Exact Solutions and Perturbative Calculations of n-Point Function in Finite Φ^3-Φ^4 Hybrid-Matrix-Model

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Introduction

- Quantum field theories on noncommutative spaces such as Moyal spaces have given a new perspective to matrix models.

Matrix model on noncommutative spaces (Grosse-Wulkenhaar model)

- It corresponds to scalar field theories on noncommutative spaces, which is renormalizable by adding harmonic oscillator potentials to the action.
- Φ^3 matrix model [Grosse-Sako-Wulkenhaar ('17)]
- Φ^4 matrix model [Grosse-Hock-Wulkenhaar ('19)]

Approach to Φ^4 matrix model

- Φ^3-Φ^4 Hybrid-Matrix-Model [Kanomata-Sako ('23)]

In order to mathematically formulate quantum field theories as a toy model, it is necessary to clarify the properties of the matrix model on noncommutative spaces (Grosse-Wulkenhaar model).

Scalar QFT on the Moyal Plane ($\Omega = 1$ case)

Definition(Action of the N.C. Real Scalar ϕ_2^3 QFT)

$$S[\phi] := \frac{1}{8\pi}\int_{\mathbb{R}_\theta^2} \frac{1}{2}\partial_i\phi\partial_i\phi + \frac{\mu_0^2}{2}\phi^2 + \Omega^2(\widetilde{x}_i\phi)(\widetilde{x}_i\phi) + \frac{\lambda}{3}\phi^3 + \kappa\phi$$

- $\phi = \phi^\dagger$: scalar field
- θ : N.C. parameter
- $[x_1, x_2] = i\theta \leftrightarrow [\widetilde{x}_1, \widetilde{x}_2] = i\frac{1}{\theta}$

From $\partial_i f = i[\widetilde{x}_i, f]$ and $\Omega = 1$,

$$\begin{aligned} S[\phi] =& \frac{1}{8\pi}\int_{\mathbb{R}_\theta^2} -\frac{1}{2}[\widetilde{x}_i, \phi][\widetilde{x}_i, \phi] + \widetilde{x}_i\phi\widetilde{x}_i\phi + \frac{\mu_0^2}{2}\phi^2 + \frac{\lambda}{3}\phi^3 + \kappa\phi \\ =& \frac{1}{8\pi}\int_{\mathbb{R}_\theta^2} (\widetilde{x}_i\widetilde{x}_i)\phi\phi + \frac{\mu_0^2}{2}\phi^2 + \frac{\lambda}{3}\phi^3 + \kappa\phi \end{aligned}$$

From $\int = (2\pi\theta)\mathrm{Tr}$,

$$S[\phi] = \frac{\theta}{4} \sum_{i,l,k=1}^{N} \left\{ \left(\widetilde{x}_i \widetilde{x}_i + \frac{\mu_0^2}{2} \right) \Phi_{il} \Phi_{li} + \kappa \Phi_{ii} + \frac{\lambda}{3} \Phi_{il} \Phi_{lk} \Phi_{ki} \right\}$$

From $V = \frac{\theta}{4}$ and $\widetilde{x}_i \widetilde{x}_i = i + \frac{1}{2}$,

$$S[\phi] = V \sum_{i,l,k=1}^{N} \left\{ \left(i + \frac{1 + \mu_0^2}{2} \right) \Phi_{il} \Phi_{li} + \kappa \Phi_{ii} + \frac{\lambda}{3} \Phi_{il} \Phi_{lk} \Phi_{ki} \right\}$$

From $\mu^2 = 1 + \mu_0^2$ and $E_{il} = \left(\frac{\mu^2}{2} + i \right) \delta_{il}$,

$$S[\phi] = V \mathrm{tr} \left(E \Phi^2 + \kappa \Phi + \frac{\lambda}{3} \Phi^3 \right)$$

Kontsevich model (Φ^3 matrix model)

Definition(Action of Kontsevich model)

$$S[\Phi] = iV\mathrm{tr}\left(E\Phi^2 + \kappa\Phi + \frac{\lambda}{3}\Phi^3\right)$$

- $\Phi = (\Phi_{ij}), \quad i,j = 1,\cdots,N$: Hermitian matrix
- $E = (E_{k-1}\delta_{km}), \quad k,m = 1,\cdots,N$: diagonal matrix
- $\kappa \in \mathbb{R}$

