Correlation functions involving Dirac fields from homotopy algebras

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based on arXiv 2305.11634 **K.K.**, Y. Okawa
arXiv 2305.13103 **K.K.**

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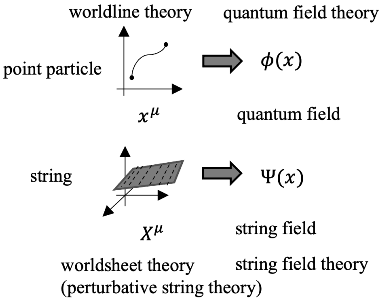
Motivations

We want to know the non-perturbative definition of string theory!

Candidates

- matrix models
- string field theory

etc....



Constructing string field theory

How to construct the string field theory?

To construct consistent theory, we need to introduce the infinite numbers of interaction terms in the action (except several examples).

Homotopy algebras such as A_∞ algebras and L_∞ algebras are related to the Batalin-Vilkovisky formalism, which is one of the method of the path integral quantization of gauge theories, and have contributed to the construction of the action of string field theory.

e.g.) hep-th/9206084, B.Zwiebach

Are there other things we can do using homotopy algebras?

Homotopy algebras and string field theory

There are many things we can do using homotopy algebras!

- relating covariant and light-cone string field theories

  [Erler and Matsunaga, arXiv:2012.09521]

- calculating scattering amplitudes

  [Kajiura, math/0306332] etc.

- integrating out fields

  [Sen, arXiv:1609.00459],

  [Erbin, Maccaferri, Schnabl and Vošmera, arXiv:2006.16270],

  [Koyama, Okawa and Suzuki, arXiv:2006.16710] etc.

etc....

In particular, homotopy algebras reproduces the ordinary calculations of Feynman diagrams.

It is difficult to deal with the string field theory, however, descriptions using homotopy algebras are systematic, so we would use this to exploit the string field theory.

Homotopy algebras and quantum field theory

We can also use homotopy algebras to describe the quantum field theory.

In fact, the descriptions are essentially the same (universal) in any theory.

Therefore, exploring the descriptions of quantum field theory may help to understand string field theory!

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A_∞ algebras

We consider the vector space $\mathcal{H}$. It is decomposed as

$$\begin{aligned} \mathcal{H} &= \ldots \oplus \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \ldots \\ &= \bigoplus_{i \in \mathbb{Z}} \mathcal{H}_i \,. \end{aligned}$$

The space $\mathcal{H}$ is usually the space of (string) fields.
In string field theory, i is the ghost number of string fields.
We denote the degree of Φ by deg (Φ):

$$\deg(\Phi) = \begin{cases} 0 & (\Phi : \text{ degree even}) \qquad (\text{mod } 2) \\ 1 & (\Phi : \text{ degree odd}) \qquad (\text{mod } 2) \,. \end{cases}$$

We consider an action of the form

$$S = -\frac{1}{2}\,\omega(\Phi, Q\,\Phi) - \sum_{n=0}^{\infty} \frac{1}{n+1}\,\omega(\Phi, m_n(\Phi \otimes \ldots \otimes \Phi))\,.$$

The classical action is written in terms of degree-even elements of $\mathcal{H}_1$.

A_∞ algebras

$$S = -\frac{1}{2}\,\omega\,(\,\Phi, Q\,\Phi\,) - \sum_{n=0}^{\infty} \frac{1}{n+1}\,\omega\,(\,\Phi\,, m_n\,(\,\Phi \otimes \ldots \otimes \Phi\,)\,)$$

the operator Q : degree-odd map from $\mathcal{H}$ to $\mathcal{H}$
the operators m_n : degree-odd maps from $\mathcal{H}^{\otimes n}$ to $\mathcal{H}$

$$\mathcal{H}^{\otimes n} = \underbrace{\mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}}_{n}$$

for $n > 0$.
The space $\mathcal{H}^{\otimes 0}$ is a one-dimensional vector space equipped with a single basis vector $\mathbf{1}$ which is degree even and satisfies

$$\mathbf{1} \otimes \Phi = \Phi\,, \quad \Phi \otimes \mathbf{1} = \Phi$$

for any Φ in $\mathcal{H}$, and elements of $\mathcal{H}^{\otimes 0}$ are given by multiplying $\mathbf{1}$ by complex numbers.

A_∞ algebras

The symplectic form :

$$\omega\,(\,\Phi_1\,,\Phi_2\,) = \,-\,(-1)^{\deg(\Phi_1)\deg(\Phi_2)}\,\omega\,(\,\Phi_2\,,\Phi_1\,)\,.$$

