GWW Phase Transition in Induced QCD on the Graph

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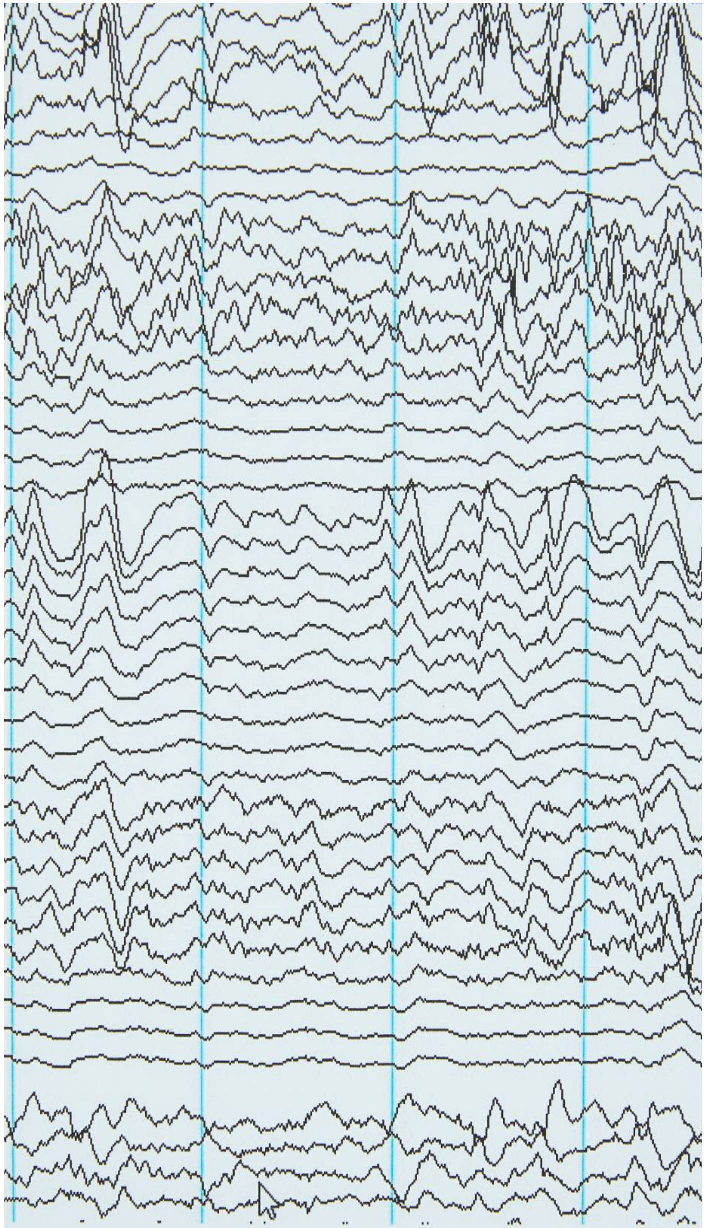
Based on work with K. Ohta
arXiv:2303.03692
arXiv:2308.*****

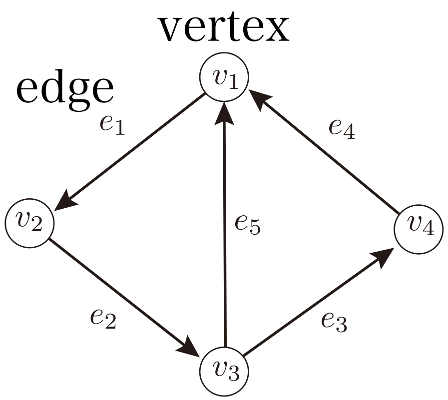


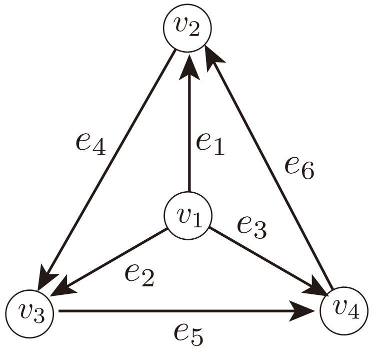
Plan of the talk

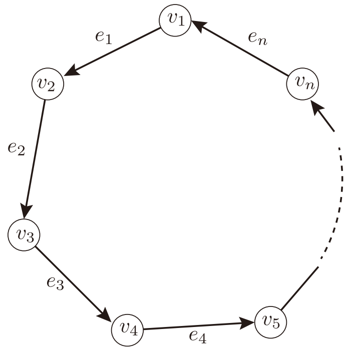
- Graph zeta functions
- Fundamental Kazakov-Migdal model on the Graph and graph zeta functions
- Duality
- GWW phase transitions in the FKM model
- Numerical results
- Conclusion

