Curved Shamir Type Domain-wall Fermion

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arXiv:2203.03782, arXiv:2212.11583,
arXiv:2304.13954 (SA, H. Fukaya, N. Kan, M. Koshino and Y. Matsuki)

⟶ Kan's Talk on Wed, August 9th

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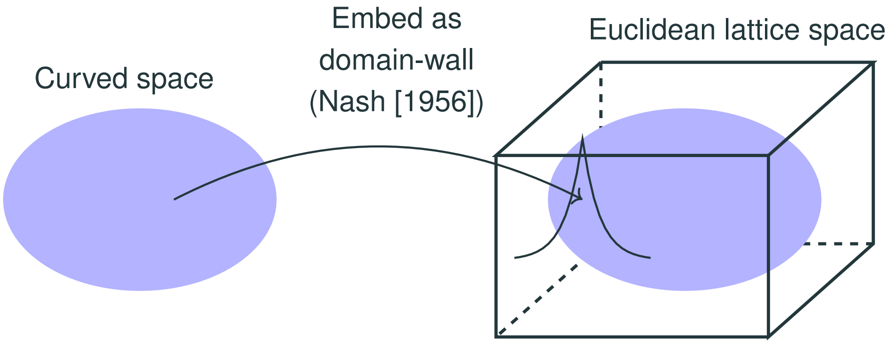
Motivation

Every curved manifold can be isometrically embedded into some higher-dimensional Euclidean spaces.



Localize the edge modes of the curved domain-wall fermion.
= they feel "gravity" by the equivalence principle.

Embedding a curved space

For any n-dim. Riemann space (Y, g), there is an embedding $f : Y \to \mathbb{R}^m$ $(m \gg n)$ such that Y is identified as

$$x^\mu = x^\mu(y^1, \cdots, y^n) \ (\mu = 1, \cdots, m)$$

$$\left(\begin{array}{ll} x^\mu & : \text{Cartesian coordinates of } \mathbb{R}^m \\ y^i & : \text{coordinates of } Y \end{array} \right.$$

and the metric is written as

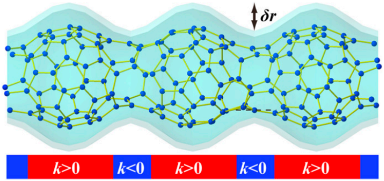
$$g_{ij} = \sum_{\mu\nu} \delta_{\mu\nu} \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j}.$$

$\longrightarrow$ vielbein and spin connection are also induced!

Any Riemannian manifold can be identified as a submanifold of a flat Euclidean space!

Cf. Nash [1956].

"Gravity" in Condensed Matter Physics

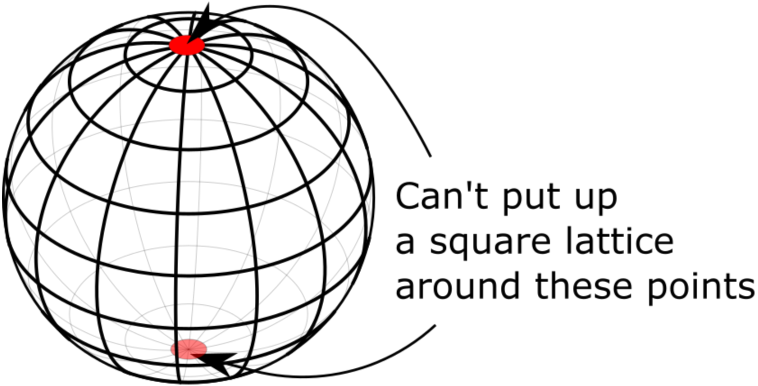


J. Onoe et al. observed a gravitational effect on $1D$ uneven peanut-shaped C_{60} polymer.

$$H = -\frac{\hbar^2}{2m_*}\left[\frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right) + h^2 - k\right], \quad \begin{cases} h\text{: mean curvature} \\ k\text{: Gaussian curvature} \end{cases}$$

$\longrightarrow$ Density of states depends on the curvatures.

Lattice gauge theory on a curved space



We can not approximate d-dim space with d-dim square lattice.

$\longrightarrow$ Can't handle gravity!

Triangular Lattice

Arbitrary space can be discretezed by a triangular lattice. Their length and angle represent the gravity. [Regge [1961]; Ambjørn et al. [2001]; Brower et al. [2017]]

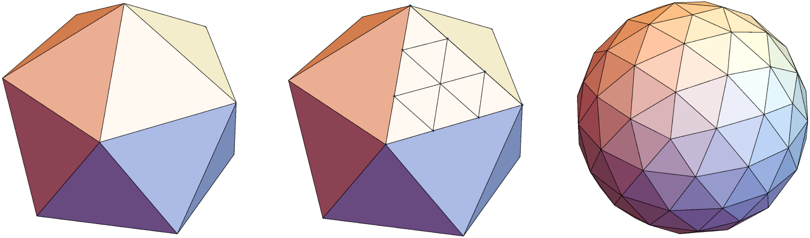


Fig 1: Triangular lattice on 2-dim sphere[Brower et al. [2017]]

However, continuum limit is not unique and symmetry restoration is non-trivial.

Dirac operator With Curved DW

We consider a FREE Dirac operator

$$D = \sum_{i=1}^{2n+1} \gamma^i \frac{\partial}{\partial x^i} + m\text{sign}(f) = \not{D} + m\text{sign}(f)$$

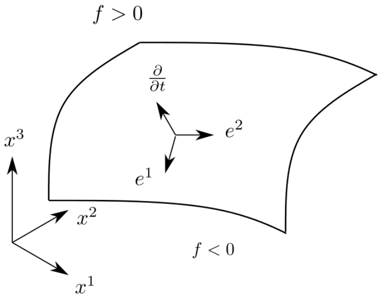
$$\{\gamma^a, \gamma^b\} = 2\delta^{a,b},\ (a, b = 1, \cdots, 2n+1)$$

where the smooth function $f : \mathbb{R}^{2n+1} \to \mathbb{R}$. The edge modes are

- localized at the domain-wall $Y = \{f = 0\}$,
- the chiral eigenstate of $\gamma_{\text{normal}} = \boldsymbol{n} \cdot \boldsymbol{\gamma}$,
- and feel gravity throguth the spin connection on Y.

Chiral Fermion

$$D = \sum_{i=1}^{2n+1} \gamma^i \frac{\partial}{\partial x^i} + m\text{sign}(f)$$
$$\simeq \gamma^{2n+1}\left(\frac{\partial}{\partial t} + F\right) + m\text{sign}(f)$$
$$+ \underbrace{\gamma^a\left(e_a + \frac{1}{4}\sum_{bc}\omega_{bc,a}\gamma^b\gamma^c\right)}_{\not{D}^Y}$$



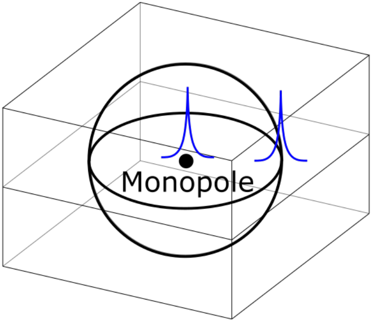
In the large m limit, $D \to \not{D}^Y_+ = \not{D}^Y \frac{1}{2}(1 + \gamma^{2n+1})$.