The following A_∞ *relations*:

$$(Q+m_1)(m_0(\mathbf{1}))\;=\;0\,,$$

$$(Q+m_1)((Q+m_1)(\Phi_1)) + m_2(m_0(\mathbf{1})\otimes\Phi_1) + (-1)^{\deg(\Phi_1)}m_2(\Phi_1\otimes m_0(\mathbf{1}))\;=\;0\,,$$

$$\cdots\,,$$

The cyclic properties:

$$\omega\,(\,\Phi_1\,,Q\,(\,\Phi_2\,)\,) = \,-\,(-1)^{\deg(\Phi_1)}\,\omega\,(\,Q(\,\Phi_1\,)\,,\Phi_2\,)\,,$$

$$\omega\,(\,\Phi_1\,,M_n\,(\,\Phi_2\otimes\ldots\otimes\Phi_{n+1}\,)\,) = -(-1)^{\deg(\Phi_1)}\,\omega\,(\,M_n(\,\Phi_1\otimes\ldots\otimes\Phi_n\,)\,,\Phi_{n+1}\,)\,,$$

$$\Phi_1,\ldots,\Phi_n,\Phi_{n+1}\in\mathcal{H}$$

Then, we call this algebra a cyclic A_∞ *algebra*.

coalgebra representation

It is convenient to use the *coalgebra representation*.
In the coalgebra representation, we consider linear operators acting on $T\mathcal{H}$ defined by

$$T\mathcal{H} = \mathcal{H}^{\otimes 0} \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \mathcal{H}^{\otimes 3} \oplus \ldots .$$

We introduce the coderivations $\mathbf{Q}$ and $\boldsymbol{m}_n$ associated with Q and m_n, respectively.

We define $\boldsymbol{m}$ by

$$\boldsymbol{m} = \sum_{n=0}^{\infty} \boldsymbol{m}_n .$$

The action is described by the coderivation $\mathbf{Q} + \boldsymbol{m}$, and the gauge invariance of the action follows from the relation

$$(\mathbf{Q} + \boldsymbol{m})^2 = 0 .$$

projection operators

When we consider physics in terms of homotopy algebras, we usually consider degree-even projection operator P from $\mathcal{H}$ to its subspace.

- calculating scattering amplitudes
  P is defined to be on-shell region.
- integrating out fields
  P is defined to be unintegrated region.
  e.g.) If we want to calculate effective action for massless sector, we take P to be massless sector.

P is defined to satisfy the following relations:

$$P^2 = P\,, \qquad P\,Q = Q\,P\,.$$

In the coalgebra representation, we use the projection operator **P** acting on $T\mathcal{H}$.

contracting homotopy

We also introduce the *contracting homotopy h* which is a degree-odd map from $\mathcal{H}$ to $\mathcal{H}$ and satisfies

$$Q\,h + h\,Q = \mathbb{I} - P\,, \quad h\,P = 0\,, \quad P\,h = 0\,, \quad h^2 = 0\,.$$

Roughly, the contracting homotopy h is the propagator.
We then promote h to the linear operator $\boldsymbol{h}$ acting on $T\mathcal{H}$.

The last ingredient to describe the formula for correlation functions is the operator $\mathbf{U}$. The operator $\mathbf{U}$ is normalized by

$$(\,\omega \otimes \mathbb{I}\,)\,(\,\mathbb{I} \otimes U\,) = \mathbb{I}\,,$$

where U is a map from $\mathcal{H}^{\otimes 0}$ to $\mathcal{H}^{\otimes 2}$ given by

$$U = \pi_2\,\mathbf{U}\,\pi_0$$

and ω is a map from $\mathcal{H}^{\otimes 2}$ to $\mathcal{H}^{\otimes 0}$ with

$$\omega\,(\,\Phi_1 \otimes \Phi_2\,) = \omega\,(\,\Phi_1\,,\,\Phi_2\,)\,\mathbf{1}$$

for Φ_1 and Φ_2 in $\mathcal{H}$.

How to reproduce Feynman diagrams

The A_∞ structure of the action is written by

$$\pi_1 \left(\boldsymbol{Q} \, + \, \boldsymbol{m} \right).$$

When we consider tree-level on-shell amplitudes (effective action), we use the projection onto on-shell (unintegrated) region.
Then, we can calculate them using

$$\pi_1 \, \mathbf{P} \, \boldsymbol{Q} \, \mathbf{P} + \pi_1 \, \mathbf{P} \, \boldsymbol{m} \, \frac{1}{\mathbf{I} \, + \, \boldsymbol{h} \, \boldsymbol{m}} \, \mathbf{P} \, ,$$

which preserves the A_∞ structure.
When we consider loop-corrections, we use

$$\pi_1 \, \mathbf{P} \, \boldsymbol{Q} \, \mathbf{P} + \pi_1 \, \mathbf{P} \, \boldsymbol{m} \, \frac{1}{\mathbf{I} \, + \, \boldsymbol{h} \, \boldsymbol{m} \, + i\hbar \, \boldsymbol{h} \, \mathbf{U}} \, \mathbf{P} \, ,$$

if we ignore the subtlety.
Mathematically, this operation correspond to transfer one A_∞ algebra to another A_∞ algebra.

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formula for correlation functions

The A_∞ structure of the action is written by

$$\pi_1 \left(\boldsymbol{Q} + \boldsymbol{m} \right).$$

where π_n is the projection from $T\mathcal{H}$ onto $\mathcal{H}^{\otimes n}$.