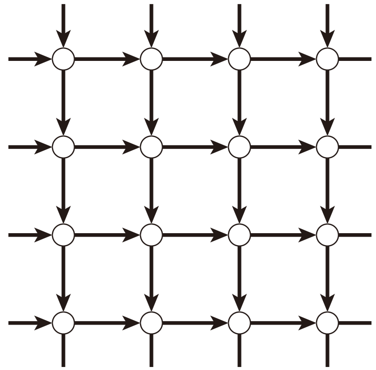
Graph zeta functions

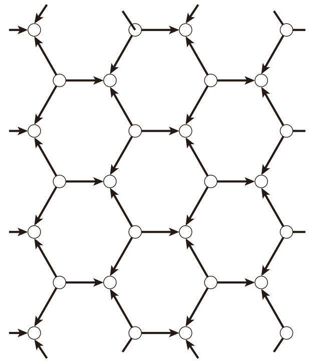
グラフとサイクル



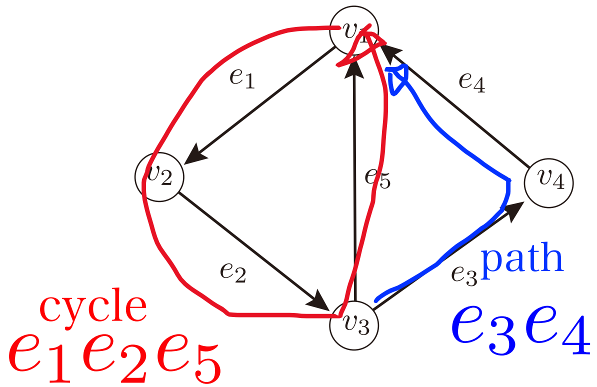


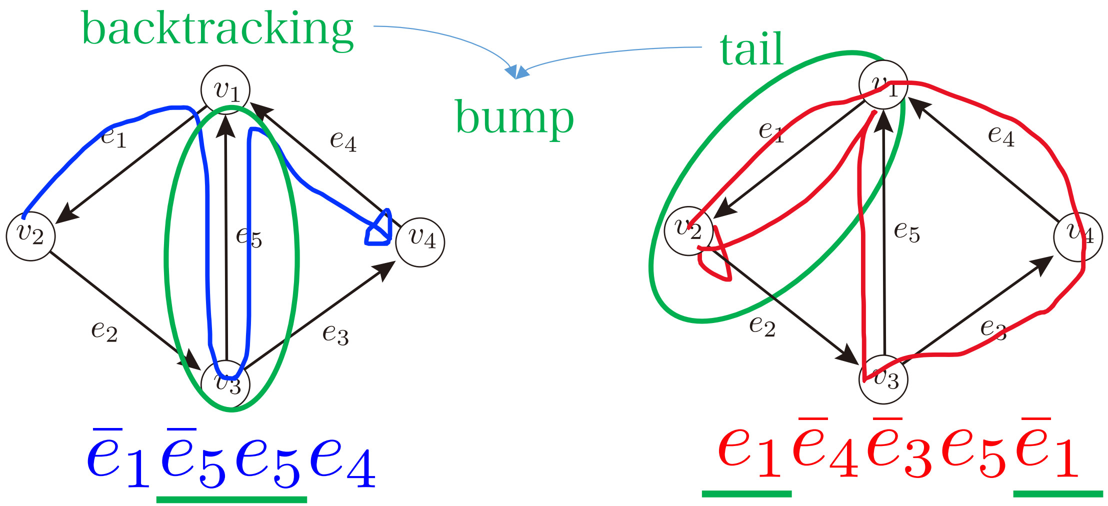






基本的な用語

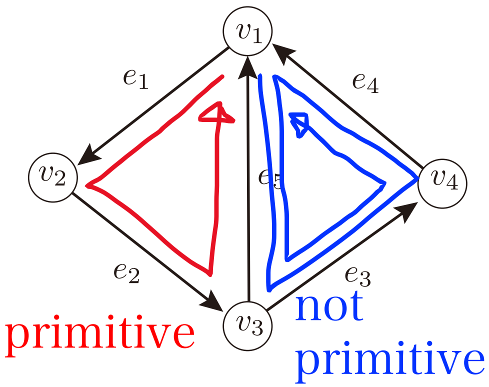




サイクルの分類と伊原ゼータ関数

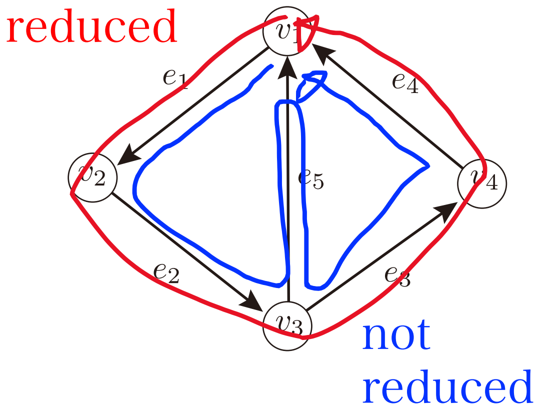
<u>primitive cycle</u>

$$C \neq B^n$$



<u>reduced cycle</u>

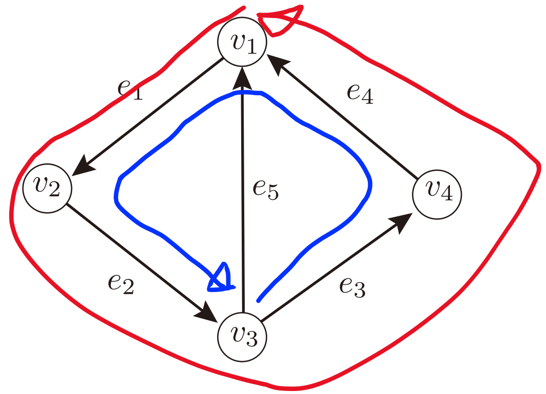
bumpを持たないcycle



<u>同値なcycle</u>

$$C_1 \sim C_2 \qquad \begin{array}{l} C_1 = e_1 \cdots e_l \\ C_2 = e'_1 \cdots e'_l \end{array}$$

$$\Leftrightarrow \ {}^{\exists} n \in \mathbb{N} \text{ s.t. } e'_i = e_{n+i} \text{ for } {}^{\forall} i$$



$$q \equiv e^{-s}$$
$$p_C \equiv e^{|C|}$$

⬇

$$\zeta_G(q) = \prod_{[C]:PR} \frac{1}{1 - p_C^{-s}}$$

<u>伊原ゼータ関数</u>

$$\zeta_G(q) \equiv \prod_{[C]:\text{primitive reduced}} \frac{1}{1 - q^{|C|}}$$

cf) リーマンゼータ関数

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

伊原ゼータ関数の行列式表示

<u>頂点表示</u>

伊原ゼータ関数は行列式（多項式）の逆数で書ける

Ihara 1966

$$\zeta_G(q) = (1-q^2)^{-(n_E - v_V)} \det\Big(I - qA + q^2(D-I)\Big)^{-1}$$

D: 度数行列
A: 頂点隣接行列
$(n_V \times n_V\ matrix)$

n_V: 頂点の数
n_E: 辺の数

<u>辺表示</u>

伊原ゼータ関数の辺表示

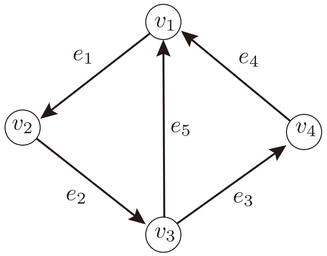
Bass 1992

$$\zeta_G(q) = \det(1 - qW)^{-1}$$

<u>辺隣接行列</u> $\boldsymbol{e} = \{e, e^{-1} | e \in E\}$

$$W_{\boldsymbol{e}\boldsymbol{e}'} = \begin{cases} 1 & \text{if } t(\boldsymbol{e}) = s(\boldsymbol{e}') \text{ and } \boldsymbol{e}'^{-1} \neq \boldsymbol{e} \\ 0 & \text{others} \end{cases}$$

(例) Double Triangle



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$W =$

	e_1	e_2	e_3	e_4	e_5	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$\bar{e}_4$	$\bar{e}_5$
e_1		1								
e_2			1		1					
e_3				1						
e_4	1									1
e_5	1								1	
$\bar{e}_1$									1	1
$\bar{e}_2$						1				
$\bar{e}_3$					1		1			
$\bar{e}_4$								1		
$\bar{e}_5$			1				1			

Bartholdiゼータ関数

Bartholdi 2000

$$\zeta_G(q,u) \equiv \prod_{[C]:\text{primitive}} \frac{1}{1-q^{|C|}u^{b(C)}}$$

bumpの数

$u=0$の時、Bartholdiゼータ関数は伊原ゼータ関数になる

頂点表示

$$\zeta_G(q,u) = \left(1-(1-u)^2q^2\right)^{-(n_E-n_V)} \det\left(1-qA+(1-u)q^2(D-(1-u)1)\right)^{-1},$$

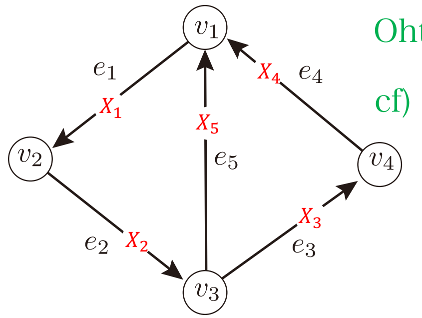
辺表示

$$\zeta_G(q,u) = \det(1-q(W+uJ))^{-1}$$

$$J_{\boldsymbol{e}\boldsymbol{e}'} = \begin{cases} 1 & (\boldsymbol{e}'^{-1}=\boldsymbol{e}) \\ 0 & (\text{others}) \end{cases}$$

行列重み付きBartholdiゼータ関数

- それぞれの辺eに正則行列X_e（サイズK）を配置する
- $X_{e^{-1}} = X_e^{-1}$とする
- $C = e_{i_1} \cdots e_{i_n}$に対し$X_C \equiv X_{e_{i_1}} \cdots X_{e_{i_n}}$
- 行列重み付き隣接行列



Ohta-S.M. 2022

cf) Mizuno, Sato 2003,2006

$$A(X)_{vv'} = \begin{cases} X_e & \langle v, v' \rangle = e \\ 0 & \text{others} \end{cases} \qquad (W_X)_{\boldsymbol{ee'}} = \begin{cases} X_{\boldsymbol{e}} & \text{if } t(\boldsymbol{e}) = s(\boldsymbol{e'}) \text{ and } \boldsymbol{e'}^{-1} \neq \boldsymbol{e} \\ 0 & \text{others} \end{cases}, \quad (J_X)_{\boldsymbol{ee'}} = \begin{cases} X_{\boldsymbol{e}} & \text{if } \boldsymbol{e'}^{-1} = \boldsymbol{e} \\ 0 & \text{others} \end{cases}.$$

<ins>行列重み付き伊原ゼータ関数</ins>　通常の伊原ゼータ関数と同じく行列式表示を持つ

$$\zeta_G(q, u; X) \equiv \prod_{C \in [\mathcal{P}]} \det\left(1_K - q^{|C|} u^{b(C)} X_C\right)^{-1},$$

$$= \left(1 - (1-u)^2 q^2\right)^{-K(n_E - n_V)} \det\left(1_{Kv_N} - qA_X + (1-u)q^2(D - (1-u)1_{Kv_N})\right)^{-1}$$

$$= \det\left(1_{2Kn_E} - q(W_X + uJ_X)\right)^{-1}$$

リーマンゼータ関数についてざっくりと

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

ゼータ関数を特徴付ける大切な３つの性質：

(1) オイラー積表示を持つ

$$\zeta(s) = \prod_{p} \frac{1}{1-p^{-s}}$$

(2) 関数等式が成り立つ

完備ゼータ関数：$\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ ➡ 関数等式： $\xi(1-s) = \xi(s)$