Similarly, $D^\dagger \to \not{D}^Y_- = \not{D}^Y \frac{1}{2}(1 - \gamma^{2n+1})$.

$\longrightarrow$ Is it possible to formulate a Chiral fermion on the wall?

We analyze a spectrum of $D^\dagger D$ and $DD^\dagger$.

Anomaly inflow on S^2 domain-wall System



The center localized mode cancels chiral anomaly on the edge.

This mode becomes an obstacle to constructing a chiral gauge theory on the square lattice.

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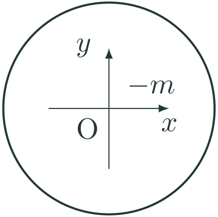
Shamir type S^2 domain-wall

We consider a Dirac operator on B^3.

The boundary is S^2 with the radius r_0

Shamir Type Domain-wall



Dirac operator:

$$D = \sum_{i=1}^{3} \sigma_i \frac{\partial}{\partial x^i} - m = \sigma_r \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) - m - \sigma_r \frac{D^{S^2}}{r},$$

$$D^{S^2} = \sum_{i=1}^{3} \sigma^i L_i + 1, \ \left(L_i = -i\epsilon_{ijk} x_j \frac{\partial}{\partial x^k} \right)$$

$$\sigma_r = \frac{x}{r}\sigma_1 + \frac{y}{r}\sigma_2 + \frac{z}{r}\sigma_3$$

Effective Dirac operator on S^2

A spin rotation

$$R = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) & -e^{-i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) & e^{i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} e^{i\frac{\phi}{2}}$$

changes $\chi \to R^{-1}\chi$ and

$$D^{S^2} \to i\left(\sigma_1 \frac{\partial}{\partial\theta} + \frac{\sigma_2}{\sin\theta}\left(\frac{\partial}{\partial\phi} + \underbrace{\frac{i}{2} - \frac{\cos\theta}{2}\sigma_1\sigma_2}\right)\right),$$

Spin^c connection on S^2

$$\sigma_r \to \sigma_3$$

Edge states feel gravity through the induced connection!
[Takane and Imura [2013]].

Boundary Condition

At least, physical states should satisfy

$$\int_{B^3} dx^3 \phi^\dagger D^\dagger D\psi = \int_{B^3} dx^3 (D\phi)^\dagger D\psi.$$

The boundary term should be zero.

$$\longrightarrow \int_{S^2} d\Omega^2 \psi^\dagger \sigma_r D\psi(r_0) = -\int_{S^2} d\Omega^2 \psi^\dagger \frac{D^{S^2}}{r_0} \psi(r_0) = 0$$

Since $\left\{ D^{S^2}, \sigma_r \right\} = 0$, $\psi(r_0)$ is an eigenstate of σ_r.

Eigenstate of $D^\dagger D$ and $DD^\dagger$

Let $\chi_\pm$ satisfy

$$D^{S^2}\chi_\pm = \lambda\chi_\mp,\ (\lambda = 1, 2, \cdots)$$
$$\sigma_r\chi_\pm = \pm\chi_\pm.$$

In the large m limit, we assume $\psi_\pm = \frac{1}{r}e^{-m|r-r_0|}\chi_\pm$, then we get

$$D\psi_+ = \left(\sigma_r\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) - m - \sigma_r\frac{D^{S^2}}{r}\right)\psi_+ \simeq \frac{\lambda}{r_0}\psi_-.$$
$$D^\dagger\psi_- = \left(-\sigma_r\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) + -m + \sigma_r\frac{D^{S^2}}{r}\right)\psi_- \simeq \frac{\lambda}{r_0}\psi_+.$$

$\longrightarrow D^\dagger D\psi_+ = \left(\frac{\lambda}{r_0}\right)^2\psi_+$ and $DD^\dagger\psi_- = \left(\frac{\lambda}{r_0}\right)^2\psi_-$

Chiral modes appear at the boundary

S^2 Domain-wall Fermion

We consider a lattice on B^3 with the radius r_0.

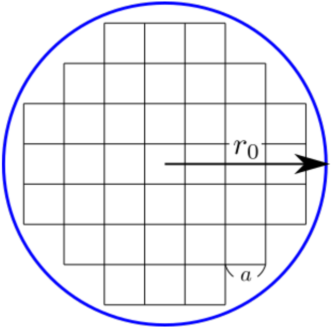
The (Wilson) Dirac op is

$$D = \frac{1}{a}\left(\sum_{i=1}^{3}\left[\sigma^i \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2}\nabla_i\nabla_i^\dagger\right] - m\right).$$

$$(\nabla_i\psi)_x = \psi_{x+\hat{\imath}} - \psi_x,\ (\nabla_i^\dagger\psi)_x = \psi_{x-\hat{\imath}} - \psi_x$$

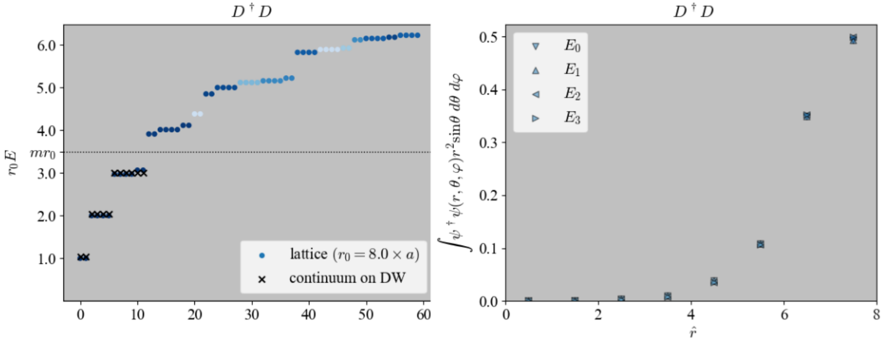
+OBC

We analyze $D^\dagger D$ and $DD^\dagger$.



Spectrum and Edge modes of $D^\dagger D$

We solve $D^\dagger D\psi = E^2\psi$.