When we consider effective action, we use the projection onto unintegrated region. Then, we can calculate them using

$$\pi_1 \, \mathbf{P} \, \boldsymbol{Q} \, \mathbf{P} + \pi_1 \, \mathbf{P} \; \boldsymbol{m} \, \frac{1}{\mathbf{I} + \boldsymbol{h} \, \boldsymbol{m} + i\hbar \, \boldsymbol{h} \, \mathbf{U}} \, \mathbf{P} \, .$$

This means we can integrate the region projected by $\mathbb{I} - P$.
If we want to integrate all the region, we need to take $P = 0$??

In that case, the above operator after the homological perturbation becomes trivial.

Is it meaningless to take $P = 0$?

formula for correlation functions

Notice that if $P = 0$,

$$\mathbf{P} = \pi_0 \neq 0\,.$$

Then, the below red part is non-zero.

$$\mathbf{P}\,\boldsymbol{Q}\,\mathbf{P} + \mathbf{P}\;\boldsymbol{m}\;\frac{1}{\mathbf{I} + \boldsymbol{h}\;\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}}\,\mathbf{P}\,.$$

In fact, the red part contains the information of correlation functions.

[Okawa, arXiv:2203.05366]

Correlation functions are given by

$$\langle\, \Phi^{\otimes n}\,\rangle = \pi_n\,\boldsymbol{f}\;\mathbf{1}$$

with

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\;\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}}\,.$$

$$\Phi^{\otimes n} = \underbrace{\Phi \otimes \Phi \otimes \ldots \otimes \Phi}_{n}\,,$$

$$\Phi = \int d^d x\, \varphi(x)\, c(x) + \int d^d x\, (\,\overline{\theta}_\alpha(x)\, \Psi_\alpha(x) + \overline{\Psi}_\alpha(x)\, \theta_\alpha(x)\,)\,.$$

Does this formula really reproduce the correlation functions in ordinary quantum field theory?

the Schwinger-Dyson equations

Does this formula really reproduce the correlation functions in ordinary quantum field theory?

Correlation functions are solutions of the Schwinger-Dyson equation such as

$$\sum_{i=1}^{n-1} \delta^d(x_i - x_n) \langle \varphi(x_1) \ldots \varphi(x_{i-1}) \varphi(x_{i+1}) \ldots \varphi(x_{n-1}) \rangle + \frac{i}{\hbar} \langle \varphi(x_1) \ldots \varphi(x_{n-1}) \frac{\delta S}{\delta \varphi(x_n)} \rangle = 0 \,.$$

the Schwinger-Dyson equations

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We do not discuss the detailed proof, but we can directly prove that correlation functions from our formula satisfy the Schwinger-Dyson equations using the trivial identity:

$$(\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}) \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}} \mathbf{1} = \mathbf{1} \,,$$

where

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}} \,.$$

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Problems to deal with fermions

In the usual framework, we consider the real scalar field $\varphi(x)$ and take

$$\Phi = \varphi(x)\,.$$

With the appropriate definition, we can calculate the correlation functions as follows: [Okawa, arXiv:2203.05366]

$$\begin{aligned}&\langle\, \varphi(x_1)\, \varphi(x_2)\, \ldots\, \varphi(x_n)\,\rangle \\ &\qquad = \omega_n\,(\,\pi_n\, \boldsymbol{f}\, \mathbf{1}\,, \delta^d(x-x_1) \otimes \delta^d(x-x_2) \otimes \ldots \otimes \delta^d(x-x_n)\,)\,,\end{aligned}$$

where

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\, \boldsymbol{m} + i\hbar\, \boldsymbol{h}\, \mathbf{U}}\,.$$

To extend this description, we naively take

$$\Phi \sim \Psi(x)\,,$$

but this description is the same as that of scalar field theory, and we cannot describe the antisymmetry of the Dirac fields under the exchange of fermions.

Problems to deal with fermions

To resolve this problem, there are two approaches.

- super A_∞ algebras
  In addition to the grading from A_∞ algebras, we introduce the $\mathbb{Z}_2$ grading from the super vector space to distinguish bosons and fermions.
- introducing string-field-theory-like field
  In open superstring field theory, string field Φ is described by degree-even string fields in $\mathcal{H}_1$, but Φ is schematically expanded as

$$\Phi = \underbrace{\sum_i \int d^{10}k\, \varphi_i(k)\,|\, i\,;k\,\rangle}_{\text{degree even} \times \text{degree even}} + \underbrace{\sum_\alpha \int d^{10}k\, \psi_\alpha(k)\,|\,\alpha\,;k\,\rangle}_{\text{degree odd} \times \text{degree odd}},$$

  where $\varphi_i(k)$ are bosonic fields and $\psi_\alpha(k)$ are fermionic fields with i and α collectively labeling various fields.

We use the latter approach and introduce string-field-theory-like field to describe Dirac fields.

Dirac field theory

Let us describe Dirac fields using A_∞ algebras.
We consider theories without gauge symmetries so the vector space $\mathcal{H}$ is given by

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \,.$$

We define the degree-odd basis vector of $\mathcal{H}_1$ by $\theta_\alpha(x)$.
We also use the Dirac adjoint $\overline{\theta}_\alpha(x)$ of $\theta_\alpha(x)$.

