Fermionic CFTs from classical codes

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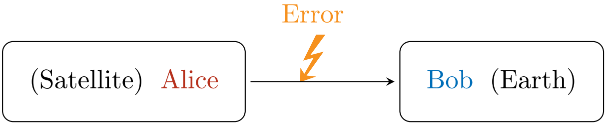
Based on [2303.11613] with S. Yahagi (UTokyo)

Error-correcting codes

Let us consider the situation when

- Alice in satellite wants to transmit information to the Earth.
- Bob on Earth wants to receive the information without error.



$$0 \longmapsto \begin{cases} 0 & \text{(without error)} \\ 1 & \text{(with error)} \end{cases}$$

The problem can be solved by error-correcting codes.

[Hamming, 1947] [Shannon, 1948] [Golay, 1949]

Alice adds some redundancy to the original message,
which Bob can use to check consistency and recover the message.

Aspects of Golay code

The Golay code was discovered in 1949 and can correct up to 3 bit errors.

It plays an important role in engineering, mathematics, and physics.

Engineering Spacecraft Voyager 1&2 (1977) transmitted information from Jupiter and Saturn using Golay code.

Math Golay code contains Mathieu group symmetry, which inspires
- understanding of monstrous moonshine phenomena
- development of vertex operator algebra.

[Conway-Norton, 1979] [Borcherds, 1992]

Physics The CFT from the Golay code was conjectured to be dual to pure AdS_3 gravity. [Witten, 0706.3359]

Furthermore, stimulated by the Golay code case,

construction of CFTs from codes has been developed.

History of codes and CFTs

The construction of CFTs from classical codes has a long history:

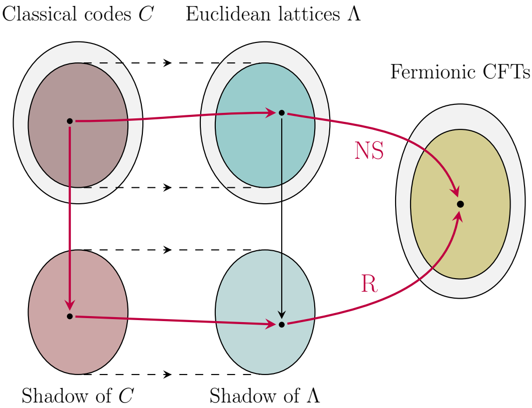
- classical binary $(0,1)$ codes to construct chiral bosonic CFTs.
  [Frenkel, Lepowsky, Meurman, 1984]
  [Dolan, Goddard, Montague, hep-th/9410029]

- classical ternary $(0,1,2)$ codes to construct chiral fermionic CFTs.
  [Gaiotto, Johnson-Freyd, 1811.00589]

In terms of coding theory, it seems natural to
generalize the construction to p-ary $(0,1,2,\cdots,p-1)$ codes.

We will construct chiral fermionic CFTs from classical p-ary codes.
[KK, Yahagi, 2303.11613]

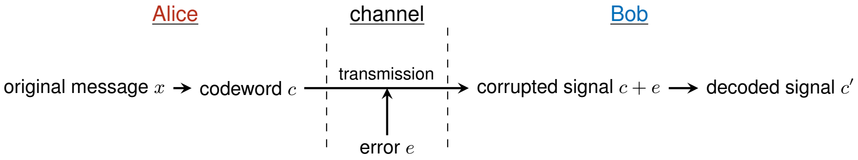
Goal of this talk



Error-correcting codes

Error-correcting codes

a framework that protects the original message from noise
by translating the message into a signal with redundancy.