(3) リーマン予想が成り立つと思われている

$\zeta(s)$の（非自明な）ゼロ点は、$Re(s) = \frac{1}{2}$のみに存在するに違いない

伊原ゼータ関数はゼータたり得るか？

(1) オイラー積表示を持つか？

$$\zeta_G(q) \equiv \prod_{[C]:PR} \frac{1}{1-q^{|C|}}$$

(2) 関数等式は成り立つか？

- グラフを(t+1)-正則グラフとすると、$\zeta_G(q) = (1-q^2)^{-(n_E-n_V)} \det\left((1+tq^2)\mathbf{1}_{n_V} - qA\right)^{-1}$
- 完備伊原ゼータ関数：$\xi_G(q) \equiv (1-q^2)^{n_E-\frac{n_V}{2}}(1-t^2q^2)^{\frac{n_V}{2}}\zeta_G(q) = (1-q^2)^{\frac{n_V}{2}}(1-t^2q^2)^{\frac{n_V}{2}} \det\left((1+tq^2)\mathbf{1}_{n_V} - qA\right)$

$$\xi_G\left(\frac{1}{tq}\right) = (-1)^{n_V}\xi_G(q)$$

(3) リーマン予想の類似物は？

グラフがラマヌジャンならば、$\zeta_G(t^{-s})$のゼロ点は$Re(s) = \frac{1}{2}$上のみに存在する。

（グラフがラマヌジャン：(t+1)-正則グラフで、隣接行列の固有値λが$\lambda^2 - 4\lambda < 0$ を満たすグラフ）

<u>証明</u>

グラフが(t+1)正則の時の伊原ゼータ関数：

$\zeta_G(q) = (1-q^2)^{n_V-n_E} \det\left((1-tq^2)\mathbf{1}_{n_V} - qA\right) = (1-q^2)^{n_V-n_E}\prod_{\lambda}(tq^2 - \lambda q + 1)$ ➡ ゼロ点：$q = \frac{\lambda \pm \sqrt{\lambda^2-4t}}{2t} \equiv t^{-s_\pm}$

$\lambda^2 - 4\lambda < 0$のとき$s_- = s_+^*$なので、$t^{-s_+} \cdot t^{-s_-} = t^{-s_+-s_-} = t^{-2Re(s_+)} \stackrel{\lambda}{=} t^{-1}$　（2次方程式の解と係数の関係より）

Fundamental Kazakov-Migdal model on the Graph and graph zeta functions

FKM model

Fundamental Kazakov-Migdal (FKM) model on a general graph

Arefeva 1993
Ohta-S.M. 2023

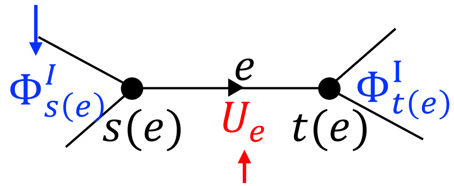
$$S = \sum_{v\in V} m_v^2 \Phi_v^{\dagger I}\Phi_{vI} - q\sum_{e\in E}\left(\Phi^{\dagger}_{s(e)}{}^{I} U_e \Phi_{t(e)I} + \Phi^{\dagger}_{t(e)}{}^{I} U_e^{\dagger}\Phi_{s(e)I}\right)$$

(cf) KM model on the graph

Kazakov-Migdal 1992
Ohta-S.M. 2022

$$S_{\mathrm{KM}} = \mathrm{Tr}\left\{\frac{m_0^2}{2}\sum_{v\in V}\Phi_v^2 + q\sum_{e\in E}\left(\frac{r}{2}\left(\Phi_{s(e)}^2+\Phi_{t(e)}^2\right) - \Phi_{s(e)}U_e\Phi_{t(e)}U_e^{\dagger}\right)\right\}$$
)

fundamental representation
$(I = 1,\cdots,N_f)$



unitary matrix
(color N_c)

Partition function as a Gaussian integral

$$Z = \prod_{I=1}^{N_f}\int \prod_{v\in V} d\Phi_{vI}d\Phi^{\dagger vI}\prod_{e\in E} dU_e\, e^{-\Phi^{\dagger vI}\Delta(U)_v{}^{v'}\Phi_{v'I}},$$

$$\Delta(U)_v{}^{v'} \equiv m_v^2\delta_v{}^{v'}\mathbf{1}_{N_c} - q(A_U)_v{}^{v'} \qquad (A_U)_v{}^{v'} = \sum_{\boldsymbol{e}\in E_D} U_{\boldsymbol{e}}\,\delta_{\langle v,v'\rangle,\boldsymbol{e}}$$

Partition function as a graph zeta function

[Recall] Bartholdi zeta function in vertex representation

$$\zeta_G(q,u;U) = \left(1-(1-u)^2q^2\right)^{-N_c(n_E-n_V)} \det\left(\mathbf{1}_{N_c n_V} - qA_U + (1-u)q^2(D-(1-u)\mathbf{1}_{N_c n_V})\right)^{-1}$$

tunning the mass parameter

$$\Delta(U)_v{}^{v'} \equiv m_v^2 \delta_v{}^{v'} \mathbf{1}_{N_c} - q(A_U)_v{}^{v'} \qquad m_v^2 = 1 - q^2(1-u)^2 + q^2(1-u)\deg v$$

$$Z_G = (2\pi)^{N_f N_c n_V} \left(1-(1-u)^2q^2\right)^{N_f N_c(n_E-n_V)} \int \prod_{e\in E} dU_e \boxed{\zeta_G(q,u;U)^{N_f}}$$

FKM model is described by the unitary matrix weighted Graph zeta function

Effective action and relation to Wilson action

(For simplicity, $u = 0$ in the following)

$$Z_G = (2\pi)^{N_f N_c n_V} \left(1 - q^2\right)^{N_f N_c (n_E - n_V)} \int \prod_{e \in E} dU_e \, \zeta_G(q;U)^{N_f}$$

$$\downarrow$$

$$\zeta_G(q;U) \equiv \prod_{C \in [\mathcal{P}_R]} \det\left(\mathbf{1}_{N_c} - q^{|C|} U_C\right)^{-1} = \exp\left(\sum_{C \in [\mathcal{P}_R]} \sum_{n=1}^{\infty} \frac{q^n}{n} \left(\mathrm{Tr}\, U_C^n + \mathrm{Tr}\, U_C^{\dagger\, n}\right)\right)$$

$$S_{\mathrm{eff}}(U) = -N_f \sum_{C \in [\Pi_+]} \sum_{n=1}^{\infty} \frac{q^n}{n} \left(\mathrm{Tr}\, U_C^n + \mathrm{Tr}\, U_C^{\dagger\, n}\right)$$

valid only for small $|q|$
(at most |q|<1)

$\gamma \equiv N_f / N_c \quad q \to 0, \quad \gamma \to \infty, \quad \lambda \equiv \frac{1}{\gamma q^l}$: fixed (l : minimal length of the cycles)

$$S_{\mathrm{eff}}(U) \to -\frac{N_c}{\lambda} \sum_{C \in [\Pi_+^l]} \left(\mathrm{Tr}\, U_C + \mathrm{Tr}\, U_C^{\dagger}\right)$$

FKM model includes the usual lattice gauge theory with Wilson action

Duality

When the graph is regular

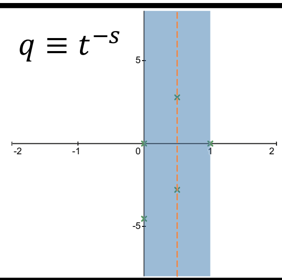
$$S_{\text{eff}}(q;U) = -N_c \log \zeta_G(q;U)$$

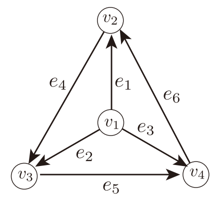
<u>fact</u> Kotani-Sunada

When G is d-regular, $\zeta_G(q)$ has poles only in $\frac{1}{d-1} \leq Re(q) \leq 1$.