The element Φ of $\mathcal{H}_1$ can be expanded in this basis as

$$\Phi = \int d^d x \, (\, \overline{\theta}_\alpha(x) \, \Psi_\alpha(x) + \overline{\Psi}_\alpha(x) \, \theta_\alpha(x) \,) \,,$$

where $\Psi_\alpha(x)$ is the Dirac field we define $\Psi_\alpha(x)$ to be degree odd. In this expansion, $\overline{\Psi}_\alpha(x)$ has to be the Dirac adjoint of $\Psi_\alpha(x)$.
Note that Φ is degree even.

For the vector space $\mathcal{H}_2$, we define the degree-even basis vector by $\lambda_\alpha(x)$.

Dirac field theory

We then define the following operators:

$$Q\,\theta_\alpha(x) = (\,-i\,\partial\!\!\!/ + m\,)_{\alpha\beta}\,\lambda_\beta(x)\,, \quad Q\,\lambda_\alpha(x) = 0\,,$$

$$Q\,\overline{\theta}_\alpha(x) = -\,\overline{\lambda}_\beta(x)\,(\,i\,\overleftarrow{\partial\!\!\!/} + m\,)_{\beta\alpha}\,, \quad Q\,\overline{\lambda}_\alpha(x) = 0\,,$$

$$\begin{aligned}
\omega\,(\,\theta_{\alpha_1}(x_1)\,,\overline{\lambda}_{\alpha_2}(x_2)\,) &= \delta_{\alpha_1\alpha_2}\,\delta^d(x_1 - x_2)\,,\\
\omega\,(\,\overline{\theta}_{\alpha_1}(x_1)\,,\lambda_{\alpha_2}(x_2)\,) &= \delta_{\alpha_1\alpha_2}\,\delta^d(x_1 - x_2)\,,\\
\omega\,(\,\overline{\lambda}_{\alpha_1}(x_1)\,,\theta_{\alpha_2}(x_2)\,) &= -\,\delta_{\alpha_1\alpha_2}\,\delta^d(x_1 - x_2)\,,\\
\omega\,(\,\lambda_{\alpha_1}(x_1)\,,\overline{\theta}_{\alpha_2}(x_2)\,) &= -\,\delta_{\alpha_1\alpha_2}\,\delta^d(x_1 - x_2)\,,
\end{aligned}$$

and the symplectic form vanishes for all other cases.
Then, we obtain

$$\begin{aligned}
S &= \int d^d x\,\big[\,i\,\overline{\Psi}(x)\,\partial\!\!\!/\,\Psi(x) - m\,\overline{\Psi}(x)\,\Psi(x)\,\big]\\
&= -\,\frac{1}{2}\,\omega\,(\,\Phi,\,Q\,\Phi\,)\,.
\end{aligned}$$

When we calculate correlation functions, we consider the projection with

$$P = 0\,.$$

Dirac field theory

Then, the contracting homotopy is constructed as follows:

$$h\,\theta_\alpha(x) = 0\,, \qquad h\,\lambda_\alpha(x) = \int d^d y\, S(x-y)_{\alpha\beta}\,\theta_\beta(y)\,,$$
$$h\,\overline{\theta}_\alpha(x) = 0\,, \qquad h\,\overline{\lambda}_\alpha(x) = -\int d^d y\,\overline{\theta}_\beta(y)\,S(y-x)_{\beta\alpha}\,,$$

where $S(x-y)_{\alpha\beta}$ is the Dirac propagator.
The operator **U** is defined by

$$\mathbf{U} = -\int d^d x\,(\,\overline{\boldsymbol{\theta}}_\alpha(x)\,\boldsymbol{\lambda}_\alpha(x) + \boldsymbol{\theta}_\alpha(x)\,\overline{\boldsymbol{\lambda}}_\alpha(x)\,)\,,$$

where $\boldsymbol{\theta}_\alpha(x)$, $\overline{\boldsymbol{\theta}}_\alpha(x)\,\boldsymbol{\lambda}_\alpha(x)$ and $\overline{\boldsymbol{\lambda}}_\alpha(x)$ are coderivations with

$$\pi_1\,\boldsymbol{\theta}_\alpha(x)\,\mathbf{1} = \theta_\alpha(x)\,, \quad \pi_1\,\boldsymbol{\theta}_\alpha(x)\,\pi_n = 0\,, \quad \pi_1\,\overline{\boldsymbol{\theta}}_\alpha(x)\,\mathbf{1} = \overline{\theta}_\alpha(x)\,, \quad \pi_1\,\overline{\boldsymbol{\theta}}_\alpha(x)\,\pi_n = 0$$

$$\pi_1\,\boldsymbol{\lambda}_\alpha(x)\,\mathbf{1} = \lambda_\alpha(x)\,, \quad \pi_1\,\boldsymbol{\lambda}_\alpha(x)\,\pi_n = 0\,, \quad \pi_1\,\overline{\boldsymbol{\lambda}}_\alpha(x)\,\mathbf{1} = \overline{\lambda}_\alpha(x)\,, \quad \pi_1\,\overline{\boldsymbol{\lambda}}_\alpha(x)\,\pi_n = 0$$

for $n > 0$.