Definition(Partition function of Kontsevich model)

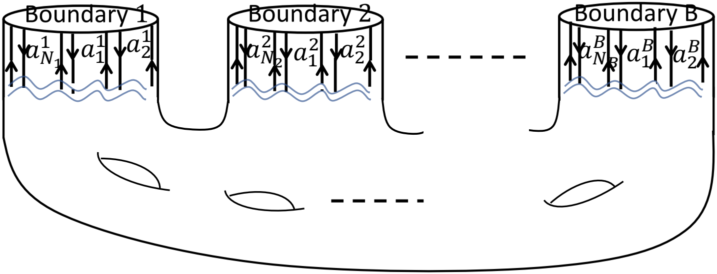
$$\mathcal{Z}[J] = \int \mathcal{D}\Phi \exp(-S[\Phi] + iV\mathrm{tr}(J\Phi))$$

- $J = (J_{mn}), \quad m,n = 1,\cdots,N$: Hermitian matrix
- $\mathcal{D}\Phi = \prod_{i<j} d\Phi^I_{ij} \prod_{i\le j} d\Phi^R_{ij}$: integral measure

$\sum_{j=1}^{B} N_j$-point function of Kontsevich model is defined as follows:

$$\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}$$
$$:= \sum_{B=1}^{\infty} \sum_{1 \le N_1 \le \cdots \le N_B}^{\infty} \sum_{p_1^1,\ldots,p_{N_B}^B=1}^{N} (iV)^{2-B} \frac{G_{|p_1^1 \ldots p_{N_1}^1|\ldots|p_1^B \ldots p_{N_B}^B|}}{S_{(N_1,\ldots,N_B)}} \prod_{\beta=1}^{B} \frac{\mathbb{J}_{p_1^\beta \ldots p_{N_\beta}^\beta}}{N_\beta}$$

- $N_i \;\; i = 1, \ldots, B$: N_i-external lines from the i-th boundary
- $\mathbb{J}_{a_1 \ldots a_{N_i}} := \prod^{N_i} J_{a_j a_{j+1}}$ Where $N_i + 1 \equiv 1$.



$$
\begin{aligned}
\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} =& 1 + iV \sum_{n=1}^{N} G_{|n|} J_{nn} \\
&+ \sum_{n,m=1}^{N} \left(\frac{iV}{2} G_{|nm|} J_{nm} J_{mn} + \left(-\frac{V^2}{2} G_{|n|} G_{|m|} J_{nm} J_{mn} \right) \right) + \cdots
\end{aligned}
$$

- Any $\sum_{j=1}^{B} N_j$-point function of Φ^3 matrix model was calculated by solving Schwinger-Dyson equation exactly by using Ward-Takahashi identity. Any $\sum_{j=1}^{B} N_j$-point function of Φ^3 matrix model in large N, V limit was calculated in the previous studies by Grosse, Wulkenhaar, and Sako.

- In this study, we obtained the exact solutions of $\sum_{j=1}^{B} N_j$-point function of the finite Kontsevich model (Φ^3 matrix model).
- It is known that any $G_{|a_1^1 \ldots a_{N_1}^1| \ldots |a_1^B \ldots a_{N_B}^B|}$ can be expressed using $G_{|a^1| \ldots |a^n|}$ type n-point function. Thus we focused on the rigorous calculation of $G_{|a^1| \ldots |a^n|}$. The formula for $G_{|a^1| \ldots |a^n|}$ was obtained, and it was achieved by using the partition function $\mathcal{Z}[J]$ calculated by Harish-Chandra-Itzykson-Zuber integral.

In the next talk, we will deal with a model in which a potential is integrable as well as Kontsevich model.

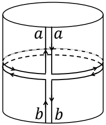
Φ^3-Φ^4 Hybrid-Matrix-Model

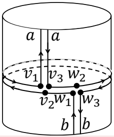
Definition(Action of Φ^3-Φ^4 Hybrid-Matrix-Model)

$$S[\Phi] = V\mathrm{tr}\left(E\Phi^2 + \kappa\Phi + \frac{1}{2}M\Phi M\Phi + \sqrt{\lambda}M\Phi^3 + \frac{\lambda}{4}\Phi^4\right)$$

- $M^2 = E$

- We constructed Feynman rules for Φ^3-Φ^4 Hybrid-Matrix-Model and calculated the perturbative expansion in ordinary methods.