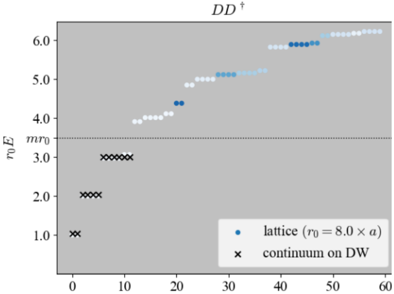


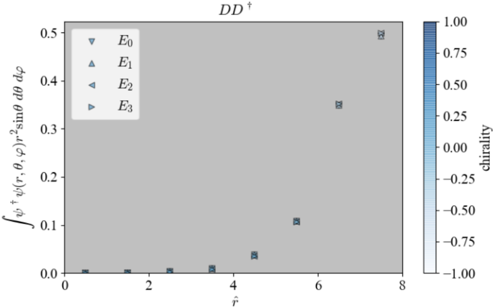
The edge modes

- are chiral: $\sigma_{\text{normal}} = \frac{x}{r}\sigma^1 + \frac{y}{r}\sigma^2 + \frac{z}{r}\sigma^3 \simeq +1$
- have a gap from zero (as a gravitational effect)
- agree well with the continuum prediction

Spectrum and Edge modes $DD^\dagger$

We solve $DD^\dagger\psi = E^2\psi$ as well.





The result is almost the same as $D^\dagger D$, but edge modes are negative chiral modes.

$$\sigma_{\text{normal}} = \frac{x}{r}\sigma^1 + \frac{y}{r}\sigma^2 + \frac{z}{r}\sigma^3 \simeq -1$$

It seems that a chiral theory is possible...

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Chiral Anomaly

$S = \int \bar{\psi} \not{D} \psi$ has a chiral symmetry.
$\longrightarrow j_A^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi$ is conserved.

$$\partial_\mu j_A^\mu = 0$$

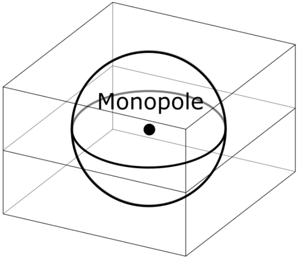
However, a quantum loop correction breaks this conservation law:

$$\int dx^2 \partial_\mu j_A^\mu = \int \frac{1}{2\pi} F = n_+ - n_- \neq 0$$

$$n_\pm = \# \left\{ \psi \;\middle|\; \not{D}\psi = 0,\ \gamma^5\psi = \pm\psi \right\}$$

This anomaly is related to the zero-mode!

S^2 with Monopole



$$A = \frac{1-\cos\theta}{2} d\phi$$

$$\text{Chiral Anomaly} = \underbrace{\frac{1}{2\pi}\int_{S^2} dA = 1}_{\text{Dirac index}}$$

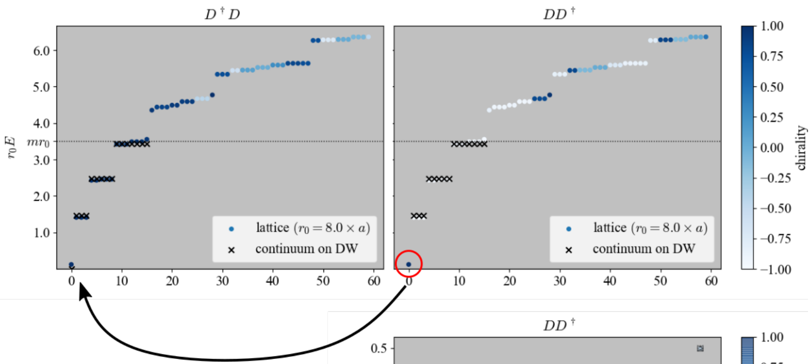
Covariant Derivative: $(\nabla_i \psi)_x = \exp\left[-i\int_{x+\hat{i}}^{x} A_i dx^i\right]\psi_{x+\hat{i}} - \psi_x$

Dirac Operator:

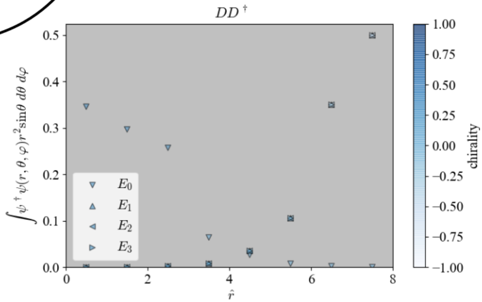
$$D = \frac{1}{a}\left(\sum_{i=1}^{3}\left[\sigma^i \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2}\nabla_i\nabla_i^\dagger\right] - ma\right).$$

$\longrightarrow$ # of zero-mode of $D^\dagger D$ should be different from that of $DD^\dagger$.

Anomaly Inflow



This mode is localised at the center

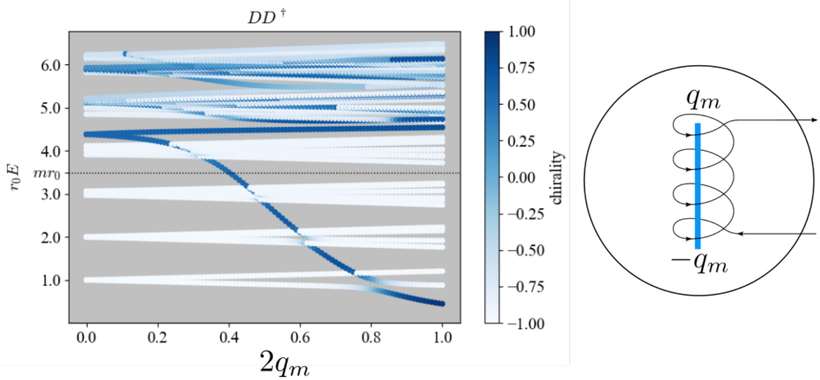


It cancels the anomaly

Kan's talk (August 9th)

Adding a solenoid

Adding a solenoid to the inside.



This mode is an obstacle to constructing a chiral gauge theory.

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Summary and Outlook

[Summary]

In cases S^2, we embodied Nash's thm in domain-wall.

- Chiral edge states feel gravity through the induced spin connection.
- New localized mode cancels chiral anomaly on the edge.
- This mode is an obstacle to formulating a chiral gauge theory.

[Outlook]

- Gravitational anomaly inflow
- Index theorem with a nontrivial curvature
- Formulate real projective space

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Fukuda, H. and Yonekura, K. (2021). Witten effect, anomaly inflow, and charge teleportation. Journal of High Energy Physics, 2021(1).

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Takane, Y. and Imura, K.-I. (2013). Unified description of dirac electrons on a curved surface of topological insulators. Journal of the Physical Society of Japan, 82(7):074712.

Witten, E. (1979). Dyons of Charge e theta/2 pi. Phys. Lett. B, 86:283–287.

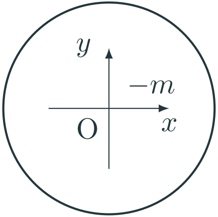
Witten, E. (2016). Fermion path integrals and topological phases. Reviews of Modern Physics, 88(3).

