{A}(s+i\varepsilon,t) - \mathcal{A}(s-i\varepsilon,t) = \mathrm{Disc}\mathcal{A}(s,t)$$





Analyticity

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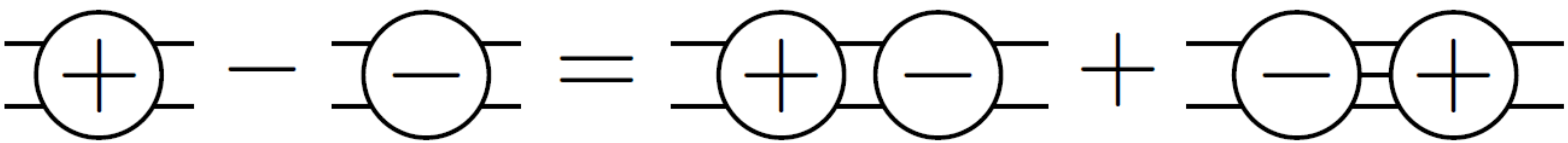
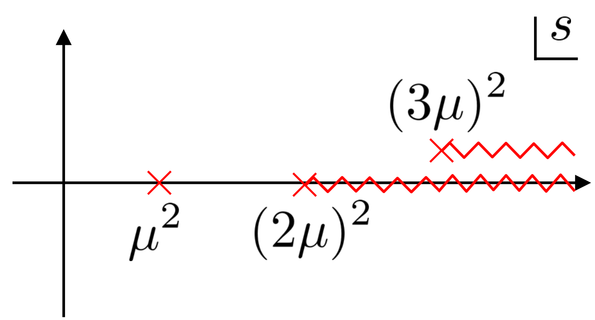
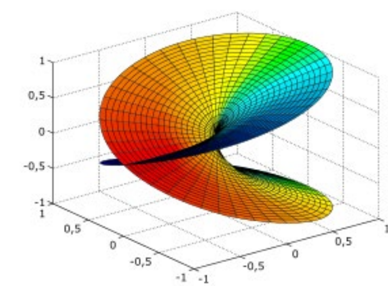


$$(2\mu)^2 < s < (4\mu)^2$$

$$\text{l.h.s.} = \mathcal{A}(s+i\varepsilon,t) - \mathcal{A}(s-i\varepsilon,t) = \text{Disc}\mathcal{A}(s,t)$$





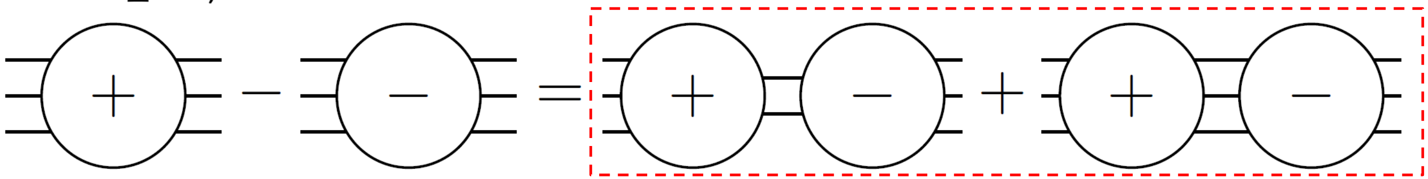
Analyticity

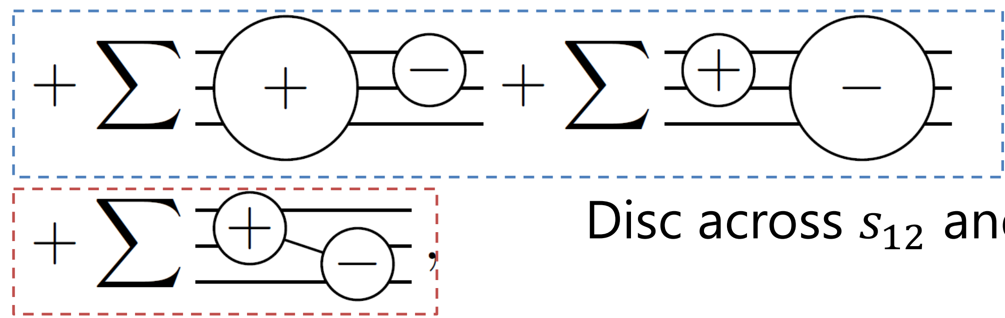
$$\langle \text{out}|S|\text{in}\rangle \to \mathcal{A}^{(+)}, \ \langle \text{out}|S^\dagger|\text{in}\rangle \to \mathcal{A}^{(-)}$$