$G_{|a|b|} = 4V^2$ 

$+ V^2 \sum_{\omega\in\mathscr{W}} \sum_{v\in\mathscr{V}}$ 

$+ \mathcal{O}(\lambda^2)$

- We calculated the path integral of the partition function $\mathcal{Z}[J]$ and used the result to compute exact solutions for 1-point function $G_{|a|}$ with 1-boundary, 2-point function $G_{|ab|}$ with 1-boundary, 2-point function $G_{|a|b|}$ with 2-boundaries, and n-point function $G_{|a^1|a^2|\cdots|a^n|}$ with n-boundaries.

Feynman Rules of Φ^3-Φ^4 Hybrid-Matrix-Model ($\kappa = 0$)

We calculate $\mathcal{Z}_{free}[J]$;

$$\mathcal{Z}_{free}[J] = \int \mathcal{D}\Phi \exp\left(-V\mathrm{tr}\left(E\Phi^2 + \frac{1}{2}M\Phi M\Phi\right)\right)\exp\left(V\mathrm{tr}(J\Phi)\right)$$

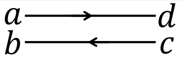
$$= \mathcal{Z}_{free}[0]\exp\left(\frac{V}{2}\sum_{n,m=1}^{N} J_{mn}\frac{1}{E_{n-1}+E_{m-1}+\sqrt{E_{n-1}}\sqrt{E_{m-1}}}J_{nm}\right)$$

- $\mathcal{Z}_{free}[0] = \left(\prod_{n=1}^{N}\sqrt{\frac{2\pi}{3VE_{n-1}}}\right)\left(\prod_{1\leq n<m\leq N}\frac{\pi}{V(E_{n-1}+E_{m-1}+\sqrt{E_{n-1}E_{m-1}})}\right)$

We introduce the free n-point functions:

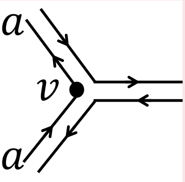
$$\left\langle \prod_{k=1}^{n}\Phi_{i_k j_k}\right\rangle_{free} := \frac{1}{\mathcal{Z}_{free}[0]}\int \mathcal{D}\Phi\ \Phi_{i_1 j_1}\cdots\Phi_{i_n j_n}\exp\left(-V\mathrm{tr}\left(E\Phi^2+\frac{1}{2}M\Phi M\Phi\right)\right)$$

The Feynman graph of the propagator (ribbon) is then defined as follows:



$$= \langle \Phi_{ba}\Phi_{dc}\rangle_{free} = \frac{1}{V}\frac{\delta_{ad}\delta_{bc}}{E_{c-1}+E_{d-1}+\sqrt{E_{c-1}}\sqrt{E_{d-1}}}$$

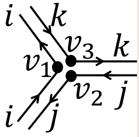
From $-V\mathrm{tr}\sqrt{\lambda}M\Phi^3 = -V\sqrt{\lambda}\sum_{k,l,m=1}^{N}\sqrt{E_{k-1}}\Phi_{kl}\Phi_{lm}\Phi_{mk}$, the vertex weight of the three-point interaction is determined:

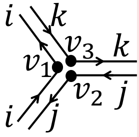


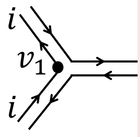
$$= -V\sqrt{\lambda E_{a-1}}$$

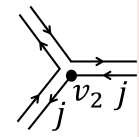
- The black dot v corresponds to $\sqrt{E_{a-1}}$.

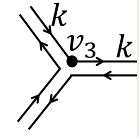
we use the following notation:

$\sum_{v\in\{\{v_1,v_2,v_3\}\}}$ 

$:= \sum_{v\in\{\{i,j,k\}\}}$ 

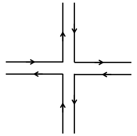
$:=$ 

$+$ 

$+$ 

- $\{\{v_1, v_2, v_3\}\}$ means multi set.