Contents

S^1 domain-wall

Domain wall:

$$\epsilon(r) = \text{sign}(r - r_0) = \begin{cases} -1 & (r < r_0) \\ 1 & (r \geq r_0) \end{cases},$$



Hermitian Dirac operator:

$$H = \sigma_3 \left(\sum_{i=1,2} \left(\sigma_i \frac{\partial}{\partial x^i} \right) + m\epsilon \right)$$

$$= \begin{pmatrix} m\epsilon & e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right) \\ -e^{i\theta}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\theta}\right) & -m\epsilon \end{pmatrix}.$$

Spectrum of Edge modes

Effective Dirac operator:

$$i\mathbb{D}^{S^1}_{eff} = \frac{1}{r_0}\left(-i\frac{\partial}{\partial\theta} + \underset{\sim}{\frac{1}{2}}\right)$$

Spin^c connection

$\longrightarrow$The edge modes is effectively anit-periodic spinor.
(trivial element of the spin bordism group)

Eigenvalue:

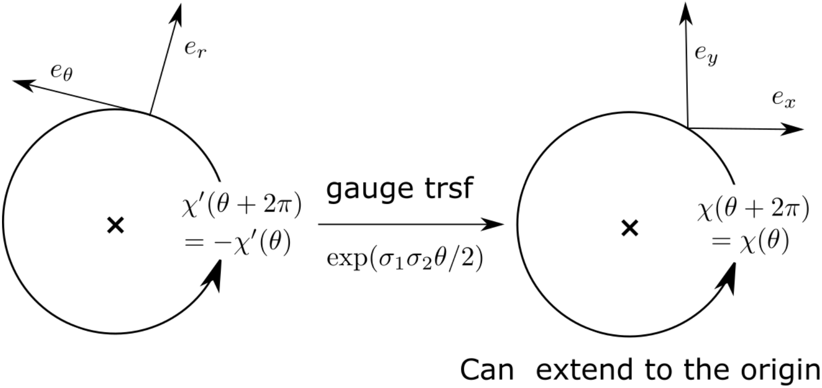
$$E = \pm\frac{n+\frac{1}{2}}{r_0}\ (n = 0, 1, \cdots).$$

$\longrightarrow$ Gravity appears as the gap of the spectrum

Periodicity of Edge modes

S^1 admits two spin structures:

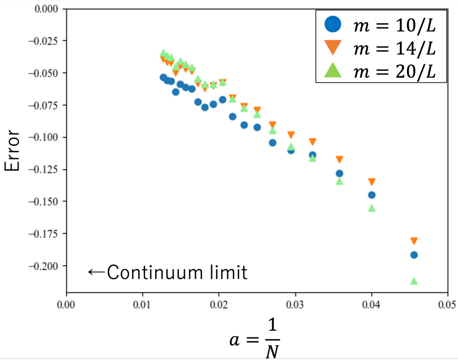
$\longrightarrow$ periodic spinor and anti-periodic spinor.



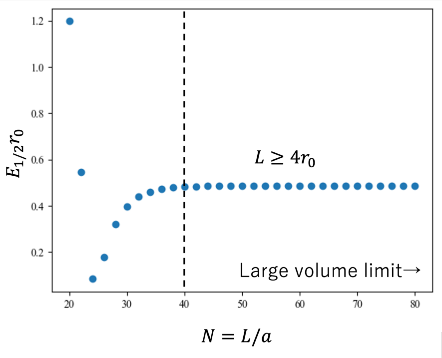
Only anti-periodic spinors appear at the boundary.

Continuum limit and Finite-volume effect

Continuum limit $a = 1/N \to 0$



Large volume limit $L = Na \to \infty$



Fixed parameter:

$L = Na, r_0 = Na/4, m = 14/L$

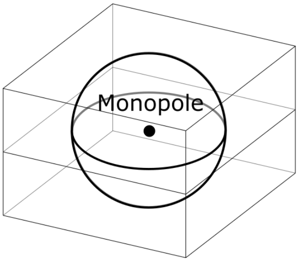
Agree well with the conti. prediction!

Fixed parameter:

$r_0 = 10a$

Saturates when $L \geq 4r_0$!

S^2 with Monopole



$$A = \frac{1-\cos\theta}{2} d\phi$$

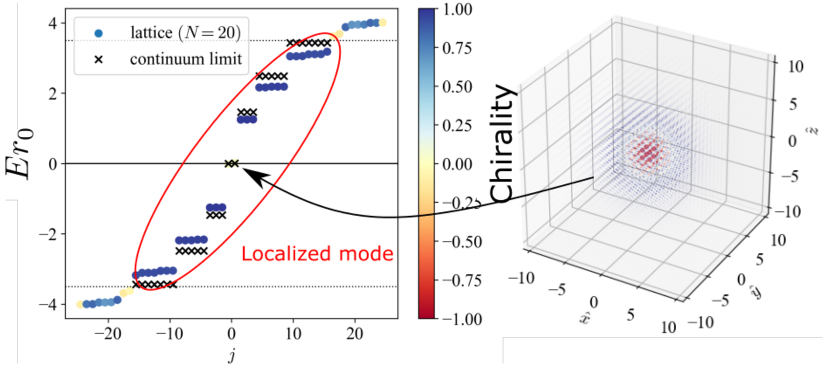
$$\text{Chiral Anomaly} = \underbrace{\frac{1}{2\pi}\int_{S^2} dA = 1}_{\text{Dirac index}}$$

Covariant Derivative: $(\nabla_i \psi)_x = \exp\left[-i \int_{x+\hat{i}}^{x} A_i dx^i\right] \psi_{x+\hat{i}} - \psi_x$

Hermitian Dirac Operator:

$$H = \frac{\bar{\gamma}}{a}\left(\sum_{i=1,2,3}\left[\gamma^i \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2}\nabla_i \nabla_i^\dagger\right] + \epsilon m a\right).$$

Spectrum ($H\psi = E\psi$)



- 0-modes appear at the wall and the center.
- Center-localized mode cancels Chiral anomaly on S^2!
- Monopole and Edge share one charge.

$\longrightarrow$ Monopole becomes a dyon with charge $\frac{1}{2}$.

(Witten [1979], Fukuda and Yonekura [2021])

Construct H

We consider matrix valued function and two action of Pauli matrix:

$$\psi = \psi_0 + \psi_i \sigma^i,$$
$$\sigma^{i,L}\psi = \sigma^i \psi,\ \sigma^{i,R}\psi = \psi \sigma^i$$

Hermitian Dirac operator H is defined by

$$D = \left(\sum_{i=1,2,3} \left[\sigma^{i,L} \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2} \nabla_i \nabla_i^\dagger \right] + \epsilon m a \right)$$
$$D_1 = D \frac{x^i \sigma^{i,R}}{r} + \frac{x^i \sigma^{i,R}}{r} D$$
$$H = \begin{pmatrix} 0 & D_1 \\ D_1^\dagger & 0 \end{pmatrix} \to (d + d^\dagger)_{S^2}$$

T-Anomaly

"Anomaly" is a phenomenon in which a partition function $Z[A]$ does not have a symmetry of the classical action $S[A]$.

