Why magnetic monopole becomes dyon in topological insulators

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Based on a collaboration with Shoto Aoki, Hidenori Fukaya, Mikito Koshino, Yoshiyuki Matsuki (Osaka U.), arXiv:2304.13954 [cond-mat.mes-hall].

What will happen to a magnetic monopole when it is put inside a topological insulator?

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EFT of topological insulator and Witten effect

A topological insulator: the bulk is the insulator (gapped), but the edge is the gapless. The effective theory of the T-symmetric topological insulator is described by the $\theta = \pi$ vacuum. In the presence of the magnetic monopole, the θ-term is given by

$$L_\theta = \frac{\theta}{8\pi^2} \int d^3x \, \boldsymbol{E} \cdot \boldsymbol{B} = -\frac{q_m}{2} \int d^3x \, A^0 \delta^{(3)}(\boldsymbol{r}).$$

This implies that there is a particle with electric charge $q_e = -q_m/2$ which is coupled to the A^0 potential.

The monopole with $q_m = 1$ obtains the electric charge $q_e = -1/2$.

The effective theory description above is quite simple, but can't answer to the following questions:

(1) what is the origin of the electric charge? (must be electrons)
(2) if the origin is the electrons, why is it bound to monopole?
(3) why is the electric charge fractional?

In this our work [Aoki, Fukaya, Kan, Koshino Matsuki ('23)], we try to give answers to the questions from in terms of a microscopic description.

A microscopic description

We put a $U(1)$ gauge flux located at the origin describing the monopole:

$$A_x = \frac{-q_m y}{r(r+z)}, \quad A_y = \frac{q_m x}{r(r+z)}, \quad A_z = 0,$$

of which field strength is

$$F_{ij} = q_m \epsilon_{ijk} \frac{x_k}{r^3} - 4\pi q_m \delta(x)\delta(y)\theta(-z)\epsilon_{ij3},$$

where the second term represents the Dirac string. Due to the Dirac quantization, we assume $q_m = n/2$ with $n \in \mathbb{Z}$.

Naively a UV description of the system is given by the Dirac Hamiltonian with a mass $m < 0$ [Yamagishi ('83)]:

$$H = \gamma_0 \left(\gamma_i \left(\partial_i - iA_i \right) + m \right),$$

where $\gamma_0 = \sigma_3 \otimes \mathbf{1}$ and $\gamma_i = \sigma_1 \otimes \sigma_i$.

In addition to J^2 and J_3, there is an operator that commutes with H; $[H, \sigma_3 \otimes D^{S^2}] = 0$, where we define the "spherical" operator

$$D^{S^2} := \sigma_i \left(L_i + \frac{n}{2} \frac{x_i}{r} \right) + 1, \quad L_i = -i\epsilon_{ijk} x_j \left(\partial_k - iA_k \right) - n \frac{x_i}{2r},$$

Physical meanings of the operator are explained later.

We find the normalizable zero-mode ($E = 0$) solution localized at the monopole ($r = 0$) with $j = |n/2| - 1/2$:

$$\psi_{j,j_3,0} = \frac{C_{j,j_3,0}}{r} \exp(-|m|r) \begin{pmatrix} 1 \\ \mathrm{sign}(m)\mathrm{sign}(n) \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta, \phi),$$

where $D^{S^2} \chi_{j,j_3,0}(\theta, \phi) = 0$.

1. No difference between the positive and negative mass in the solution. The Witten effect predicts the dyon appear only in the topological insulator ($m < 0$). The solution can't explain it w/o imposing "the chiral boundary condition" by hand.
2. Why does the electric charge become $q_e = -1/2$?

Regularized Dirac equation

The Wilson term

In order to answer these questions, we take account of the leading correction from the Pauli–Villars regularization. The partition function is expanded as

$$
\begin{aligned}
Z &= \det\left(\frac{D+m}{D+M_{\mathrm{PV}}}\right), \\
&= \det\Bigg[\frac{1}{M_{\mathrm{PV}}}\Bigg(D+m+\frac{1}{M_{\mathrm{PV}}}D_\mu^\dagger D^\mu \\
&\qquad\qquad + \mathcal{O}(1/M_{\mathrm{PV}}^2, m/M_{\mathrm{PV}}, F_{\mu\nu}/M_{\mathrm{PV}})\Bigg)\Bigg].
\end{aligned}
$$

"The Wilson term" $D_\mu^\dagger D^\mu/M_{\mathrm{PV}}$ appears as the leading correction.

Then the "regularized" Dirac Hamiltonian is given by

$$H_{\rm reg} = \gamma_0 \left(\gamma^i D_i + m + \frac{D_i^\dagger D^i}{M_{\rm PV}} \right).$$

Note that the sign of m is well-defined once the sign of $M_{\rm PV}(> 0)$ is fixed. The Dirac equation is manifestly different between positive and negative m.

Since the Laplacian $D_i^\dagger D^i$ is always positive, the mass shift due to the Wilson term is always positive when we take $M_{\rm PV}$ positive.