The physical amplitude and its hermitian conjugate are real boundary values of the same analytic function whose singularities are inferred from unitarity. In particular, the unitarity equations are supposed to be the sums of discontinuities across individual thresholds.

$$SS^\dagger = 1 \Rightarrow$$

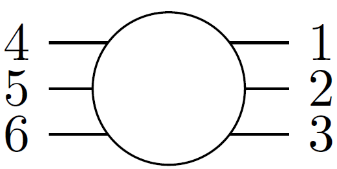
Disc across $s = s_{123}$





Disc across s_{12} and so on

Disc across s_{145} and so on



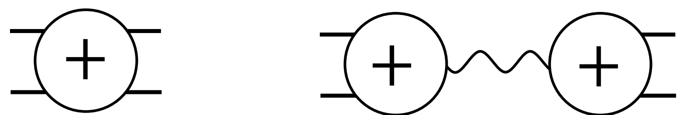
$$s_{ijk\cdots} := -(\pm p_i \pm p_j \pm p_k \pm \cdots)^2$$

+ for in momenta and – for out momenta.

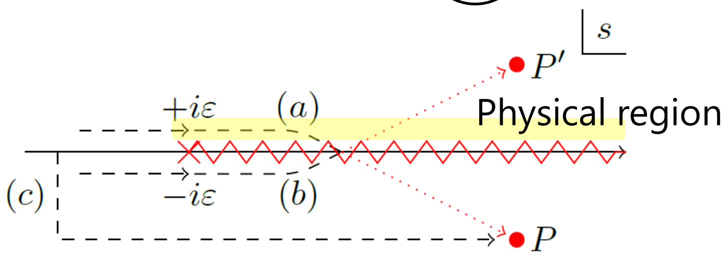
Decaying mode and growing mode

- ❑ Analyticity ⇒ There is also a growing solution (complex conjugate).

$$\mathcal{A}_{\varphi\varphi\to\varphi\varphi} \sim \mathcal{A}_{\mathcal{A}\to\varphi\varphi}\frac{1}{s-M^2}\mathcal{A}_{\varphi\varphi\to\mathcal{A}} \quad \text{at P}$$



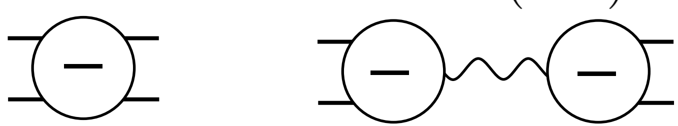
Decaying mode



Continued from $+i\varepsilon$

Continued from $-i\varepsilon$

$$\mathcal{A}_{\varphi\varphi\to\varphi\varphi} \sim \mathcal{A}_{\mathcal{A}\to\varphi\varphi}\frac{1}{s-(M^2)^*}\mathcal{A}_{\varphi\varphi\to\mathcal{A}} \quad \text{at P}'$$



Growing mode

Decaying ($+i\varepsilon$) and growing ($-i\varepsilon$) are denoted by $+$ and $-$.

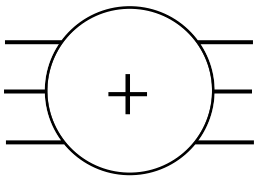
The growing mode is not a physical particle but its existence is crucial.

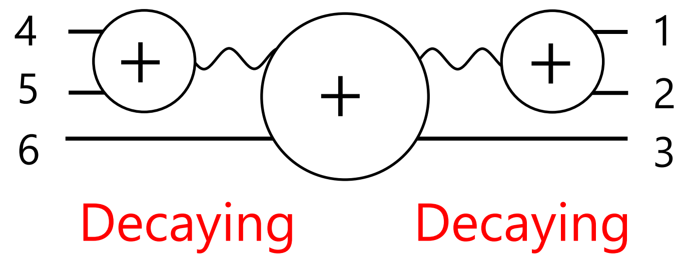
Unstable-particle amplitudes

- The 2-to-2 amplitudes are similarly defined.