From $-V\mathrm{tr}\dfrac{\lambda}{4}\Phi^4$, the vertex weight of the four-point interaction is

 $= -\dfrac{V\lambda}{4}$

Perturbative Expansion of 1-Point Function $G_{|a|}$ $(\kappa = 0)$

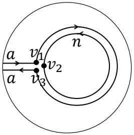
We calculate the connected 1-point function $G_{|a|}$ using perturbative expansion.

$$G_{|a|} = \frac{1}{V} \frac{\partial \log \mathcal{Z}[J]}{\partial J_{aa}} \bigg|_{J=0} = \cdots$$

$$= - V\sqrt{\lambda} \frac{\mathcal{Z}_{free}[0]}{Z[0]} \sum_{m_1, m_3, m_4 = 1}^{N} \sqrt{E_{m_1 - 1}} \langle \Phi_{aa} \Phi_{m_1 m_3} \Phi_{m_3 m_4} \Phi_{m_4 m_1} \rangle_{free} + \mathcal{O}(\lambda\sqrt{\lambda})$$

We compute each term of this expansion by drawing perturbative expansions of the 1-point function $G_{|a|}$ in Feynman diagrams.

$$G_{|a|} = \sum_{n=1}^{N} \sum_{v \in \mathscr{V}}$$



$$+ \mathcal{O}(\lambda\sqrt{\lambda})$$

$$= - \frac{\sqrt{\lambda}}{3E_{a-1}V} \sum_{n=1}^{N} \sum_{v \in \{\{a, a, n\}\}} \frac{\sqrt{E_{v-1}}}{E_{n-1} + E_{a-1} + \sqrt{E_{a-1}E_{n-1}}} + \mathcal{O}(\lambda\sqrt{\lambda})$$

Perturbative Expansion of 1-Point Function $G_{|1|}$ $(\kappa = 0)$

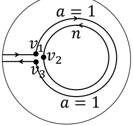
When $N = 2$, we calculate the connected 1-point function $G_{|1|}$ using perturbative expansion.

$$G_{|1|} = \frac{1}{V}\frac{\partial \log \mathcal{Z}[J]}{\partial J_{aa}}\bigg|_{J=0} = \cdots$$

$$= -V\sqrt{\lambda}\frac{\mathcal{Z}_{free}[0]}{Z[0]}\sum_{m_1,m_3,m_4=1}^{2}\sqrt{E_{m_1-1}}\langle \Phi_{aa}\Phi_{m_1m_3}\Phi_{m_3m_4}\Phi_{m_4m_1}\rangle_{free} + \mathcal{O}(\lambda\sqrt{\lambda})$$

We compute each term of this expansion by drawing perturbative expansions of the 1-point function $G_{|1|}$ in Feynman diagrams.

$$G_{|1|} = \sum_{n=1}^{2}\sum_{v\in\mathscr{V}_n}$$



$$= -\frac{\sqrt{\lambda}}{V}\frac{\sqrt{E_0}}{3E_0^2} - \frac{\sqrt{\lambda}}{V}\frac{\sqrt{E_0}}{3E_0}\frac{2}{E_0+E_1+\sqrt{E_0}\sqrt{E_1}}$$

$$-\frac{\sqrt{\lambda}}{V}\frac{\sqrt{E_1}}{3E_0}\frac{1}{E_0+E_1+\sqrt{E_0}\sqrt{E_1}} + \mathcal{O}(\lambda\sqrt{\lambda})$$

Calculation of Partition Function $\mathcal{Z}[J]$

$$\mathcal{Z}[J] = \int \mathcal{D}\Phi \exp\left(-V\mathrm{tr}\left(E\Phi^2 + \kappa\Phi + \frac{1}{2}M\Phi M\Phi + \sqrt{\lambda}M\Phi^3 + \frac{\lambda}{4}\Phi^4\right)\right)$$
$$\times \exp\left(V\mathrm{tr}\left(J\Phi\right)\right)$$

↓

$$\mathcal{Z}[J] = \exp\left(-V\mathrm{tr}\left(\frac{3}{4\lambda}M^3 - \frac{\kappa}{\sqrt{\lambda}}I + \frac{1}{\sqrt{\lambda}}J\right)M\right)$$
$$\int\left(\prod_{i=1}^{N} dx_i \exp\left(-\frac{\lambda V}{4}x_i^4\right)\right)\left(\prod_{1\le k<l\le N}(x_l - x_k)^2\right)$$
$$\int_{U(N)} dU \exp\left(V\mathrm{tr}\left\{\left(\frac{1}{\sqrt{\lambda}}M^3 - \kappa I + J\right)U\widetilde{X}U^*\right\}\right)$$

The integration is divided into diagonal elements and off-diagonal elements.