Dirac field theory

We claim that the formula for correlation functions takes the same form as in the scalar field theory when it is expressed in terms of Φ:

$$\langle \Phi^{\otimes n} \rangle = \pi_n \, \boldsymbol{f} \, \mathbf{1} \,,$$

where

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\, \boldsymbol{m} + i\hbar\, \boldsymbol{h}\, \mathbf{U}} \,,$$

$$\Phi = \int d^d x \, (\bar{\theta}_\alpha(x) \, \Psi_\alpha(x) + \overline{\Psi}_\alpha(x) \, \theta_\alpha(x)) \,.$$

$$\begin{aligned} \langle \Phi \otimes \Phi \rangle = \int d^d x_1 \, d^d x_2 \Big[& - \langle \Psi_{\alpha_1}(x_1) \, \Psi_{\alpha_2}(x_2) \rangle \, \bar{\theta}_{\alpha_1}(x_1) \otimes \bar{\theta}_{\alpha_2}(x_2) \\ & + \langle \Psi_{\alpha_1}(x_1) \, \overline{\Psi}_{\alpha_2}(x_2) \rangle \, \bar{\theta}_{\alpha_1}(x_1) \otimes \theta_{\alpha_2}(x_2) \\ & + \langle \overline{\Psi}_{\alpha_1}(x_1) \, \Psi_{\alpha_2}(x_2) \rangle \, \theta_{\alpha_1}(x_1) \otimes \bar{\theta}_{\alpha_2}(x_2) \\ & - \langle \overline{\Psi}_{\alpha_1}(x_1) \, \overline{\Psi}_{\alpha_2}(x_2) \rangle \, \theta_{\alpha_1}(x_1) \otimes \theta_{\alpha_2}(x_2) \Big] \,. \end{aligned}$$

Dirac field theory

Then, we can extract the correlation functions, for example,

$$\langle\, \Psi_{\alpha_1}(x_1)\, \overline{\Psi}_{\alpha_2}(x_2)\,\rangle = \omega_2\,(\,\pi_2\, \boldsymbol{f}\, \mathbf{1}\,, \lambda_{\alpha_1}(x_1) \otimes \overline{\lambda}_{\alpha_2}(x_2)\,)\,,$$
$$\langle\, \overline{\Psi}_{\alpha_1}(x_1)\, \Psi_{\alpha_2}(x_2)\,\rangle = \omega_2\,(\,\pi_2\, \boldsymbol{f}\, \mathbf{1}\,, \overline{\lambda}_{\alpha_1}(x_1) \otimes \lambda_{\alpha_2}(x_2)\,)\,,$$

where

$$\omega_n\,(\,\Phi_1 \otimes \Phi_2 \otimes \ldots \otimes \Phi_n\,, \widetilde{\Phi}_1 \otimes \widetilde{\Phi}_2 \otimes \ldots \otimes \widetilde{\Phi}_n\,) = \prod_{i=1}^{n}\, \omega\,(\,\Phi_i\,, \widetilde{\Phi}_i\,)\,.$$

The correlation function for example, can be extracted as

$$\langle\, \Psi_{\alpha_1}(x_1)\, \ldots \Psi_{\alpha_n}(x_n)\, \overline{\Psi}_{\beta_1}(y_1)\, \ldots \overline{\Psi}_{\beta_n}(y_n)\,\rangle$$
$$= \omega_{2n}\,(\,\pi_{2n}\, \boldsymbol{f}\, \mathbf{1}\,, \lambda_{\alpha_1}(x_1) \otimes \ldots \otimes \lambda_{\alpha_n}(x_n) \otimes \overline{\lambda}_{\beta_1}(y_1) \otimes \ldots \otimes \overline{\lambda}_{\beta_n}(y_n)\,)\,.$$

Dirac field theory

Let us calculate two-point functions.
The two-point functions can be calculated from $\pi_2 \, \boldsymbol{f} \, \mathbf{1}$.
For the free theory, it is given by

$$\pi_2 \, \boldsymbol{f} \, \mathbf{1} \; = \; - i\hbar \, \pi_2 \, \boldsymbol{h} \, \mathrm{U} \, \mathbf{1} \, .$$

The operator **U** acting on **1** generates the element of $\mathcal{H} \otimes \mathcal{H}$:

$$\mathbf{U} \, \mathbf{1} = - \int d^d x \, (\, \overline{\theta}_\alpha(x) \otimes \lambda_\alpha(x) + \lambda_\alpha(x) \otimes \overline{\theta}_\alpha(x) + \theta_\alpha(x) \otimes \overline{\lambda}_\alpha(x) + \overline{\lambda}_\alpha(x) \otimes \theta_\alpha(x) \,) \, .$$

The action of $\boldsymbol{h}$ on $\mathcal{H} \otimes \mathcal{H}$ is given by

$$\boldsymbol{h} \, \pi_2 = (\, \mathbb{I} \otimes h \,) \, \pi_2 \, .$$

Since h annihilates $\theta_\alpha(x)$ and $\overline{\theta}_\alpha(x)$, two terms survive:

$$\boldsymbol{h} \, \mathrm{U} \, \mathbf{1} = \int d^d x \, (\, \overline{\theta}_\alpha(x) \otimes h \, \lambda_\alpha(x) + \theta_\alpha(x) \otimes h \, \overline{\lambda}_\alpha(x) \,) \, .$$