In particular, there is a simple pole at $q = \frac{1}{d-1}$.

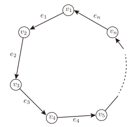
$q \equiv t^{-s}$

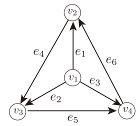


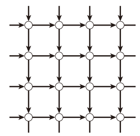


<u>functional relation for d-regular graph</u> $(t \equiv d-1)$ Ohta-S.M. coming soon

$$\zeta_G(1/tq;U) = (tq^2)^{n_V N_c} \left(\frac{-tq^2(1-q^2)}{1-t^2q^2} \right)^{(n_E - n_C)N_c} \zeta_G(q;U)$$







The FKM model on a regular graph is symmetric under the dual transformation,

$$q \leftrightarrow \frac{1}{tq} \quad \text{or} \quad s \leftrightarrow 1-s \qquad (q \equiv t^{-s})$$

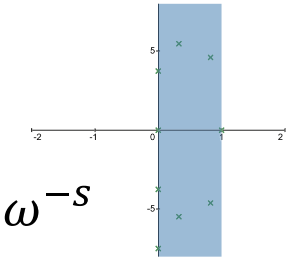
When the graph is irregular

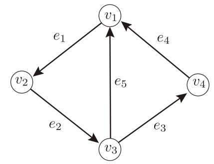
<u>fact</u> Kotani-Sunada

- All the poles of $\zeta_G(q)$ are in $\frac{1}{\omega} \leq Re(q) \leq 1$ ($1/\omega$: maximal radius of convergence)
- $\frac{1}{t_{max}} \leq \frac{1}{\omega} \leq \frac{1}{t_{min}}$
- There is a simple pole at $q = 1/\omega$

➡ $\omega \leftrightarrow t$

natural parametrization : $q = \omega^{-s}$





<u>"functional relation" for irregular graph</u> Ohta-S.M. coming soon

$$\zeta_G(1/\omega q; U) \propto \det\left(1 - \omega q \left(\tilde{Q}^{-1} W_U - (1 - \tilde{Q}^{-1}) J_U\right)\right)^{-1} \qquad \left(\tilde{Q} \equiv \mathrm{diag}_e(\deg s(e) - 1)\right)$$

matrix Bartholdi zeta function with (unfamiliar) weights

The FKM model on an irregular graph has also a dual expression in $q > 1$

GWW phase transitions

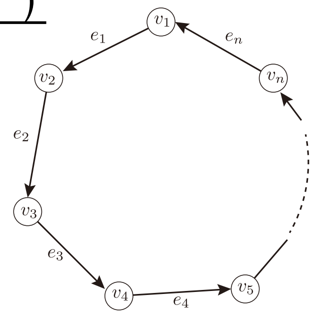
GWW phase transition in the cycle graph

cf) Watanabe-san's talk

After gauge fixing : $U_2 = \cdots = U_n = 1$ $(U_1 \equiv U, \alpha \equiv q^n)$

$$Z_{C_n} = \left(\frac{2\pi}{q^n}\right)^{N_f N_c n_V} \alpha^{N_c N_f} \int dU e^{N_f \sum_{m=1}^{\infty} \frac{\alpha^m}{m}\left(\mathrm{Tr}\, U^m + \mathrm{Tr}\, U^{-m}\right)}$$

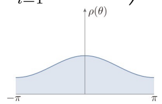
$$= \mathcal{N} \int_{-\pi}^{\pi} \prod_{i=1}^{N_c} d\theta_i \, e^{\sum_{j\neq k} \log\left|\sin\frac{\theta_j - \theta_k}{2}\right| - N_f \sum_i \log\left(1 - 2\alpha\cos\theta_i + \alpha^2\right)}$$

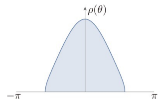


Eigenvalue density in large N_c

$$\left(\rho(\theta) \equiv \frac{1}{N_c}\sum_{i=1}^{N_c} \delta(\theta - \theta_i)\right)$$

$$\rho(\theta) = \begin{cases} \frac{1}{2\pi}\left(1 + 2\gamma \frac{\alpha\cos\theta - \alpha^2}{1 - 2\alpha\cos\theta + \alpha^2}\right), & (\theta_0 = \pi) \\ \frac{2(\gamma-1)\alpha}{\pi} \frac{\cos\frac{\theta}{2}}{1 - 2\alpha\cos\theta + \alpha^2}\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}, & (\theta_0 < \pi) \end{cases}$$





$$\left(\sin^2\frac{\theta_0}{2} = \frac{(1-\alpha)^2}{4\alpha}\frac{2\gamma - 1}{(\gamma-1)^2}\right)$$

Wilson limit

$$q \to 0, \quad \gamma \to \infty, \quad \lambda \equiv \frac{1}{\gamma q^l} \ : \ \text{fixed}$$

$$\rho(\theta) \to \begin{cases} \frac{1}{2\pi}\left(1 + \frac{2}{\lambda}\cos\theta\right), & (\theta_0 = \pi) \\ \frac{2}{\pi\lambda}\cos\frac{\theta}{2}\sqrt{\frac{\lambda}{2} - \sin^2\frac{\theta}{2}}, & (\theta_0 < \pi) \end{cases}$$

$$F_{C_n} \to \begin{cases} F^-_{C_n} \equiv -\frac{1}{\lambda^2} & (0 < \alpha \le \alpha^*) \\ F^+_{C_n} \equiv -\frac{2}{\lambda} - \frac{1}{2}\log\frac{\lambda}{2} + \frac{3}{4} & (\alpha^* < \alpha < 1) \end{cases},$$

exactly reproduces
the result of GWW model

Free energy

$$F_{C_n} \equiv -\lim_{N_c \to \infty} \frac{1}{N_c^2} \log Z_{C_n}$$

$$= -\int_0^1 dx dy \log\left|\sin\frac{\theta(x) - \theta(y)}{2}\right| + \gamma \int_0^1 dx \, \log(1 - 2\alpha\cos\theta(x) + \alpha^2)$$

$$= \begin{cases} F^-_{C_n} \equiv \gamma^2 \log\left(1 - \alpha^2\right) & (0 < \alpha \le \alpha^*) \\ F^+_{C_n} \equiv (2\gamma - 1)\log(1-\alpha) + \frac{1}{2}\log\alpha + f(\gamma) & (\alpha^* < \alpha \le 1) \end{cases} \quad \left(\alpha^* = \frac{1}{2\gamma - 1}\right)$$

3rd order GWW phase transition

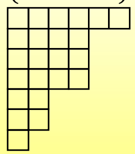
Partition function for a general graph in large N

large N decomposition of Wilson loops ($qN_f \ll N_c$)

$$\int \prod_{e\in E} dU_e \left(\prod_{C\in[\Pi_+]} \Upsilon_{\lambda_C}(U_C)\Upsilon_{\mu_C}(U_C^\dagger) \right) = \prod_{C\in[\Pi_+]} \left(\int \prod_{e\in E} dU_e\, \Upsilon_{\lambda_C}(U_C)\Upsilon_{\mu_C}(U_C^\dagger) \right) + \mathcal{O}(1/N_c)$$

$$\left(\Upsilon_\lambda(U) \equiv \prod_{i=1}^{k} \mathrm{Tr}\left(U^{l_i}\right)^{m_i} \quad \lambda = (l_1^{m_1}, l_2^{m_2}, \cdots) \right)$$

$(6^1 4^3 2^2 1^1)$



$$Z_G \to (2\pi)^{N_f N_c n_V} \left(1-q^2\right)^{N_f N_c (n_E - n_V)} \prod_{C\in[\Pi_+]} \frac{1}{\left(1-q^{2|C|}\right)^{N_f^2}}$$

saddle point approximation around $U_C = 1$ ($N_c \ll N_f$) (r : rank of the graph)

$$Z_G \to \mathcal{N}\, (2\pi)^{N_f N_c n_V} \left(1-q^2\right)^{N_f N_c (n_E - n_V)} N_f^{\frac{r}{2} N_c^2} \prod_{C\in[\Pi_+]} \frac{1}{(1-q^{|C|})^{2N_f N_c}} (\det \mathcal{M}_G)^{-\frac{N_c^2}{2}}$$