We assume $S[A] = \int_Y \bar{\psi} \not{D}^Y \psi$ has time reversal symmetry. However, the partition function

$$Z_{reg}[A] = \prod_{\lambda} \frac{i\lambda}{i\lambda + M_{PV}} = |Z[A]| \exp\left(-i\frac{\pi}{2}\eta(i\not{D}^Y)\right)$$

breaks T-symmetry ($Z[A]^* \neq Z[A]$)
since PV regulator has no T-symm.

$$\eta(i\not{D}^Y) = \lim_{\epsilon \to +0} \lim_{s \to 0} \sum_{\lambda \in \mathrm{Spec}\left(i\not{D}^Y\right)} \frac{\lambda + \epsilon}{|\lambda + \epsilon|^{1+s}}$$

$$= \sum_{\lambda \neq 0} \mathrm{sign}(\lambda) + \#\{\lambda = 0\}$$

Anomaly inflow

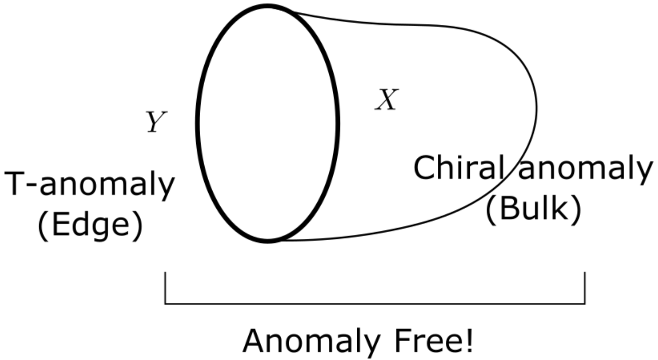
T-anomaly is cancelled by $\exp\left(i\pi \int_X ch(F)\right)$, so

$$Z[A,X] = |Z[A]| \exp\left[i\pi \left(\underbrace{\int_X ch(F) - \frac{1}{2}\eta(i\not{D}^Y)}_{\text{APS index} \in \mathbb{Z}} \right) \right]$$

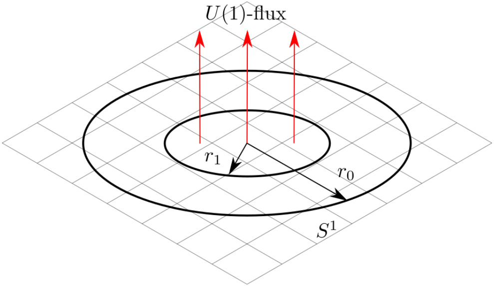
has T-symmetry!

APS index $\in \mathbb{Z}$

$\longrightarrow$ Anomaly inflow (Cf. Witten [2016])



$U(1)$ gauge field on a square lattice



$U(1)$ gauge field:

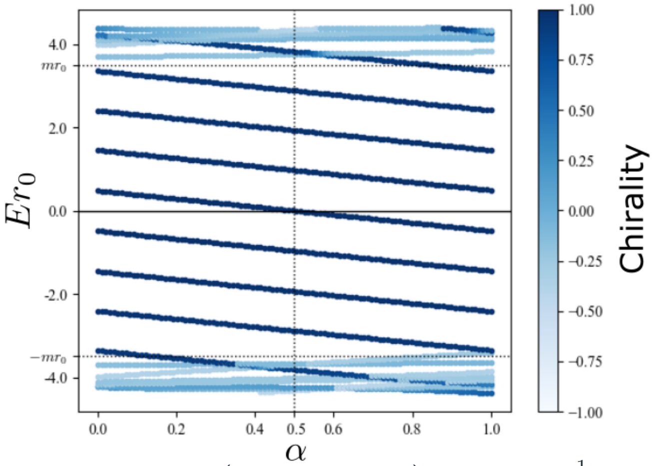
$$A = \begin{cases} \alpha\left(\frac{r}{r_1}\right)^2 d\theta & (r < r_1) \\ \alpha d\theta & (r_1 < r \leq r_0) \end{cases}$$

Covariant derivative: $(\nabla_i \psi)_x = \exp\left[-i \int_{x+\hat{i}}^{x} A_i dx^i\right] \psi_{x+\hat{i}} - \psi_x$

Hermitian Dirac Operator:

$$H = \frac{\sigma_3}{a}\left(\sum_{i=1,2}\left[\sigma_i \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2}\nabla_i \nabla_i^\dagger\right] + \epsilon m a\right)$$

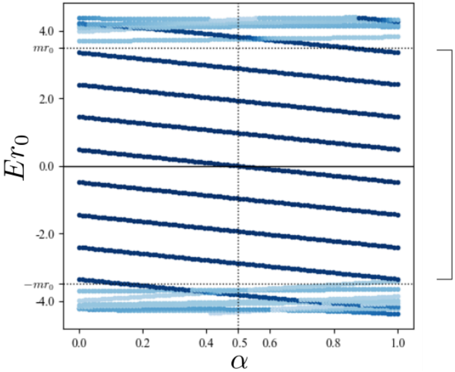
Spectrum ($H\psi = E\psi$)



$$H \to H_{eff}^{S^1} = \frac{1}{r_0}\left(-i\frac{\partial}{\partial\theta} + \frac{1}{2} - \underset{\sim}{\alpha}\right), \; E = \frac{n + \frac{1}{2} - \alpha}{r_0}$$

AB Phase

Parity (Time reversal) Anomaly



Vertical Asymmetry

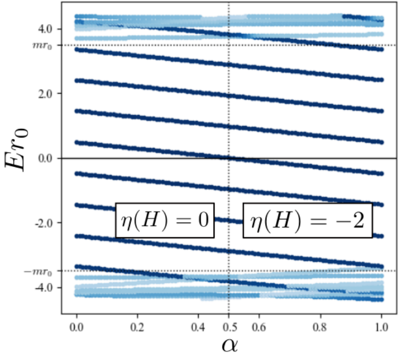
↓

T-Anomaly!

$$Z_{S^1} = \text{Det}\left(\frac{H_{eff}^{S^1}}{H_{eff}^{S^1} + i\mu}\right) \propto \exp\left(-i2\pi\frac{1}{2}\eta(H_{eff}^{S^1})\right)$$

$$-\frac{1}{2}\eta(H_{eff}^{S^1}) = -\frac{1}{2}\lim_{s\to 0}\sum_n \frac{n+\frac{1}{2}-\alpha}{\left|n+\frac{1}{2}-\alpha\right|^{1+s}} = -\alpha + [\alpha + \frac{1}{2}]$$

Anomaly Inflow



Chiral anomaly on Bluk

$$\frac{1}{2\pi}\int_{r<r_0} dA = \alpha$$

cancels the T-anomaly on Edge

$\longrightarrow$ Anomaly inflow (Witten [2016])