Dirac field theory

We thus find

$$\pi_2\, \boldsymbol{f}\, \mathbf{1} = -i\hbar\, \pi_2\, \boldsymbol{h}\, \mathrm{U}\, \mathbf{1}$$
$$= -i\hbar \int d^d x \int d^d y\, [\, \overline{\theta}_\alpha(x) \otimes S(x-y)_{\alpha\beta}\, \theta_\beta(y)$$
$$- \theta_\alpha(x) \otimes \overline{\theta}_\beta(y)\, S(y-x)_{\beta\alpha}\,],$$

and we obtain

$$\omega_2\, (\, \pi_2\, \boldsymbol{f}\, \mathbf{1}\, , \lambda_\alpha(x) \otimes \overline{\lambda}_\beta(y)\,) = -i\hbar\, S(x-y)_{\alpha\beta}\, .$$

This correctly reproduces the two-point function $\langle\, \Psi_\alpha(x)\, \overline{\Psi}_\beta(y)\, \rangle$:

$$\langle\, \Psi_\alpha(x)\, \overline{\Psi}_\beta(y)\, \rangle = \frac{\hbar}{i}\, S(x-y)_{\alpha\beta}\, .$$

We can also calculate

$$\omega_2\, (\, \pi_2\, \boldsymbol{f}\, \mathbf{1}\, , \overline{\lambda}_\beta(y) \otimes \lambda_\alpha(x)\,) = i\hbar\, S(x-y)_{\alpha\beta} = \langle\, \overline{\Psi}_\beta(y)\, \Psi_\alpha(x)\, \rangle\, .$$

Note that the antisymmetry under the exchange of fermions is realized:

$$\langle\, \overline{\Psi}_\beta(y)\, \Psi_\alpha(x)\, \rangle = -\langle\, \Psi_\alpha(x)\, \overline{\Psi}_\beta(y)\, \rangle\, .$$

Dirac field theory

We can also reproduce four-point functions.

$$\pi_4 \boldsymbol{f}\, \mathbf{1} = -\hbar^2\, \pi_4\, \boldsymbol{h}\, \mathbf{U}\, \boldsymbol{h}\, \mathbf{U}\, \mathbf{1}$$

We split $\mathbf{U}$ into two parts.

$$\mathbf{U} = \mathbf{V} + \overline{\mathbf{V}}\,,$$

where

$$\mathbf{V} = -\int d^d x\, \overline{\boldsymbol{\theta}}_\alpha(x)\, \boldsymbol{\lambda}_\alpha(x)\,, \qquad \overline{\mathbf{V}} = -\int d^d x\, \boldsymbol{\theta}_\alpha(x)\, \overline{\boldsymbol{\lambda}}_\alpha(x)\,,$$

For example, we consider $\boldsymbol{h}\mathbf{V}\boldsymbol{h}\mathbf{V}\,\mathbf{1}$. Since

$$\boldsymbol{h}\, \pi_4 = (\, \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} \otimes h\,)\, \pi_4\,,$$

and h annihilates $\overline{\theta}_\alpha(x)$ and $h\, \lambda_\alpha(x)$ so that the following three terms survive:

$$\begin{aligned} \boldsymbol{h}\, \mathbf{V}\, \boldsymbol{h}\, \mathbf{V}\, \mathbf{1} = \int d^d x \int d^d x'\, (\, & \overline{\theta}_{\alpha'}(x') \otimes \overline{\theta}_\alpha(x) \otimes h\, \lambda_\alpha(x) \otimes h\, \lambda_{\alpha'}(x') \\ & - \overline{\theta}_\alpha(x) \otimes \overline{\theta}_{\alpha'}(x') \otimes h\, \lambda_\alpha(x) \otimes h\, \lambda_{\alpha'}(x') \\ & + \overline{\theta}_\alpha(x) \otimes h\, \lambda_\alpha(x) \otimes \overline{\theta}_{\alpha'}(x') \otimes h\, \lambda_{\alpha'}(x')\,)\,. \end{aligned}$$

This should be contrasted with $\boldsymbol{h}\, \mathbf{U}\, \boldsymbol{h}\, \mathbf{U}\, \mathbf{1}$ for the scalar field. This reflects the degree of fermions and this minus sign is necessary for the antisymmetry of fermions in correlation functions.