Free energy in the both limits

different analytic expressions in the both region

$$F_G \equiv -\frac{1}{N_c^2} \log Z_G = \begin{cases} -\frac{\gamma^2}{2} \log \zeta_G(q^2)\,, & (q \ll 1) \\ -\gamma \log \zeta_G(q) + \frac{1}{2}\mathrm{Tr}\log \mathcal{M}_G + \frac{r}{2}\log\gamma + f(\gamma) & (q \lesssim 1,\ \gamma \gg 1) \end{cases}$$

Check in the cycle graph

【recall】

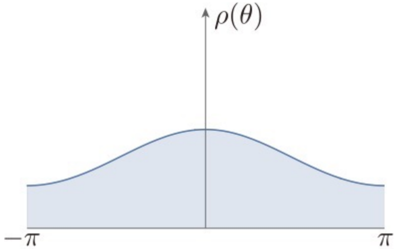
$$\zeta_{C_n}(q) = (1-q^n)^{-2} = (1-\alpha)^{-2}$$

$$\mathcal{M}_{C_n} = \frac{2\alpha}{(1-\alpha)^2}$$

【recall 2】 exact solution for C_n

$$F_{C_n} = \begin{cases} \gamma^2 \log\left(1-\alpha^2\right) & (0 < \alpha \le \alpha^*) \\ (2\gamma-1)\log(1-\alpha) + \frac{1}{2}\log\alpha + f(\gamma) & (\alpha^* < \alpha \le 1) \end{cases}$$

$$F_{C_n} = \begin{cases} -\frac{\gamma^2}{2}\log\zeta_{C_n}(q^2)\,, \\ -\gamma\log\zeta_{C_n}(q) + \frac{1}{2}\operatorname{Tr}\log\mathcal{M}_{C_n} + \frac{r}{2}\log\gamma + f(\gamma) \end{cases} = \begin{cases} \gamma^2\log(1-\alpha^2) \\ (2\gamma-1)\log(1-\alpha) + \frac{1}{2}\log\alpha + f(\gamma) \end{cases}$$

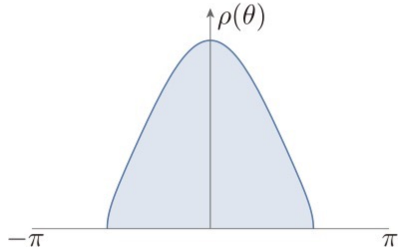


small q phase

$$\gamma q_c^n = \frac{1}{2}$$

GWW

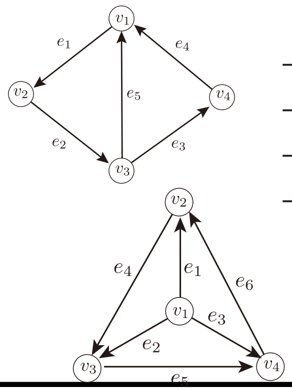
phase transition

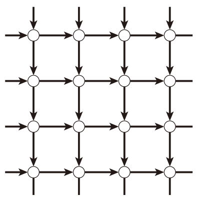


large q phase

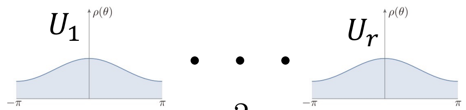
Phase structure of the FKM model

when all fundamental cycles are symmetric:



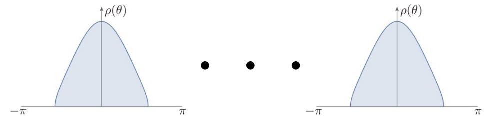


The system should have two phases



$$F_G = -\frac{\gamma^2}{2}\log\zeta_G(q^2)$$

ρ_a are all connected



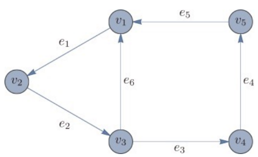
$$F_G = -\gamma\log\zeta_G(q) + \frac{1}{2}\mathrm{Tr}\log\mathcal{M}_G + \frac{r}{2}\log\gamma + f(\gamma)$$

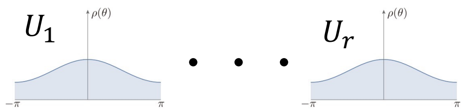
ρ_a are all disconnected

0 GWW 1 q

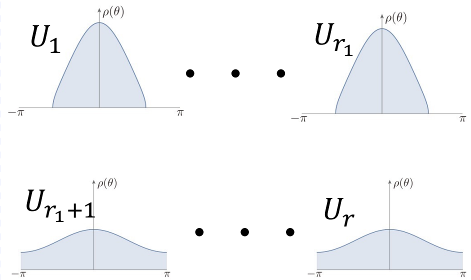
when the fundamental cycles are composed of different types of cycles:

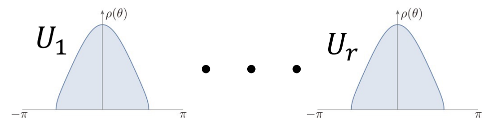
The system should have intermediate phases





$$F_G = -\frac{\gamma^2}{2}\log\zeta_G(q^2)$$





$$F_G = -\gamma\log\zeta_G(q) + \frac{1}{2}\mathrm{Tr}\log\mathcal{M}_G + \frac{r}{2}\log\gamma + f(\gamma)$$

0 … 1 q

Numerical results

Observables

``temperature''

$$Z_G = \int \prod_{e \in E} dU_e \, \zeta_G(q;U)^{N_f} = \int \prod_{e \in E} dU_e \, e^{-\gamma S_{\text{eff}}(q;U)}$$

$$S_{\text{eff}}(q;U) = -N_c \log \zeta_G(q;U)$$

free energy

$$F \equiv -\frac{1}{N_c^2} \log Z_G$$

internal energy

$$E = \gamma \frac{\partial}{\partial \gamma} F = \frac{\gamma}{N_c^2} \langle S_{\text{eff}} \rangle$$

specific heat

$$C = -\gamma^2 \frac{\partial^2}{\partial \gamma^2} F = \frac{\gamma^2}{N_c^2} \left(\langle S_{\text{eff}}^2 \rangle - \langle S_{\text{eff}} \rangle^2 \right)$$