- $\mathcal{D}X = \prod_{i=1}^{N} dx_i \prod_{1\le k<l\le N}(\mathrm{x}_l - \mathrm{x}_k)^2 dU$

Itzykson-Zuber integral

$$\int_{U(N)} \exp\left(t\,\mathrm{tr}\left(AUBU^*\right)\right) dU = c_N \frac{\det\limits_{1\le i,j\le N}\left(\exp\left(t\lambda_i(A)\lambda_j(B)\right)\right)}{t^{\frac{N^2-N}{2}}\Delta(\lambda(A))\Delta(\lambda(B))}$$

- A, B : Hermitian matrix
- $\lambda_i(A), \lambda_i(B)\ \ i = 1, \cdots, N$: Eigenvelues of A, B
- $dU = \prod\limits_{1\le i<j\le N} (U^* dU)^R_{ij} (U^* dU)^I_{ij}$
- $t \in \mathbb{C}\backslash\{0\}$
- $\Delta(\lambda(A)) := \prod\limits_{1\le i<j\le N} (\lambda_j(A) - \lambda_i(A))$: Vandermonde determinant
- $c_N := \prod\limits_{i=1}^{N-1} i! \times \pi^{\frac{N(N-1)}{2}}$: normalization constant

The off-diagonal elements are integrated using Itzykson-Zuber integral.

$$\int_{U(N)} dU \exp\left(V \mathrm{tr} \left\{ \left(\frac{1}{\sqrt{\lambda}} M^3 - \kappa I + J \right) U \widetilde{X} U^* \right\} \right) = \frac{C}{N!} \frac{\det_{1 \le i,j \le N} \exp(V x_i s_j)}{\prod_{i<j} (x_j - x_i) \prod_{i<j} (s_j - s_i)}$$

The diagonal elements are integrated using a function $P(z)$.

$$\mathcal{Z}[J] = C' \frac{e^{\frac{-V}{\sqrt{\lambda}} \mathrm{tr}(JM)} P_N(s_1, \cdots, s_N)}{\prod_{1 \le t < u \le N} (s_u - s_t)}$$

- s_i : Eigenvelues of $\frac{1}{\sqrt{\lambda}} M^3 - \kappa I + J$
- $P(s_i) = \int_{-\infty}^{\infty} dx \exp\left(-\frac{\lambda V}{4} x^4 + V x s_i \right)$
- $P_N(s_1, \cdots, s_N) = \left(\prod_{1 \le i < j \le N} (\partial_{s_i} - \partial_{s_j}) \right) P(s_1) \cdots P(s_N)$

Calculation of 1-Point Function $G_{|a|}$ $(\kappa = 0)$

In the following, J is treated as a diagonal matrix and $\kappa = 0$. In the calculation of the 1-point function $G_{|a|}$,

$$
\begin{aligned}
G_{|a|} &= \frac{1}{V}\frac{\partial \log \mathcal{Z}[J]}{\partial J_{aa}}\bigg|_{J=0} = \frac{1}{V}\frac{1}{\mathcal{Z}[0]}\frac{\partial \mathcal{Z}[J]}{\partial J_{aa}}\bigg|_{J=0} \\
&= \cdots \\
&= -\frac{\sqrt{E_{a-1}}}{\sqrt{\lambda}} - \frac{1}{V}\sum_{i=1, i\neq a}^{N} \frac{\sqrt{\lambda}}{E_{a-1}\sqrt{E_{a-1}} - E_{i-1}\sqrt{E_{i-1}}} \\
&\quad + \frac{1}{V}\partial_a \log P_N(z_1, \cdots, z_N)
\end{aligned}
$$

- $z_j = \dfrac{E_{j-1}\sqrt{E_{j-1}}}{\sqrt{\lambda}},\ j = 1, \cdots, N$
- $\partial_a = \dfrac{\partial}{\partial z_a}$