For $m < 0$ (or inside topological insulators), it is possible to locally flip the sign of the "effective" mass

$$m < 0 \quad \to \quad m_{\rm eff} = m + \frac{D_i^\dagger D^i}{M_{\rm PV}} \sim m + \frac{1}{M_{\rm PV} r_1^2} > 0,$$

when the magnetic flux is concentrated in the region $r < r_1$.

It's implies that the inside region $r < r_1$ becomes a normal insulator, and the spherical domain-wall is dynamically created and the chiral edge-mode appears on it! (It doesn't happen in the normal insulator with $m > 0$.)

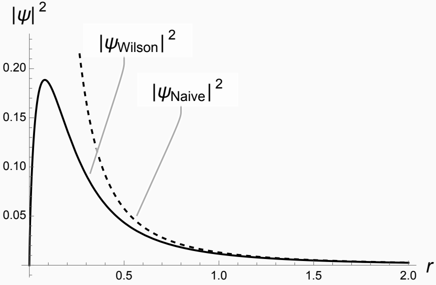
The regularized Hamiltonian is

$$H_{\text{reg}} = \begin{pmatrix} m - D_i D_i / M_{\text{PV}} & \sigma_i (\partial_i - iA_i) \\ -\sigma_i (\partial_i - iA_i) & -m + D_i D_i / M_{\text{PV}} \end{pmatrix},$$

The analytical solution of the zero-mode for $r_1 \to 0$ is given by

$$\psi^{\text{mono}}_{j,j_3} = \frac{B e^{-M_{\text{PV}} r/2}}{\sqrt{r}} I_\nu(\kappa r) \begin{pmatrix} 1 \\ -\text{sign}(n) \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta, \phi),$$

where $\nu = \sqrt{2|n| + 1}/2$, and $\kappa = M_{\text{PV}} \sqrt{1 + 4m/M_{\text{PV}}}/2$.



The plot with $n = 1$, $m = -0.1$, $M_{\rm PV} = 10$.

- The solution $\psi_{\rm Wilson}$ coincides with $\psi_{\rm Naive}$ for large r.
- A peak at $r = |n|/(2M_{\rm PV}) \sim 1/M_{\rm PV}$ is the (spherical) domain-wall.
- D^{S^2} is the Dirac operator on the spherical domain-wall created around the monopole. (cf. Shoto's talk)

The Atiyah–Singer index theorem and the half-integral charge

Because of

$$\sigma_r \chi_{j,j_3,0}(\theta,\phi) = s\chi_{j,j_3,0}(\theta,\phi), \quad s := \mathrm{sign}(n),$$

so # of the degeneracy is $2j+1=|n|$. Then the Dirac index is

$$\mathrm{Ind}\, D^{S^2} = n.$$

On the other hand, the topological index is

$$\frac{1}{4\pi}\int_{S^2} d^2x \epsilon^{\mu\nu} F_{\mu\nu} = n.$$

Stability of the zero modes on the domain-wall is topologically protected by the AS index theorem.

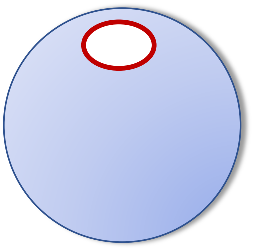
So far, we considered a $\mathbb{R}^3$ space, but in order to discuss topological feature of the fermion zero mode, we also need an IR regularization, such as the one-point compactification, S^3.

Then the topological insulator region with $(m_{\text{eff}} < 0)$ has topology of a disk with a small S^2 boundary at $r = r_1$.

However, due to the cobordism invariance of the AS index,

$$\int_{\partial M} F = \int_M dF = 0,$$

the disk is not possible.

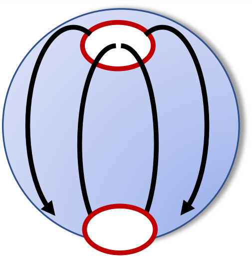


A resolution is: to create another domain-wall at, say, $r = r_0$, outside of the topological insulator.

Another zero-modes are localized at the outside domain-wall, and the index is kept trivial:

$$0 = \int_M dF = \int_{\Sigma_{\text{mono}}} F + \int_{\Sigma_{\text{out}}} F,$$

where $\partial M = \Sigma_{\text{mono}} \cup \Sigma_{\text{out}}$.



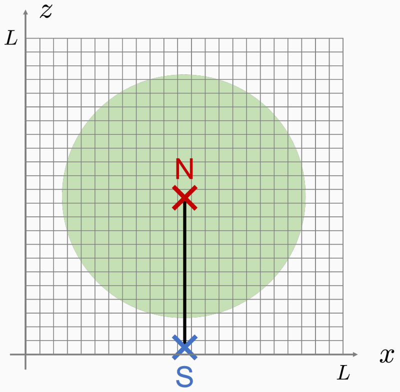
Then the 50% of the zero-mode state is located at the monopole, while the other 50% is sit at the domain-wall.

Thus the dressed electric charge of the monopole becomes $-1/2$!