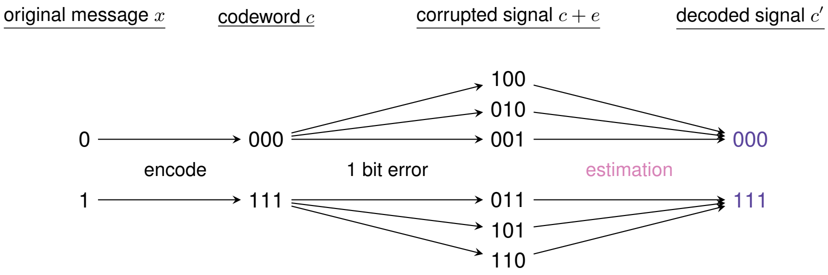
- Alice encodes a message to a codeword with redundancy.
- Bob has to decide which codeword was transmitted from $c+e$.
- Bob's strategy is an estimation by choosing the most likely error e.

Example: Repetition code

Consider a binary code that only repeats an original message three times:

$$\text{encoding} \;:\; \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} \mapsto \left\{ \begin{matrix} 000 \\ 111 \end{matrix} \right\}$$

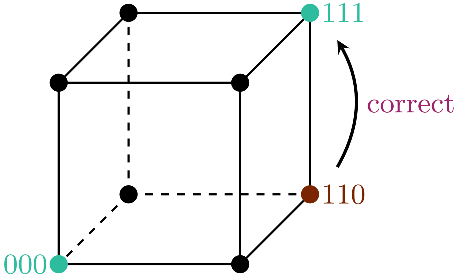
- Assume that an error occurs with a relatively small probability.
- We can expect only one error out of three bits to occur.



- A codeword transmitted can be estimated by taking the majority vote.

Geometric structure of error correction

Let us show the geometry $\mathbb{F}_2^3 = \{0,1\}^3$ and code $C = \{000, 111\}$.



- Estimate transmitted codeword by choosing the closest codeword.
- If you receive $(\mathbf{110})$, the original codeword is most likely to be $(\mathbf{111})$.

"Error-correcting code as subspace of vector space $\mathbb{F}_p^n$"

Construction of Euclidean lattices

Construction A [Leech-Sloane, 1971]

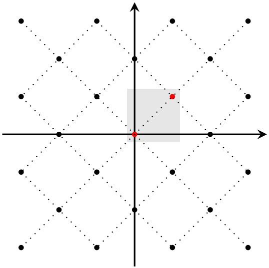
A classical code $C \in \mathcal{M}_p$ gives an odd lattice

$$\Lambda(C) = \frac{C + p\,\mathbb{Z}^n}{\sqrt{p}}$$

(Example)

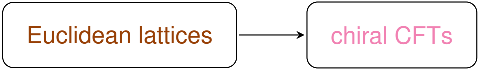
For a binary code $C = \{00, 11\}$,

$$\Lambda(C) \cong \mathbb{Z}^2\,.$$



Chiral CFTs from lattices

Chiral CFTs can be constructed by using chiral boson $X^i(z)$.



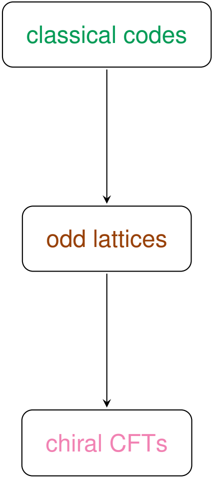
- fundamental operators:
  - $\partial X^i(z)$, $T(z) = \partial X(z) \cdot \partial X(z)$.
  - vertex operators $V_{\lambda}(z) = e^{i\lambda \cdot X(z)}$ $(\lambda \in \Lambda)$.
- Using bosonic oscillators α^i_k, the Hilbert space is

$$\mathcal{H}(\Lambda) = \left\{ \alpha^{i_1}_{-k_1} \cdots \alpha^{i_r}_{-k_r} \left|\lambda\right\rangle \;\middle|\; \lambda \in \Lambda\,,\; r \in \mathbb{Z}_{\geq 0} \right\} \,.$$

Mathematicians formulate chiral CFTs as vertex operator algebra (VOA).

Above construction is an important class called lattice VOA.