Calculation of 1-Point Function $G_{|1|}$ $(\kappa = 0)$

In the following, J is treated as a diagonal matrix and $\kappa = 0$. When $N = 2$, calculating of the 1-point function $G_{|1|}$ is

$$G_{|1|} = \frac{1}{V}\frac{\partial \log \mathcal{Z}[J]}{\partial J_{aa}}\bigg|_{J=0} = \frac{1}{V}\frac{1}{\mathcal{Z}[0]}\frac{\partial \mathcal{Z}[J]}{\partial J_{aa}}\bigg|_{J=0} = \cdots$$
$$= -\frac{\sqrt{E_0}}{\sqrt{\lambda}} + \frac{1}{V}\frac{\partial_1 P_2(z_1, z_2)}{P_2(z_1, z_2)} - \frac{1}{V}\frac{\sqrt{\lambda}}{E_0\sqrt{E_0} - E_1\sqrt{E_1}}.$$

We approximate 1-Point Function $G_{|1|}$ by a saddle point method.

$$G_{|1|} = -\frac{\sqrt{\lambda}}{V}\frac{\sqrt{E_0}}{3E_0^2} - \frac{\sqrt{\lambda}}{V}\frac{\sqrt{E_0}}{3E_0}\frac{2}{E_0 + E_1 + \sqrt{E_0}\sqrt{E_1}}$$
$$- \frac{\sqrt{\lambda}}{V}\frac{\sqrt{E_1}}{3E_0}\frac{1}{E_0 + E_1 + \sqrt{E_0}\sqrt{E_1}} + \mathcal{O}(\lambda\sqrt{\lambda})$$

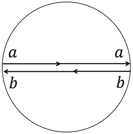
This is consistent with the calculation of the 1-point function $G_{|1|}$ $(N = 2)$ using perturbative expansion.

Perturbative Expansion of 2-Point Function $G_{|ab|}$ $(\kappa = 0)$

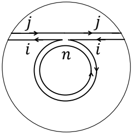
We calculate the connected 2-point function $G_{|ab|}(a \neq b)$ using perturbative expansion.

$$G_{|ab|} = \frac{1}{V} \frac{\partial^2 \log \mathcal{Z}[J]}{\partial J_{ab} \partial J_{ba}} \bigg|_{J=0} = \cdots$$

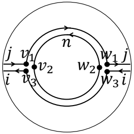
$$= V$$



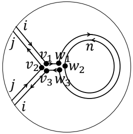
$$+ 4V \sum_{(i,j)\in\{(a,b),(b,a)\}} \sum_{n=1}^{N}$$



$$+ \frac{V}{2!} \sum_{(i,j)\in\{(a,b),(b,a)\}} \sum_{n=1}^{N} \sum_{\omega\in\mathscr{W}} \sum_{v\in\mathscr{V}}$$



$$+ 2 \times \frac{V}{2!} \sum_{(i,j)\in\{(a,b),(b,a)\}} \sum_{n=1}^{N} \sum_{\omega\in\mathscr{W}} \sum_{v\in\mathscr{V}}$$



$$+ \mathcal{O}(\lambda^2)$$

Calculation of 2-Point Function $G_{|ab|}$ $(\kappa = 0)$

In the calculation of the 2-point function $G_{|ab|}$,

$$G_{|ab|} = \frac{1}{V} \frac{\partial^2 \log \mathcal{Z}[J]}{\partial J_{ab} \partial J_{ba}} \bigg|_{J=0} = \cdots$$

$$= \frac{\sqrt{\lambda}}{V(E_{a-1}\sqrt{E_{a-1}} - E_{b-1}\sqrt{E_{b-1}})} \left\{ \frac{\partial_a P_N(z_1, \cdots, z_N) - \partial_b P_N(z_1, \cdots, z_N)}{P_N(z_1, \cdots, z_N)} \right.$$

$$\left. - \sum_{i=1, i \neq a}^{N} \frac{\sqrt{\lambda}}{E_{a-1}\sqrt{E_{a-1}} - E_{i-1}\sqrt{E_{i-1}}} + \sum_{i=1, i \neq b}^{N} \frac{\sqrt{\lambda}}{E_{b-1}\sqrt{E_{b-1}} - E_{i-1}\sqrt{E_{i-1}}} \right\}$$

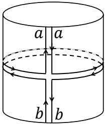
- $z_j = \dfrac{E_{j-1}\sqrt{E_{j-1}}}{\sqrt{\lambda}},\ j = 1, \cdots, N$
- $\partial_a = \dfrac{\partial}{\partial z_a}$

Perturbative Expansion of 2-Point Function $G_{|a|b|}$ $(\kappa = 0)$

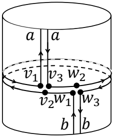
We calculate the connected 2-point function $G_{|a|b|}$ using perturbative expansion.