Numerical analysis

Lattice setup

On a three-dimensional hyper-cubic lattice with size L with open boundary conditions, we put a monopole at $\boldsymbol{x}_m = (L/2, L/2, L/2)$ with a magnetic charge $n/2$. We also put an antimonopole at $\boldsymbol{x}_a = (L/2, L/2, 1/2)$ with the opposite charge $-n/2$.



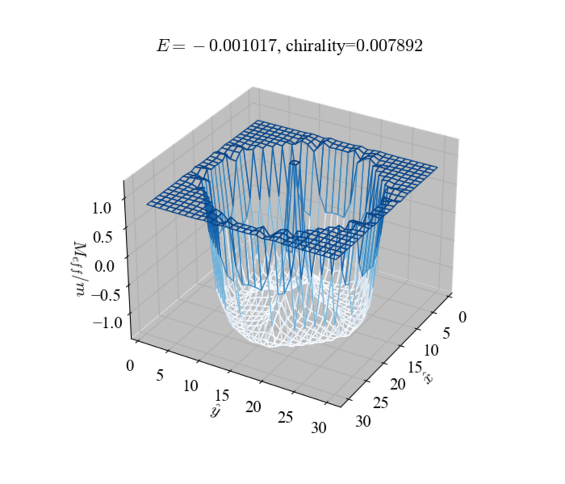
Domain-wall creation

To directly confirm creation of the domain-wall near the monopole, we plot distribution of the "effective mass" (normalized by m_0),

$$m_{\text{eff}}(\boldsymbol{x}) = \phi(\boldsymbol{x})^\dagger \left[-\sum_{i=1,2,3} \frac{1}{2} \nabla_i^f \nabla_i^b + m(\boldsymbol{x}) \right] \phi(\boldsymbol{x}) \Big/ \phi(\boldsymbol{x})^\dagger \phi(\boldsymbol{x}),$$

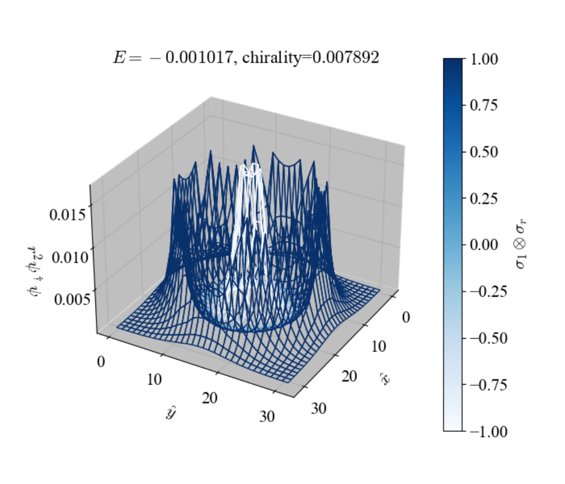
on the $z = (L+1)/2$ slice.

The effective mass of the nearest zeromode with $n = 1$ on $z = 16$ slice:



Amplitude

The amplitude of the nearest-zero mode for $n = 1$ in $z = 16$ slice:



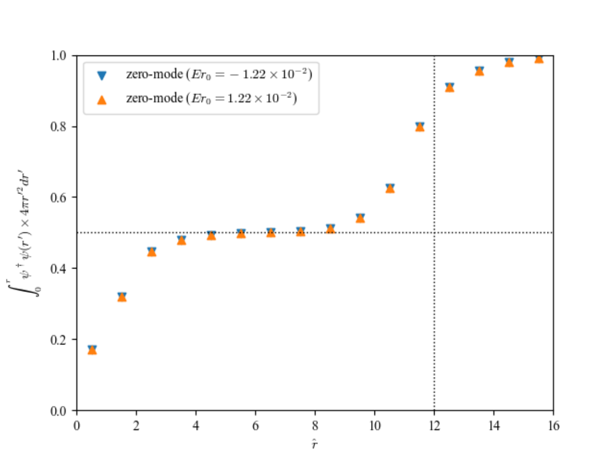
Electric charge

Let's quantify the electric charge that the monopole gains.

We plot the cumulative distribution of the nearest zero modes:

$$C_k(r) = \int_{|\boldsymbol{x}|<r} d^3x \, \phi_k(\boldsymbol{x})^\dagger \phi_k(\boldsymbol{x}).$$

For $n = 1$:



zero-mode ($Er_0 = -1.22 \times 10^{-2}$)
zero-mode ($Er_0 = 1.22 \times 10^{-2}$)
$\int_0^r \psi^\dagger \psi(r') \times 4\pi r'^2 dr'$
$\hat{r}$
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
12
14
16

Summary

We discussed a microscopic description of the Witten effect with the Wilson term.

How do we distinguish between the normal insulator ($m > 0$) and topological insulator ($m < 0$)?

- It is the topological insulator if the mass is relatively negative compared to the PV mass.

Why are electrons bound to monopole?

- Because of the positive mass correction from the magnetic field of the monopole, the domain-wall is dynamically created (only for the negative mass).

Why do the stable chiral zero modes appear?

- Because the zero modes localized at the domain-wall are protected by the AS index.

Why is the electric charge fractional?

- Because the 50% of the wavefunction is located around the monopole (the other 50% is located at the surface of the topological insulator).