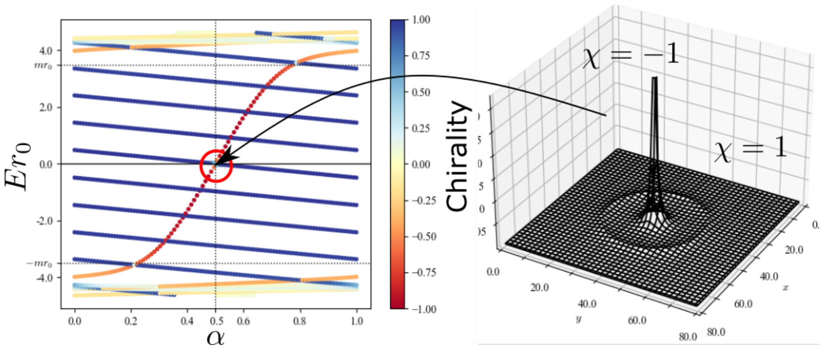
APS-index (Fukaya et al. [2020]) describes Anomaly inflow!

$$\mathrm{Ind}_{\mathrm{APS}} = \underbrace{\frac{1}{2\pi}\int_{r<r_0} dA}_{\text{buld}} - \underbrace{\frac{1}{2}\eta(H^{S^1}_{eff})}_{\text{edge}} = -\frac{1}{2}\eta(H) = [\alpha + \frac{1}{2}]$$

When $U(1)$ flux is singular ($r_1 \sim a$)

Chiral anomaly on Bluk is not well-defined.

$\longrightarrow$Another localized mode canceled the T-anomaly!

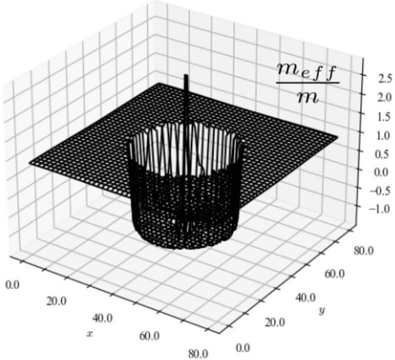


It is related to “Witten Effect" [Witten [1979]]

(cf. Naoto’s talk)

Creation of Domain-wall

$$H = \frac{\sigma_3}{a}\left(\sum_{i=1,2}\sigma_i \frac{\nabla_i - \nabla_i^\dagger}{2} + \underbrace{\sum_{i=1,2}\frac{1}{2}\nabla_i\nabla_i^\dagger + \epsilon m a}_{=:\, m_{eff}a}\right)$$



Wilson term and the $U(1)$ gauge generate a new domain-wall !

S^1 domain-wall on a square lattice

Let $(\mathbb{Z}/N\mathbb{Z})^2$ be a two-dim. lattice.

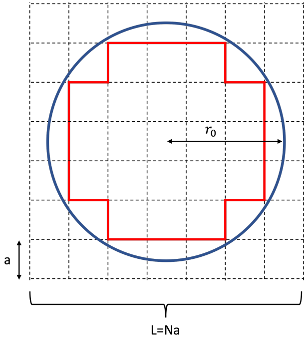
The domain-wall is given by

$$\epsilon(x) = \begin{cases} -1 & (r < r_0) \\ 1 & (r \geq r_0) \end{cases},$$

and the (Wilson) Dirac op is

$$H = \sigma_3 \left(\sum_{i=1,2} \left[\sigma_i \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2} \nabla_i \nabla_i^\dagger \right] + \epsilon m a \right),$$

$$(\nabla_i \psi)_x = \psi_{x+\hat{i}} - \psi_x, \; (\nabla_i^\dagger \psi)_x = \psi_{x-\hat{i}} - \psi_x$$



$+$ PBC for all direction.

Cf. Kaplan [1992] studied a flat domain-wall in $\mathbb{R}^{2m+1}$

Spectrum ($H\psi = E\psi$)

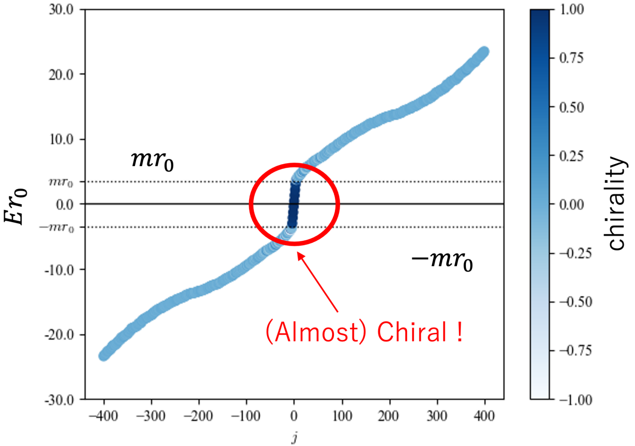
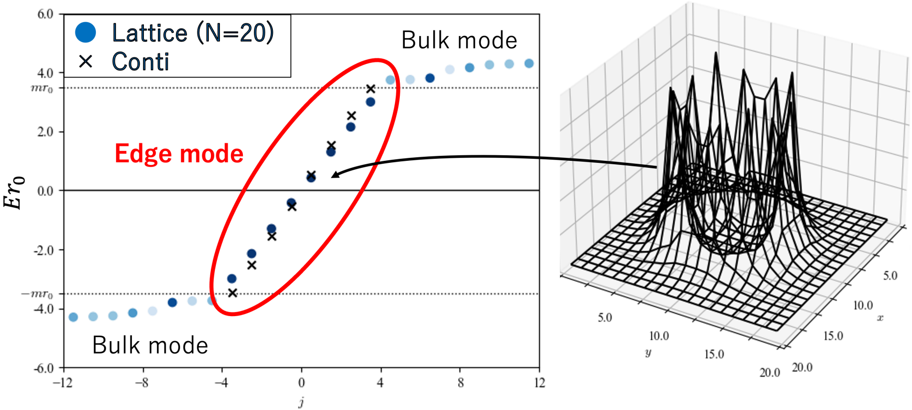


Fig 2: The Dirac eigenvalue spectrum: $ma = 0.7, r_0 = L/4, N = 20$

The color = chirality: $\gamma_{\text{normal}} = \frac{x}{r}\sigma_1 + \frac{y}{r}\sigma_2$