Dirac field theory

Similarly, we can calculate $\boldsymbol{h}\,\mathbf{V}\,\boldsymbol{h}\,\overline{\mathbf{V}}\,\mathbf{1}$, $\boldsymbol{h}\,\overline{\mathbf{V}}\,\boldsymbol{h}\,\mathbf{V}\,\mathbf{1}$, and $\boldsymbol{h}\,\overline{\mathbf{V}}\,\boldsymbol{h}\,\overline{\mathbf{V}}\,\mathbf{1}$.
Then, we obtain

$$\pi_4\,\boldsymbol{f}\,\mathbf{1} = -\hbar^2\,\pi_4\,\boldsymbol{h}\,\mathbf{U}\,\boldsymbol{h}\,\mathbf{U}\,\mathbf{1} = -\hbar^2 \int d^d x \int d^d x' \int d^d y \int d^d y'\,\mathcal{F}(x,y,x',y'),$$

where

$$\begin{aligned}
\mathcal{F}(x,y,x',y') = {} & \overline{\theta}_{\alpha'}(x') \otimes \overline{\theta}_{\alpha}(x) \otimes S(x-y)_{\alpha\beta}\,\theta_{\beta}(y) \otimes S(x'-y')_{\alpha'\beta'}\,\theta_{\beta'}(y') \\
& - \overline{\theta}_{\alpha}(x) \otimes \overline{\theta}_{\alpha'}(x') \otimes S(x-y)_{\alpha\beta}\,\theta_{\beta}(y) \otimes S(x'-y')_{\alpha'\beta'}\,\theta_{\beta'}(y') \\
& + \overline{\theta}_{\alpha}(x) \otimes S(x-y)_{\alpha\beta}\,\theta_{\beta}(y) \otimes \overline{\theta}_{\alpha'}(x') \otimes S(x'-y')_{\alpha'\beta'}\,\theta_{\beta'}(y') \\
& - \overline{\theta}_{\alpha'}(x') \otimes \theta_{\alpha}(x) \otimes \overline{\theta}_{\beta}(y)\,S(y-x)_{\beta\alpha} \otimes S(x'-y')_{\alpha'\beta'}\,\theta_{\beta'}(y') \\
& + \theta_{\alpha}(x) \otimes \overline{\theta}_{\alpha'}(x') \otimes \overline{\theta}_{\beta}(y)\,S(y-x)_{\beta\alpha} \otimes S(x'-y')_{\alpha'\beta'}\,\theta_{\beta'}(y') \\
& - \theta_{\alpha}(x) \otimes \overline{\theta}_{\beta}(y)\,S(y-x)_{\beta\alpha} \otimes \overline{\theta}_{\alpha'}(x') \otimes S(x'-y')_{\alpha'\beta'}\,\theta_{\beta'}(y') \\
& - \theta_{\alpha'}(x') \otimes \overline{\theta}_{\alpha}(x) \otimes S(x-y)_{\alpha\beta}\,\theta_{\beta}(y) \otimes \overline{\theta}_{\beta'}(y')\,S(y'-x')_{\beta'\alpha'} \\
& + \overline{\theta}_{\alpha}(x) \otimes \theta_{\alpha'}(x') \otimes S(x-y)_{\alpha\beta}\,\theta_{\beta}(y) \otimes \overline{\theta}_{\beta'}(y')\,S(y'-x')_{\beta'\alpha'} \\
& - \overline{\theta}_{\alpha}(x) \otimes S(x-y)_{\alpha\beta}\,\theta_{\beta}(y) \otimes \theta_{\alpha'}(x') \otimes \overline{\theta}_{\beta'}(y')\,S(y'-x')_{\beta'\alpha'} \\
& + \theta_{\alpha'}(x') \otimes \theta_{\alpha}(x) \otimes \overline{\theta}_{\beta}(y)\,S(y-x)_{\beta\alpha} \otimes \overline{\theta}_{\beta'}(y')\,S(y'-x')_{\beta'\alpha'} \\
& - \theta_{\alpha}(x) \otimes \theta_{\alpha'}(x') \otimes \overline{\theta}_{\beta}(y)\,S(y-x)_{\beta\alpha} \otimes \overline{\theta}_{\beta'}(y')\,S(y'-x')_{\beta'\alpha'} \\
& + \theta_{\alpha}(x) \otimes \overline{\theta}_{\beta}(y)\,S(y-x)_{\beta\alpha} \otimes \theta_{\alpha'}(x') \otimes \overline{\theta}_{\beta'}(y')\,S(y'-x')_{\beta'\alpha'}\,.
\end{aligned}$$