$$s_{ij} = -(p_i + p_j)^2$$

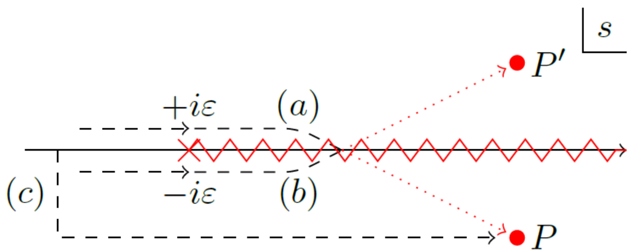
$$\mathcal{A}_{\varphi\varphi\varphi\to\varphi\varphi\varphi} \sim \mathcal{A}_{\mathcal{A}\to\varphi\varphi} \frac{1}{s_{45} - M^2} \mathcal{A}_{\varphi\mathcal{A}\to\varphi\mathcal{A}} \frac{1}{s_{12} - M^2} \mathcal{A}_{\varphi\varphi\to\mathcal{A}}$$

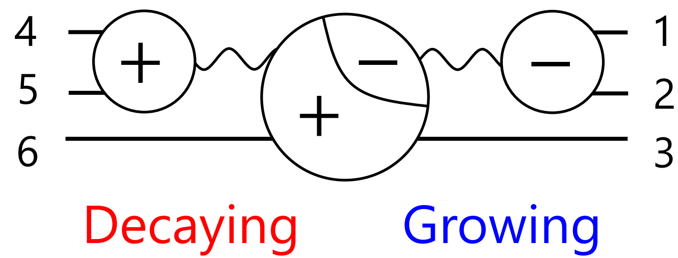




or

$$\sim \mathcal{A}_{\mathcal{A}\to\varphi\varphi} \frac{1}{s_{45} - M^2} \mathcal{A}_{\varphi\mathcal{A}\to\varphi\mathcal{A}} \frac{1}{s_{12} - (M^2)^*} \mathcal{A}_{\varphi\varphi\to\mathcal{A}}$$





Unstable-particle amplitudes are defined by residues of higher-pt amps.

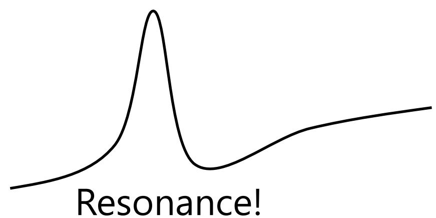
Unitarity of unstable particles

Unstable-particle amplitudes are defined by residues of higher-pt amps.
⇒ Unitarity constraints arise from unitarity of higher-pt amps.

KA 2212.07659, KA and Yu-tin Huang, in prep.

Unitarity of 3-to-3 amp ⇒ Unitarity of 2-to-2 with two unstable legs.

❑ Some technical comments:

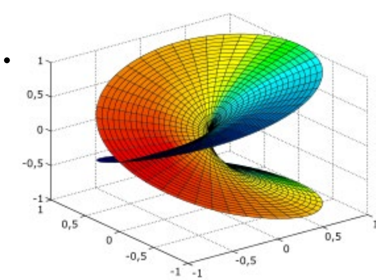


1. We assume that complex poles of unstable particles are the singularities "closest" to the physical region.

   Precisely, the momentum integrals of unitarity eq. need not be deformed.

2. We should be careful about the multiplicity of amplitudes.

   Unitarity & analyticity are used to make sure/move the positions on Riemann sheets.



Optical theorem for unstable particle

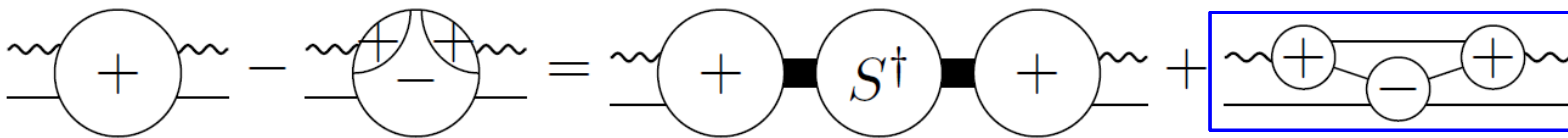
KA 2212.07659, KA and Yu-tin Huang, in prep.

Different in/out states give different optical theorems.