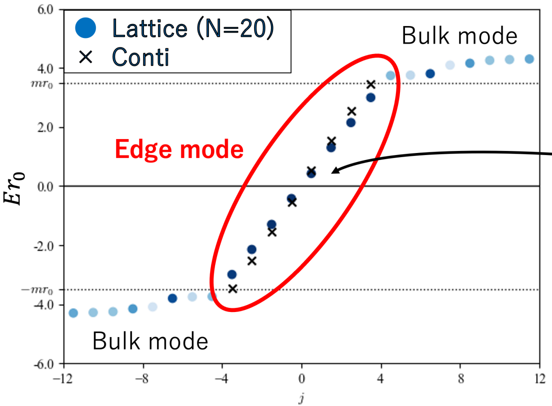
Edge modes

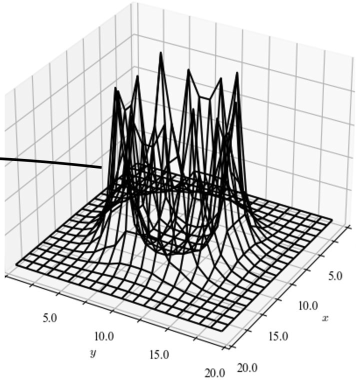


The edge modes

- are chiral: $\gamma_{\text{normal}} = \frac{x}{r}\sigma_1 + \frac{y}{r}\sigma_2$
- have a gap from zero (as a gravitational effect)
- agree well with the continuum prediction

Induced connection and Eigenvalue of the Edge modes

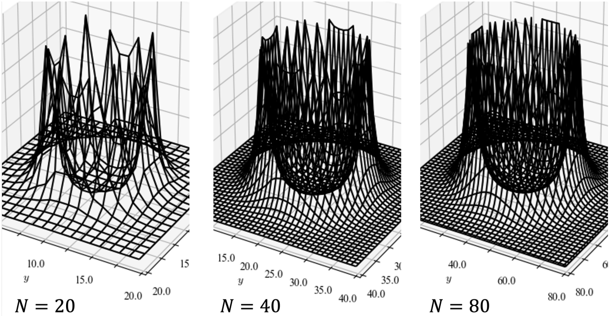


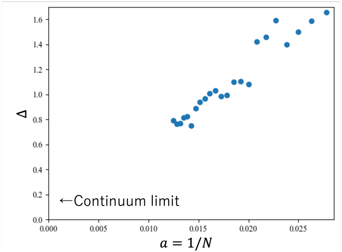


$$H \to H_{eff}^{S^1} = \frac{1}{r_0}\left(-i\frac{\partial}{\partial\theta} + \frac{1}{2}\right),\ E = \frac{n+\frac{1}{2}}{r_0}$$

Spin^c connection

Recovery of Rotational symmetry in the continuum limit (S^1)





$$\Delta = (\max(\text{peak}) - \min(\text{peak}))/a^2$$

The rotational symmetry automatically recovers in the continuum limit!

Effective Dirac op and Dirac op. of S^2

The spin rotation using

$$R = 1 \otimes \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) & -e^{-i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) \\ e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) & e^{i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} e^{i\frac{\phi}{2}}$$

changes $\chi \to R^{-1}\chi$ and

$$\begin{pmatrix} m\epsilon & \sigma^j\partial_j \\ -\sigma^j\partial_j & -m\epsilon \end{pmatrix} \to \begin{pmatrix} \epsilon m & \sigma_3\left(\frac{\partial}{\partial r} + \frac{1}{r} + \frac{1}{r}\sigma_3 \not{D}_{S^2}\right) \\ -\sigma^3\left(\frac{\partial}{\partial r} + \frac{1}{r} + \frac{1}{r}\sigma^3 \not{D}_{S^2}\right) & -\epsilon m \end{pmatrix}$$

$$\not{D}_{S^2} = \left(\sigma_1 \frac{\partial}{\partial\theta} + \frac{\sigma_2}{\sin\theta}\left(\frac{\partial}{\partial\phi} + \underbrace{\frac{i}{2} - \frac{\cos\theta}{2}\sigma_1\sigma_2}\right)\right)$$

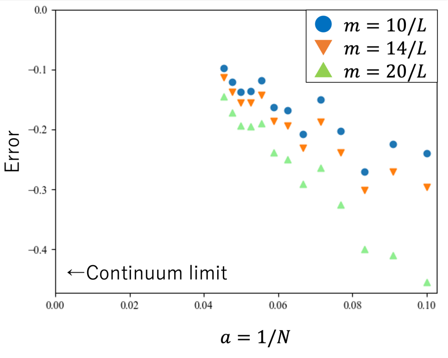
Spin^c connection on S^2

Edge states feel gravity through the induced connection!

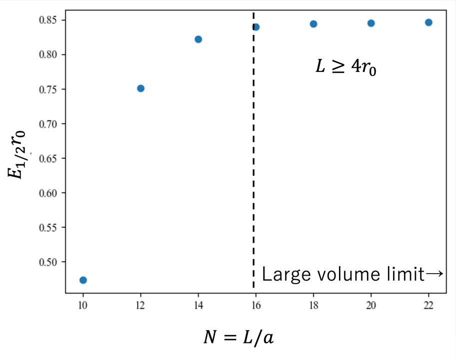
[Takane and Imura [2013]].

Continuum limit and Finite volume effect

Continuum limit $a = 1/N \to 0$



Large volume limit $L = Na \to \infty$



Fixed parameter:

$$L = Na, r_0 = Na/4, m = 14/L$$

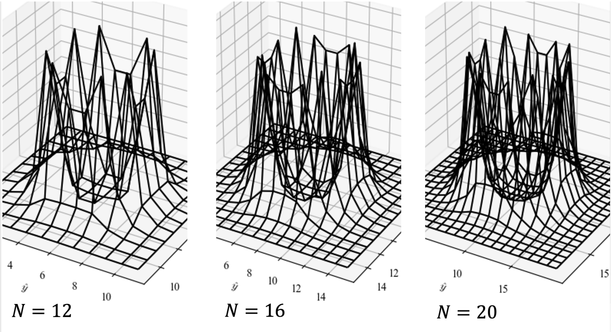
Agree well with
the conti. prediction!

Fixed parameter:

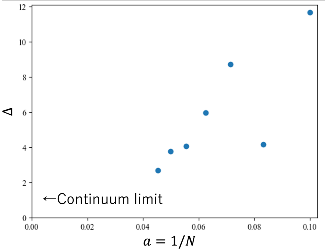
$$r_0 = 4a$$

Saturates when $L \geq 4r_0$!

Recovery of Rotational symmetry in the continuum limit (S^2)



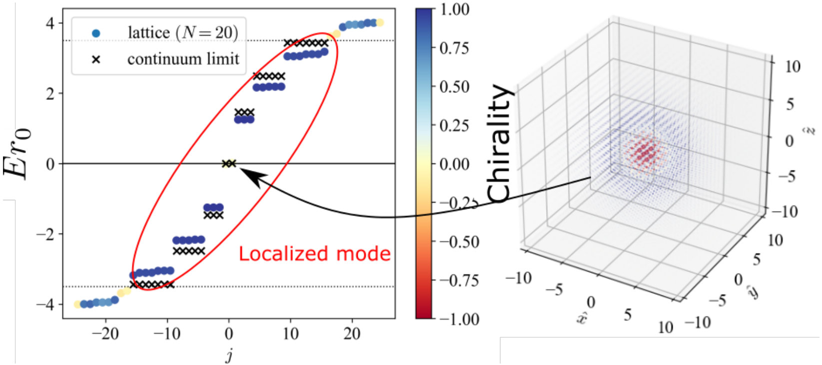
(slice at $z = N/2$)



$$\Delta = (\max(\text{peak}) - \min(\text{peak}))/a^3$$

The rotational symmetry automatically recovers in the continuum limit!