Dirac field theory

Then, we obtain

$$
\begin{aligned}
&\omega_4\,(\pi_4 \boldsymbol{f}\,\mathbf{1}\,,\lambda_{\alpha_1}(x_1)\otimes\lambda_{\alpha_2}(x_2)\otimes\overline{\lambda}_{\beta_1}(y_1)\otimes\overline{\lambda}_{\beta_2}(y_2))\\
&= -\hbar^2\,[S_{\alpha_1\beta_2}\,(x_1 - y_2)\,S_{\alpha_2\beta_1}\,(x_2 - y_1) - S_{\alpha_1\beta_1}\,(x_1 - y_1)\,S_{\alpha_2\beta_2}\,(x_2 - y_2)]\\
&= \langle\,\Psi_{\alpha_1}(x_1)\,\overline{\Psi}_{\beta_2}(y_2)\,\rangle\,\langle\,\Psi_{\alpha_2}(x_2)\,\overline{\Psi}_{\beta_1}(y_1)\,\rangle - \langle\,\Psi_{\alpha_1}(x_1)\,\overline{\Psi}_{\beta_1}(y_1)\,\rangle\,\langle\,\Psi_{\alpha_2}(x_2)\,\overline{\Psi}_{\beta_2}(y_2)\,\rangle\\
&= \langle\,\Psi_{\alpha_1}(x_1)\,\Psi_{\alpha_2}(x_2)\,\overline{\Psi}_{\beta_1}(y_1)\,\overline{\Psi}_{\beta_2}(y_2)\,\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\omega_4\,(\pi_4 \boldsymbol{f}\,\mathbf{1}\,,\lambda_{\alpha_2}(x_2)\otimes\lambda_{\alpha_1}(x_1)\otimes\overline{\lambda}_{\beta_1}(y_1)\otimes\overline{\lambda}_{\beta_2}(y_2))\\
&= -\hbar^2\,[S_{\alpha_2\beta_2}\,(x_2 - y_2)\,S_{\alpha_1\beta_1}\,(x_1 - y_1) - S_{\alpha_2\beta_1}\,(x_2 - y_1)\,S_{\alpha_1\beta_2}\,(x_1 - y_2)]\\
&= \langle\,\Psi_{\alpha_2}(x_2)\,\overline{\Psi}_{\beta_2}(y_2)\,\rangle\,\langle\,\Psi_{\alpha_1}(x_1)\,\overline{\Psi}_{\beta_1}(y_1)\,\rangle - \langle\,\Psi_{\alpha_2}(x_2)\,\overline{\Psi}_{\beta_1}(y_1)\,\rangle\,\langle\,\Psi_{\alpha_1}(x_1)\,\overline{\Psi}_{\beta_2}(y_2)\,\rangle\\
&= \langle\,\Psi_{\alpha_2}(x_2)\,\Psi_{\alpha_1}(x_1)\,\overline{\Psi}_{\beta_1}(y_1)\,\overline{\Psi}_{\beta_2}(y_2)\,\rangle\\
&= -\langle\,\Psi_{\alpha_1}(x_1)\,\Psi_{\alpha_2}(x_2)\,\overline{\Psi}_{\beta_1}(y_1)\,\overline{\Psi}_{\beta_2}(y_2)\,\rangle
\end{aligned}
$$

Note that the antisymmetry under the exchange of fermions is realized.

We can reproduce higher-point functions.

the Schwinger-Dyson equations

In general, we claim

$$\langle \Phi^{\otimes n} \rangle = \pi_n \, \boldsymbol{f} \, \mathbf{1} \, ,$$

where

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h} \, \boldsymbol{m} + i\hbar \, \boldsymbol{h} \, \mathbf{U}} \, ,$$

$$\Phi = \int d^d x \, \varphi(x) \, c(x) + \int d^d x \, (\, \overline{\theta}_\alpha(x) \, \Psi_\alpha(x) + \overline{\Psi}_\alpha(x) \, \theta_\alpha(x) \,) \, .$$

$$\begin{aligned} &\langle \, \Psi_{\alpha_1}(y_1) \, \ldots \, \Psi_{\alpha_m}(y_m) \, \overline{\Psi}_{\beta_1}(z_1) \, \ldots \, \overline{\Psi}_{\beta_m}(z_m) \, \varphi(x_1) \, \ldots \, \varphi(x_n) \, \rangle \\ &= \omega_{2m+n} \, (\, \pi_{2m+n} \, \boldsymbol{f} \, \mathbf{1} \, , \\ &\quad \lambda_{\alpha_1}(y_1) \otimes \ldots \otimes \lambda_{\alpha_m}(y_m) \otimes \overline{\lambda}_{\beta_1}(z_1) \otimes \ldots \otimes \overline{\lambda}_{\beta_l}(z_m) \otimes d(x_1) \otimes \ldots \otimes d(x_n) \,) \, . \end{aligned}$$

We can directly prove that correlation functions from our formula satisfy the Schwinger-Dyson equations using the trivial identity as in the scalar field theory:

$$(\, \mathbf{I} \, + \, \boldsymbol{h} \, \boldsymbol{m} \, + \, i \, \hbar \, \boldsymbol{h} \, \mathbf{U} \,) \, \frac{1}{\mathbf{I} \, + \, \boldsymbol{h} \, \boldsymbol{m} \, + i\hbar \, \boldsymbol{h} \, \mathbf{U}} \, \mathbf{1} \, = \, \mathbf{1} \, ,$$

where

$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h} \, \boldsymbol{m} + i\hbar \, \boldsymbol{h} \, \mathbf{U}} \, .$$

Contents

Summary

We extend the Okawa's result to general scalar-Dirac systems.
The formula is given by

$$\langle \Phi^{\otimes n} \rangle = \pi_n \, \boldsymbol{f} \, \mathbf{1} \,,$$

where

$$\Phi = \int d^d x \, \varphi(x) \, c(x) + \int d^d x \, (\overline{\theta}_\alpha(x) \, \Psi_\alpha(x) + \overline{\Psi}_\alpha(x) \, \theta_\alpha(x)) \, .$$

The future work is as follows:

- application to string field theory
- non-perturbative effect

etc....