$$G_{|a|b|} = \left.\frac{\partial^2 \log \mathcal{Z}[J]}{\partial J_{aa} \partial J_{bb}}\right|_{J=0} = \cdots$$

$$= 4V^2$$



$$+ V^2 \sum_{\omega \in \mathscr{W}} \sum_{v \in \mathscr{V}}$$



$$+ \mathcal{O}(\lambda^2)$$

$$= -\frac{\lambda}{9E_{a-1}E_{b-1}\left(E_{a-1} + E_{b-1} + \sqrt{E_{a-1}}\sqrt{E_{b-1}}\right)}$$

$$+ \sum_{\omega \in \mathscr{W}} \sum_{v \in \mathscr{V}} \frac{\sqrt{E_{v-1}}\sqrt{E_{w-1}}\lambda}{9E_{a-1}E_{b-1}\left(E_{a-1} + E_{b-1} + \sqrt{E_{a-1}}\sqrt{E_{b-1}}\right)^2}$$

$$+ \mathcal{O}(\lambda^2),$$

Calculation of 2-Point Function $G_{|a|b|}$ $(\kappa = 0)$

In the calculation of the 2-point function $G_{|a|b|}$,

$$G_{|a|b|} = \left. \frac{\partial^2 \log \mathcal{Z}[J]}{\partial J_{aa} \partial J_{bb}} \right|_{J=0} = \cdots$$

$$= - \frac{\partial_a P_N(z_1, \cdots, z_N)}{P_N(z_1, \cdots, z_N)} \frac{\partial_b P_N(z_1, \cdots, z_N)}{P_N(z_1, \cdots, z_N)} + \frac{\partial_a \partial_b P_N(z_1, \cdots, z_N)}{P_N(z_1, \cdots, z_N)}$$

$$- \frac{\lambda}{(E_{a-1}\sqrt{E_{a-1}} - E_{b-1}\sqrt{E_{b-1}})^2}$$

- $z_j = \dfrac{E_{j-1}\sqrt{E_{j-1}}}{\sqrt{\lambda}}, \; j = 1, \cdots, N$
- $\partial_a = \dfrac{\partial}{\partial z_a}, \;\; \partial_b = \dfrac{\partial}{\partial z_b}$

Calculation of n-Point Function $G_{|a^1|a^2|\cdots|a^n|}$ $(\kappa = 0)$

In the calculation of the n-Point Function $G_{|a^1|a^2|\cdots|a^n|}$,

$$G_{|a^1|a^2|\cdots|a^n|} = V^{n-2}\frac{\partial^n}{\partial J_{a^1 a^1}\cdots\partial J_{a^n a^n}} \log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}\Bigg|_{J=0} = \cdots$$
$$= V^{n-2}\frac{\partial^n}{\partial z_{a^1}\cdots\partial z_{a^n}} \log P_N(z_1, \cdots, z_N)$$

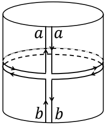
- $z_j = \dfrac{E_{j-1}\sqrt{E_{j-1}}}{\sqrt{\lambda}},\ j = 1, \cdots, N$

Overall Summary and Future Prospect

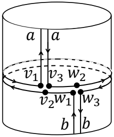
In order to mathematically formulate quantum field theories as a toy model, it is necessary to clarify the properties of the matrix model on noncommutative spaces (Grosse-Wulkenhaar model).

- We constructed Feynman rules for Φ^3-Φ^4 Hybrid-Matrix-Model and calculated the perturbative expansion in ordinary methods.

$G_{|a|b|} = 4V^2$



$+ V^2 \sum_{\omega \in \mathscr{W}} \sum_{v \in \mathscr{V}}$



$+ \mathcal{O}(\lambda^2)$

- We calculated the path integral of the partition function $\mathcal{Z}[J]$ and used the result to compute exact solutions for 1-point function $G_{|a|}$ with 1-boundary, 2-point function $G_{|ab|}$ with 1-boundary, 2-point function $G_{|a|b|}$ with 2-boundaries, and n-point function $G_{|a^1|a^2|\cdots|a^n|}$ with n-boundaries.

- In the future, we would like to clarify the solvability of Φ^4 matrix model (Grosse-Wulkenhaar model).