Chiral CFTs from classical codes



A classical code $C \in \mathcal{M}_p$ provides

$$\Lambda(C) = \frac{C + p\,\mathbb{Z}^n}{\sqrt{p}}\,.$$

We can decompose $\Lambda(C)$ into $\Lambda(C) = \Lambda_0 \cup \Lambda_2$

$$\Lambda_0 = \{\lambda \in \Lambda(C) \mid \lambda\cdot\lambda : \text{even}\}\,,$$
$$\Lambda_2 = \{\lambda \in \Lambda(C) \mid \lambda\cdot\lambda : \text{odd}\}\,.$$

The chiral CFT has the Hilbert space

$$\mathcal{H}(\Lambda(C)) = \left\{ \alpha^{i_1}_{-k_1}\cdots\alpha^{i_r}_{-k_r}\,|\lambda\rangle \;\middle|\; \lambda \in \Lambda(C) \right\}\,,$$
$$= \mathcal{H}(\Lambda_0) \cup \mathcal{H}(\Lambda_2)\,.$$

Fermionic CFTs from classical codes

Let us consider the spin of the state

$$\alpha^{i_1}_{-k_1} \cdots \alpha^{i_r}_{-k_r} |\lambda\rangle \qquad (\lambda \in \Lambda(C))$$

For an odd lattice $\Lambda(C)$,

$$\text{spin} = \frac{\lambda^2}{2} + \sum_{j=1}^{r} k_j \in \begin{cases} \mathbb{Z} & (\lambda \in \Lambda_0) \\ \mathbb{Z} + \frac{1}{2} & (\lambda \in \Lambda_2) \end{cases}$$

From the spin-statistics theorem,

$$\mathcal{H}(\Lambda(C)) = \underbrace{\mathcal{H}(\Lambda_0)}_{\text{boson}} \cup \underbrace{\mathcal{H}(\Lambda_2)}_{\text{fermion}}$$

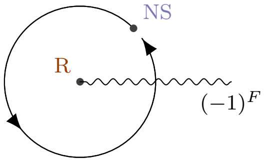
An odd lattice $\Lambda(C)$ yields the NS sector of a fermionic CFT.

Ramond sector

One characterization of the R sector is

"R sector operators are non-local operators attached to $(-1)^F$ line."

As an NS operator goes around a R sector operator,
the NS operator receives the action of $(-1)^F$.



Let us construct the Ramond sector of a fermionic code CFT.

Shadow

characteristic vector

An element $\chi \in \Lambda$ is called characteristic if it satisfies

$$\chi \cdot \lambda = \lambda \cdot \lambda \mod 2 \qquad \text{for all } \lambda \in \Lambda \,.$$

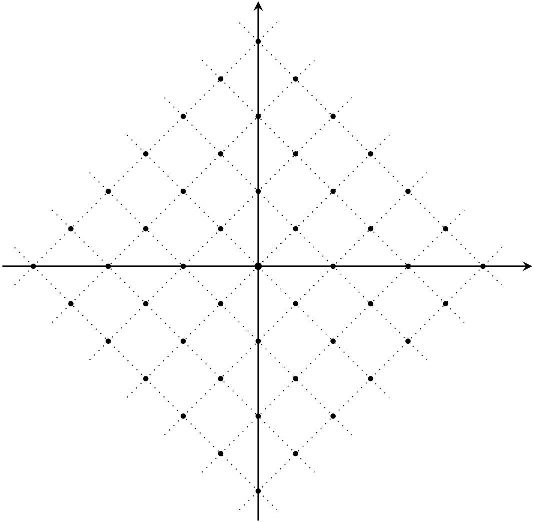
Let us introduce the following half-shift:

$$S(\Lambda(C)) = \Lambda(C) + \frac{\chi}{2} \,.$$

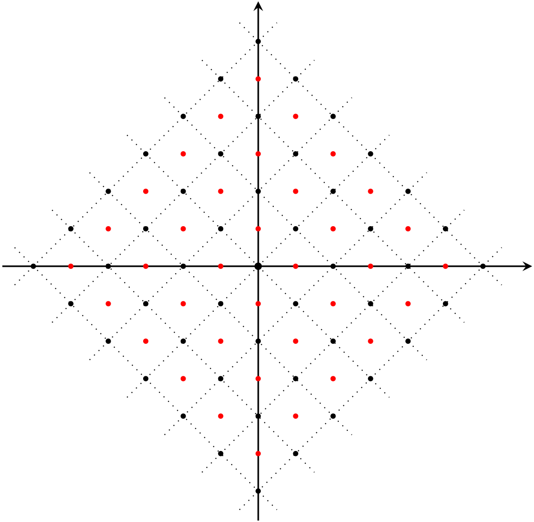
This is a special half-shift called the shadow of $\Lambda(C)$ in math.

[Conway-Sloane, 1990]

Example of shadow



Example of shadow



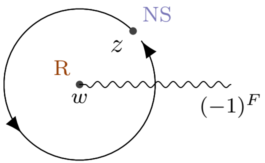
Ramond sector from shadow

Let us construct the R sector from the shadow $S(\Lambda(C))$ by

$$V_{\boldsymbol{\lambda}+\frac{\boldsymbol{\chi}}{2}}(z) = e^{\mathrm{i}(\boldsymbol{\lambda}+\frac{\boldsymbol{\chi}}{2})\cdot X(z)} \qquad (\lambda \in \Lambda(C))\,.$$

Pick up one of them and move it around the other.

$$\underbrace{V_\lambda(z)}_{\substack{\text{boson:}\lambda\in\Lambda_0\\ \text{fermion:}\lambda\in\Lambda_2}} V_{\lambda'+\frac{\chi}{2}}(w) \to (-1)^{\boldsymbol{\chi}\cdot\boldsymbol{\lambda}}\, V_\lambda(z)\, V_{\lambda'+\frac{\chi}{2}}(w)$$



$$(-1)^{\chi\cdot\lambda} = \begin{cases} +1 & (\lambda \in \Lambda_0) \\ -1 & (\lambda \in \Lambda_2) \end{cases} = (-1)^{\mathbf{F}}$$

NS operator receives the action $(-1)^{\mathbf{F}}$.

Example

Let us consider a binary code of length 36 generated by

$$
G = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
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\end{bmatrix}
$$

[Harada-Munemasa, "Database of self-dual codes"]

Example

(I) The NS partition function gives the expansion

$$Z_{\mathrm{NS}}(\tau) = q^{-\frac{3}{2}} + 108q^{-\frac{1}{2}} + \underbrace{\mathbf{1536}}_{\text{number of spin-3/2 primaries}} + 63414\,q^{\frac{1}{2}} + 2064384\,q + \cdots .$$

(II) The R sector satisfies the energy bound

$$h_R \geq \frac{c}{24} = \frac{3}{2} . \qquad \text{(unitarity bound for supersymmetry)}$$

(III) The RR partition function becomes constant (Witten index)

$$Z_{\widetilde{\mathrm{R}}}(\tau) = \mathrm{Tr}_{\mathrm{R}}\left[(-1)^F\, q^{L_0 - \frac{c}{24}}\right] = \mathbf{384} .$$

The chiral fermionic CFT with central charge 36 is very likely to be supersymmetric.

Summary

We can systematically construct chiral fermionic CFTs from classical codes.

- The NS sector is directly related to Construction A lattice from code.
- The R sector comes from the shadow of lattices and codes.

Remark on another direction

Construction of *non-chiral* CFTs from *quantum codes*.

There are several recent works:

[Dymarsky-Shapere, 2009.01244]
[Yahagi, 2203.10848]
[Furuta, 2203.11643, 2307.04190]
[KK-Nishioka-Okuda, 2212.0708]

The program started in 2020, so further development is expected.