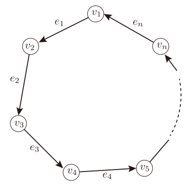
Monte-Carlo simulation

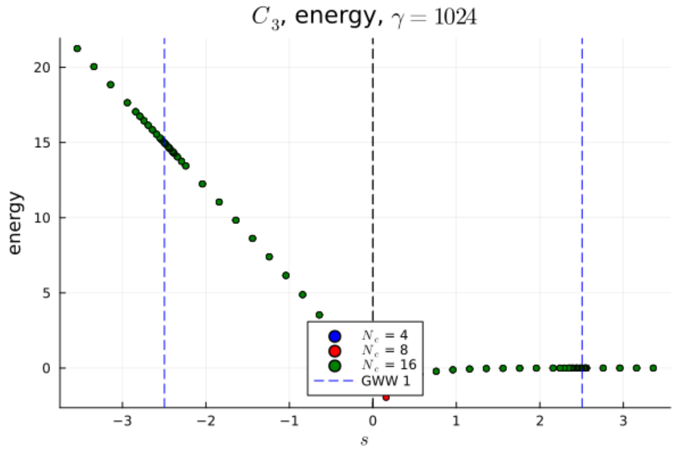
algorithm : HMC

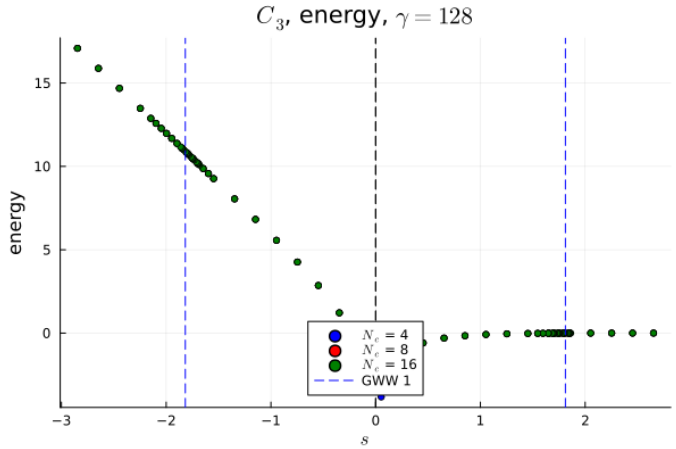
samples : 20,000 for each parameter

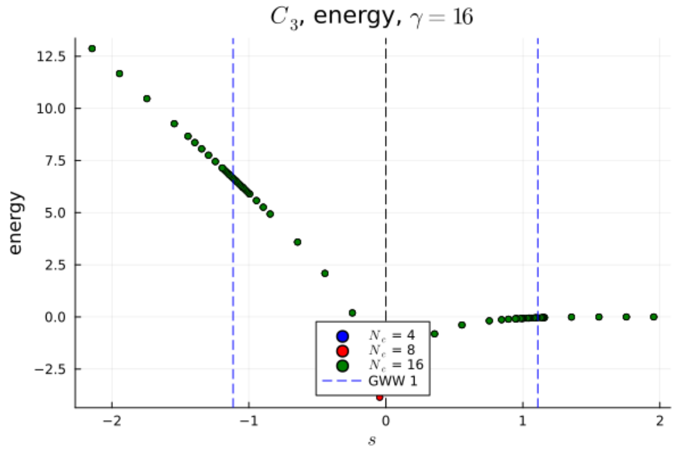
language : Julia

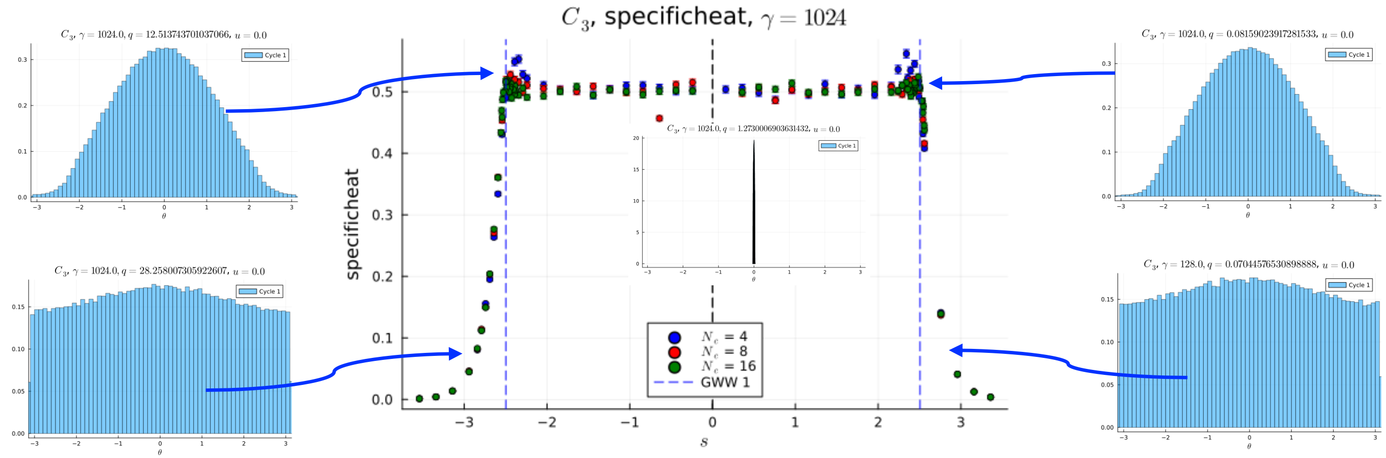
Cycle graph (n=3)