Spectrum ($H\psi = E\psi$)



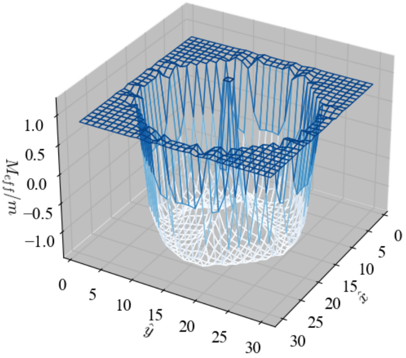
- 0-modes appear at the wall and the center.
- Center-localized mode cancels Chiral anomaly on S^2!
- Monopole and Edge share one charge.

$\longrightarrow$ Monopole becomes a dyon with charge $\frac{1}{2}$.

(Witten [1979], Naoto's Talk)

Domain-wall Creation

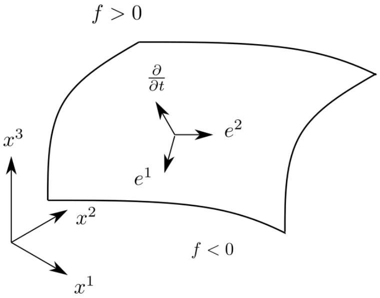
$$H = \frac{\bar{\gamma}}{a}\left(\sum_{i=1}^{3}\gamma_i\frac{\nabla_i - \nabla_i^\dagger}{2} + \underbrace{\sum_{i=1}^{3}\frac{1}{2}\nabla_i\nabla_i^\dagger + \epsilon m a}_{=:\, m_{eff}a}\right)$$



Wilson term and the $U(1)$ gauge generate a new domain-wall !

Chiral Fermion

$$
\begin{aligned}
D = & \sum_{i=1}^{2m+1} \gamma^i \frac{\partial}{\partial x^i} + m\text{sign}(f) \\
\simeq & \gamma^{2m+1} \frac{\partial}{\partial t} + F + m\text{sign}(f) \\
& \underbrace{+ \gamma^a \left(e_a + \frac{1}{4} \sum_{bc} \omega_{bc,a} \gamma^b \gamma^c \right)}_{\not{D}^Y}
\end{aligned}
$$



In the large m limit, $D \to \not{D}_+^Y = \not{D}^Y \frac{1}{2}(1 + \gamma^{2m+1})$.

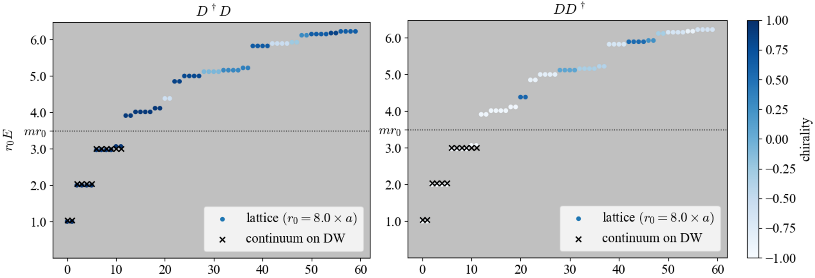
Similarly, $D^\dagger \to \not{D}_-^Y = \not{D}^Y \frac{1}{2}(1 - \gamma^{2m+1})$.

$\longrightarrow$ Is it possible to formulate a Chiral fermion on the wall?

We analyze a spectrum of $D^\dagger D$ and $D^\dagger D$.

The spectrum of $D^\dagger D$ and $DD^\dagger$ without $U(1)$ gauge field

We solve $D^\dagger D\psi = E^2\psi$ and $DD^\dagger\psi = E^2\psi$.



Weyl fermions appear at the Wall.

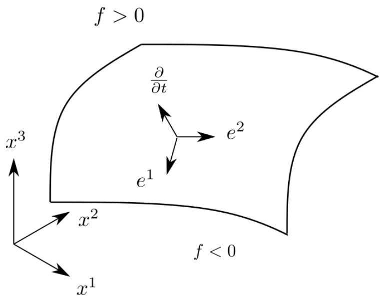
It seems that a chiral theory is possible...

Chiral Fermion

$$D = \sum_{i=1}^{2m+1} \gamma^i \frac{\partial}{\partial x^i} + m\text{sign}(f)$$

$$\simeq \gamma^{2m+1} \frac{\partial}{\partial t} + F + m\text{sign}(f)$$

$$+ \underbrace{\gamma^a \left(e_a + \frac{1}{4} \sum_{bc} \omega_{bc,a} \gamma^b \gamma^c \right)}_{\not{D}^Y}$$



In the large m limit, $D \to \not{D}^Y_+ = \not{D}^Y \frac{1}{2}(1 + \gamma^{2m+1})$.

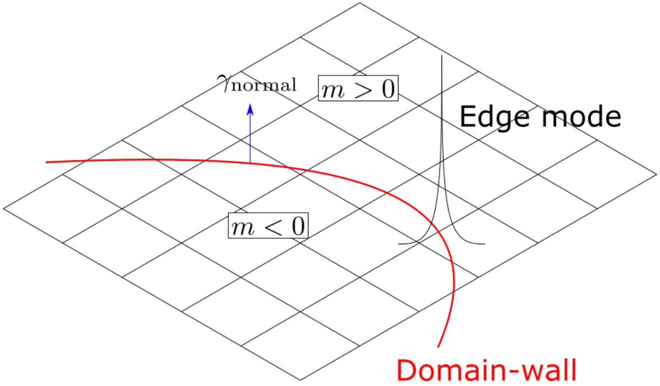
Similarly, $D^\dagger \to \not{D}^Y_- = \not{D}^Y \frac{1}{2}(1 - \gamma^{2m+1})$.

$\longrightarrow$ Is it possible to formulate a Chiral fermion on the wall?

We analyze a spectrum of $D^\dagger D$ and $D^\dagger D$.

Free Curved Domain-Wall

$$H = \frac{\bar{\gamma}}{a}\left(\sum_{i=1}^{n+1}\left[\gamma^i \frac{\nabla_i - \nabla_i^\dagger}{2} + \frac{1}{2}\nabla_i \nabla_i^\dagger\right] + ma\right), \quad \left(\begin{array}{l} \{\gamma^i, \gamma^j\} = 2\delta^{ij} \\ \{\bar{\gamma}, \gamma^J\} = 0, \ \bar{\gamma}^2 = 1 \end{array}\right)$$



- Chiral edge modes ($\gamma_{\text{normal}} = +1$) appear at the wall,
- and feel gravity through the induced spin connection.

[SA and H. Fukaya, 2022]