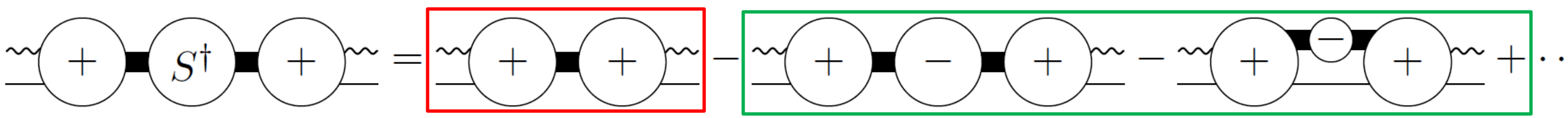
(no difference at tree level as they should be)

Decaying-decaying:

t-channel triangle cut



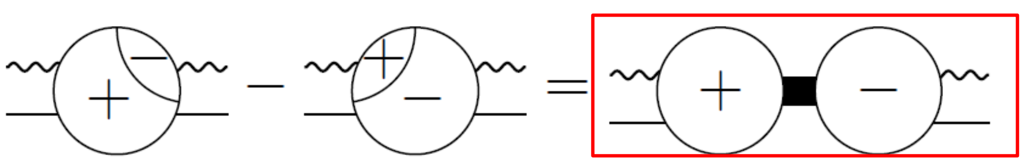
$\mathrm{Disc}\mathcal{M}^{++}(\neq \mathrm{Im}\mathcal{M}^{++})$



Similar to standard one

Additional contribution to s-channel cut

Decaying-growing:



$\mathrm{Disc}\mathcal{M}^{+-}(= \mathrm{Im}\mathcal{M}^{+-})$

Similar to standard one!

Positivity?

KA 2212.07659, KA and Yu-tin Huang, in prep.

For the decaying-growing case, we can show

$$\mathrm{Im}\mathcal{M}^{+-} > 0 \qquad \text{at a finite positive } t.$$

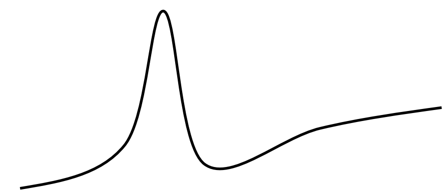
This positivity is one of the bases for EFT constraints.

$$\mathcal{L}_{\mathrm{EFT}} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + g_2(\partial\phi)^4 + g_3(\partial\phi)^2(\partial^2\phi)^2 + g_4(\partial^2\phi)^4 + \cdots \Rightarrow g_2 > 0$$

A. Adams et al. 2006 and many.

We are currently trying to understand the analytic structure more.

1. We assume that complex poles of unstable particles are the singularities "closest" to the physical region.



Resonance or Anomalous?

This assumption might be violated at least in a certain situation.

⇒ We found a particular counter-example of positivity bounds.

The positivity bounds might not be directly applied to unstable particles.

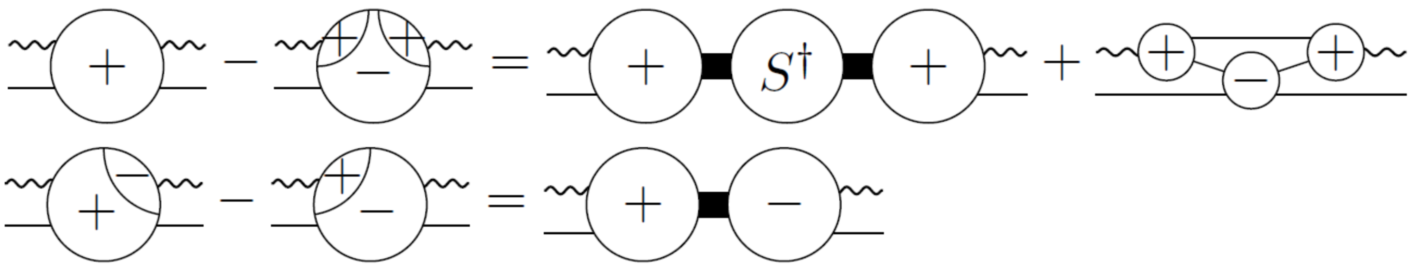
Challenges for the S-matrix programme in our world? e.g. SMEFT?

Summary and Discussions

- **Let's understand the general properties of unstable particles!**

  There are many possible applications, or it is interesting just to understand the proper meaning of "the W scattering is unitary"!

- **Unstable-particle amplitudes and their unitarity constraints are obtained from higher-point stable-particle amplitudes.**



- **There are still elephants in the room!**

  There are "new" singularities that may spoil the positivity bounds.
  We need a better understanding of "new" singularities.
  Challenges for the S-matrix programme in our world?