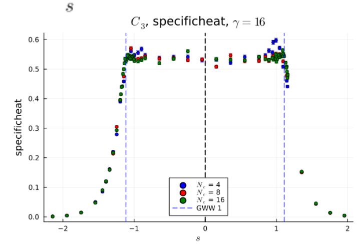


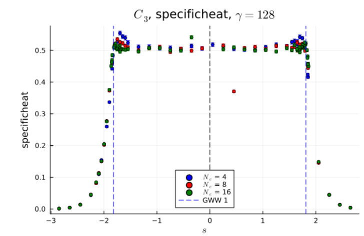




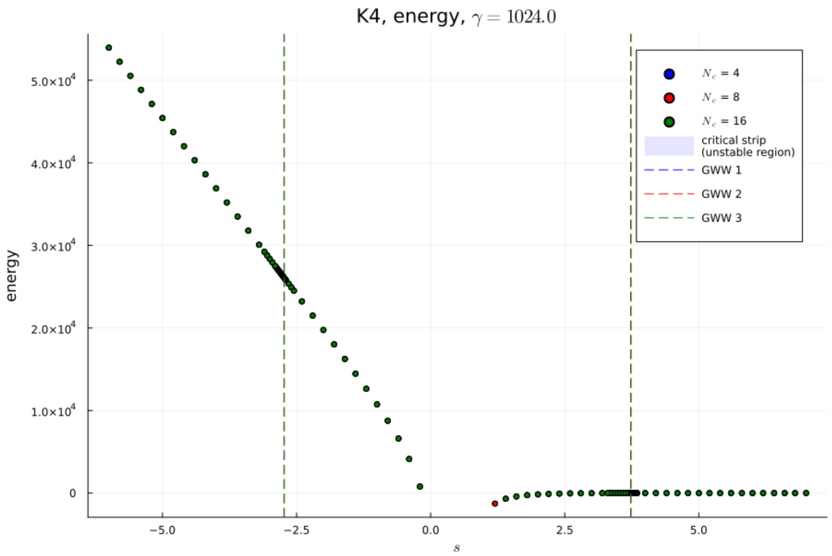


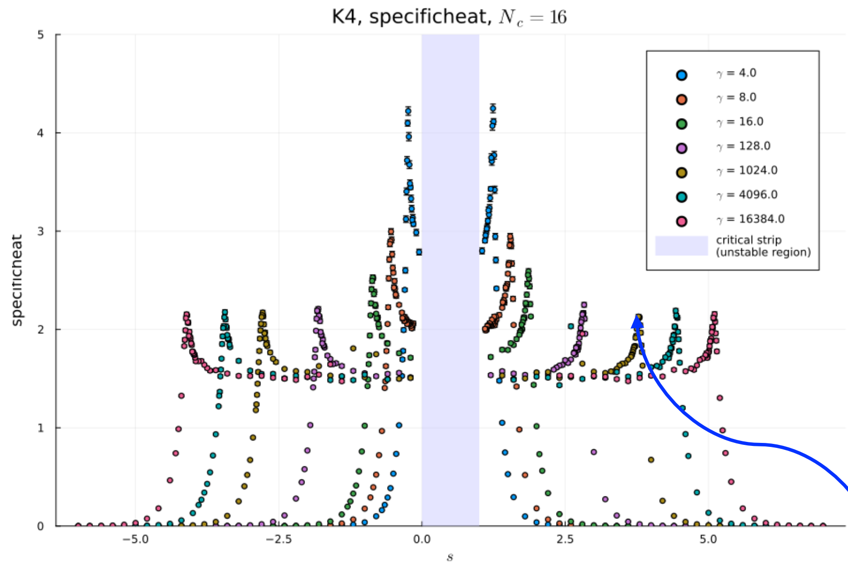


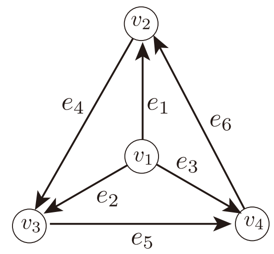


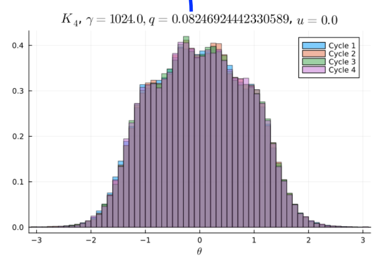


Tetrahedron



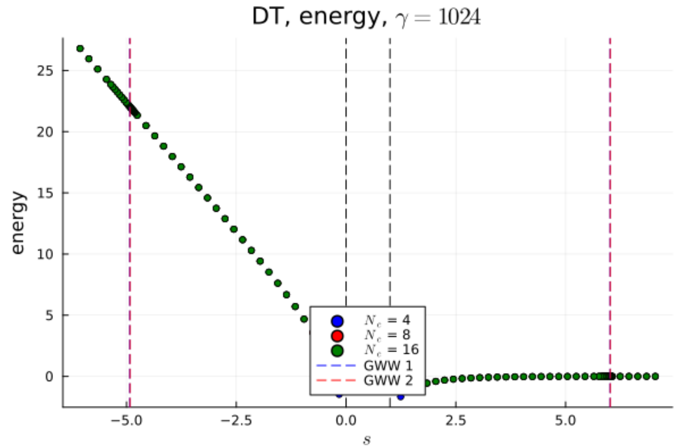




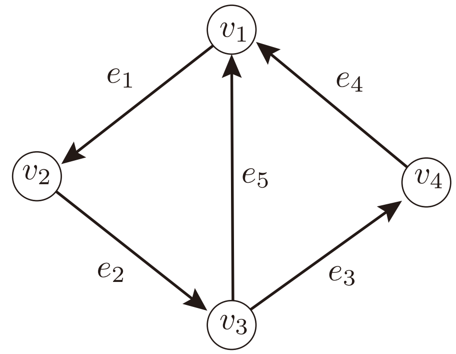


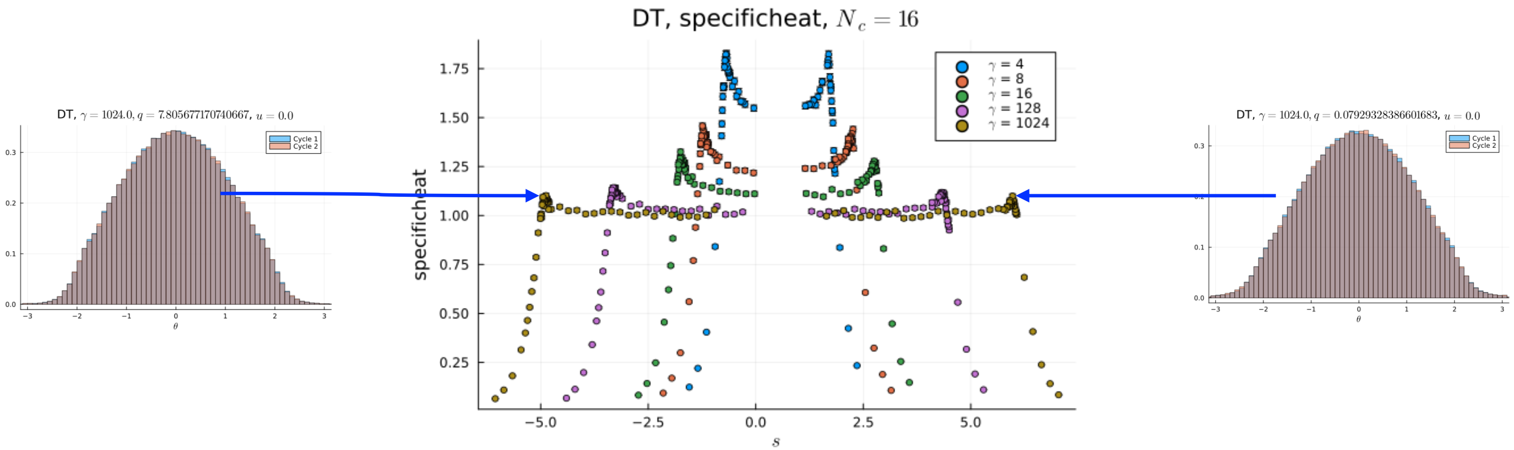
- RANK=3 (3 triangles)
- $q_c^3 \simeq 1/4\gamma$
- 2^{nd} order for all region of γ
- This is due to the difference of the number of the fundamental cycles and the number of the cycles of the minimal length.
- symmetric around s=1/2 (consistent to the duality)

Double Triangle

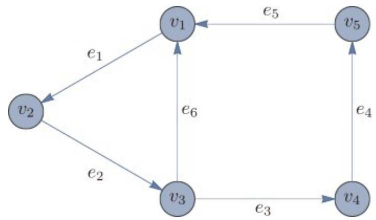


- RANK=2 (2 triangles)
- $q_c^3 \simeq 1/2\gamma$
- $2^{\rm nd}$ order for $\gamma < \infty$
- $3^{\rm rd}$ order in $\gamma \to \infty$
- slightly asymmetric (consistent to the dual description)

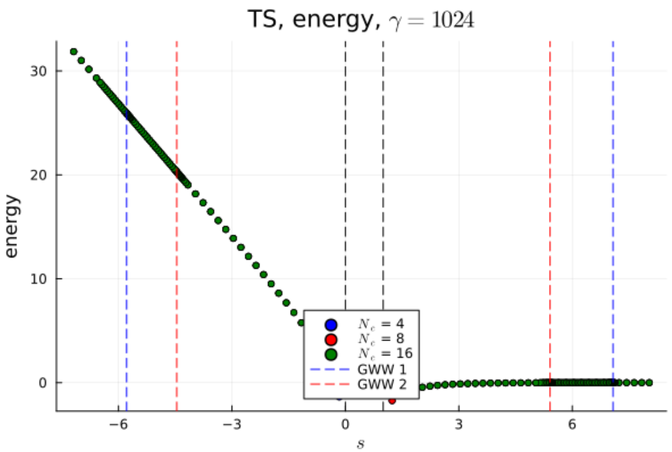


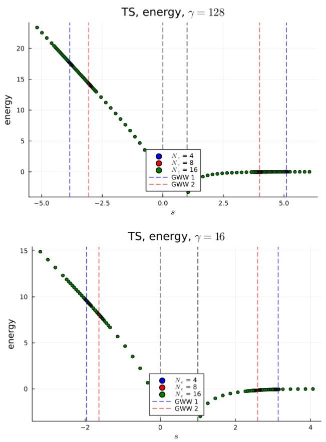


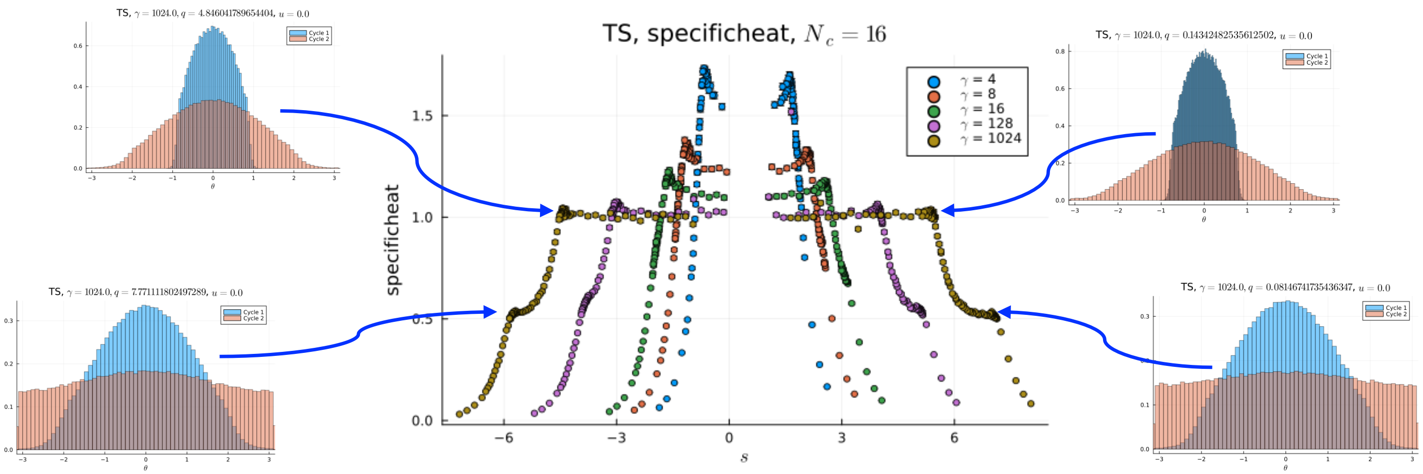
Triangle-Square



- RNAK=2 (triangle and square)
- there is an intermediate phase
- $q_{c1}^3 \simeq 1/2\gamma$, $q_{c2}^4 \simeq 1/2\gamma$
- c1: 3[rd] order for all γ
- c2: 2[nd] order for $\gamma < \infty$, 3[rd] order in $\gamma \to \infty$
- slightly asymmetric (consistent to the dual description)







Conclusion

- We have constructed the FKM model on the graph.
- The effective action of the FKM model is written by unitary matrix weighted graph zeta function.
- The FKM model reduces to Wilson's lattice gauge theory when the graph is a lattice.
- The FKM model has a strong/week coupling duality because of the functional relation of the graph zeta function.
- The FKM model enjoys the GWW phase transition in large N_c
- The phase structure of the FKM model depends on the structure of the fundamental cycles of the graph.

Future works

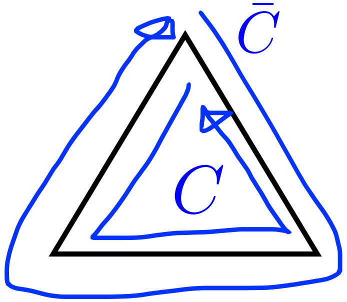
- Analytical description of the intermediate phases?
- Continuum limit?
- dynamical fermions?
- Physical meaning of the Riemann's hypothesis of graph zeta function or Ramanujan graph?
- Can graph zeta function be an observable of SUSY gauge theory on the graph?
- Relation to other zeta functions?

バックアップ

伊原ゼータ関数の例

primitive reduced cycleはふたつ

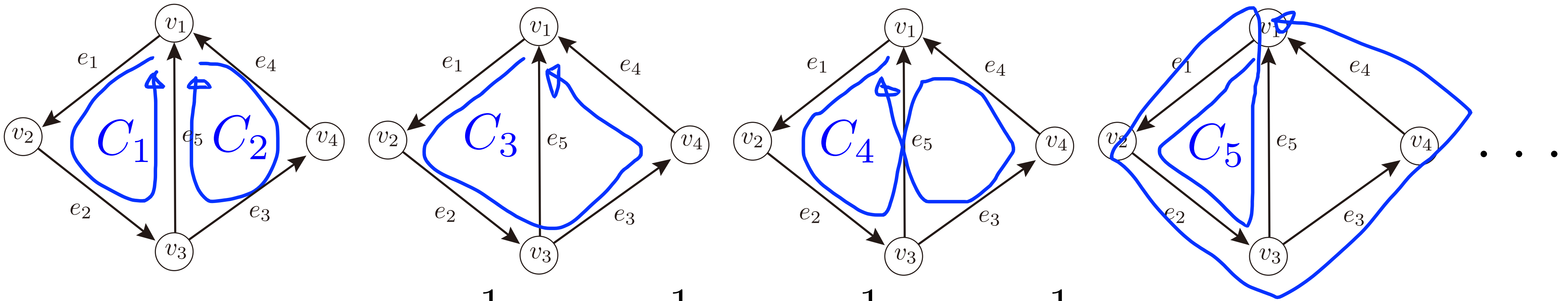
<u>Triangle</u>



$$\zeta_{C_3}(q) = \frac{1}{(1-q^3)^2} = 1 + 2q^3 + 3q^6 + 4q^9 + 5q^{12} + \cdots$$

power of q (length)	3	6	9	12	$\cdots$
coeff	2	3	4	5	$\cdots$
cycles	C, $\bar{C}$	C^2, $C\bar{C}$, $\bar{C}^2$	C^3, $C^2\bar{C}$, $C\bar{C}^2$, $\bar{C}^3$	C^4, $C^3\bar{C}$, $C^2\bar{C}^2$ $C\bar{C}^3$, $\bar{C}^4$	$\cdots$

<u>Double Triangle</u> 一般には、primitive reduced cycleは無数にあるため、素朴には閉じた形に書けない



$$\zeta_{\mathrm{DT}}(q) = \frac{1}{(1-q^3)^4}\frac{1}{(1-q^4)^2}\frac{1}{(1-q^6)^2}\frac{1}{(1-q^7)^4}\cdots$$

伊原の定理　～頂点表示～

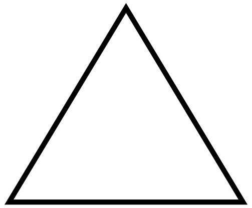
伊原ゼータ関数は行列式（多項式）の逆数で書ける

Ihara 1966

$$\zeta_G(q) = \frac{1}{(1-q^2)^{n_E - v_V} \det\Big(I - qA + q^2(D-I)\Big)}$$

n_V: 頂点の数
n_E: 辺の数
D: 度数行列
A: 頂点隣接行列
($n_V \times n_V$ matrix)

Triangle



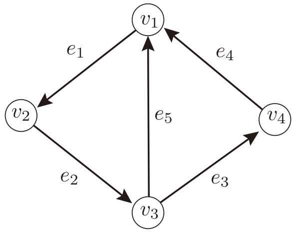
$n_V = n_E = 3$

$D = \mathrm{diag}(2,2,2)$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\zeta_{C_3}(q) = \frac{1}{\det\Big(I - qA + q^2(D-I)\Big)} = \frac{1}{(1-q^3)^2}$$

Double Triangle



$n_V = 4, n_E = 5$

$D = \mathrm{diag}(3,2,3,2)$

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\zeta_{\mathrm{DT}}(q) = \frac{1}{(1-q^2)\det\Big(I - qA + q^2(D-I)\Big)} = \frac{1}{(1-q^4)(1+q^2-2q^3)(1-q^2-2q^3)} = 1 + 4q^3 + 2q^4 + 12q^6 + 12q^7 + 3q^8 + \cdots$$

length	3	4	6	7	$\cdots$
coeff	4	2	12	12	$\cdots$
cycles	$C_1, \bar{C}_1, C_2, \bar{C}_2$	$C_3, \bar{C}_3$	$C_1^2, \bar{C}_1^2, C_2^2, \bar{C}_2^2, C_1\bar{C}_1, C_2\bar{C}_2, C_1C_2, \bar{C}_1C_2, C_1\bar{C}_2, \bar{C}_1\bar{C}_2$, $C_4, \bar{C}_4$	$C_1C_3, \bar{C}_1C_3, C_2C_3, \bar{C}_2C_3, C_1\bar{C}_3, \bar{C}_1\bar{C}_3, C_2\bar{C}_3, \bar{C}_2\bar{C}_3, C_5, \bar{C}_5, C_6, \bar{C}_6$	$\cdots$

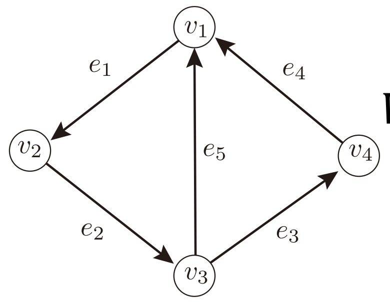
伊原ゼータ関数の辺表示

伊原ゼータ関数は辺隣接行列ベースの行列式でも表せる

<u>辺隣接行列</u>　$\boldsymbol{e} = \{e, e^{-1} | e \in E\}$

$$W_{\boldsymbol{e}\boldsymbol{e}'} = \begin{cases} 1 & \text{if } t(\boldsymbol{e}) = s(\boldsymbol{e}') \text{ and } \boldsymbol{e}'^{-1} \neq \boldsymbol{e} \\ 0 & \text{others} \end{cases}$$

(例) Double Triangle



$W =$

	e_1	e_2	e_3	e_4	e_5	$\bar{e}_1$	$\bar{e}_2$	$\bar{e}_3$	$\bar{e}_4$	$\bar{e}_5$
e_1		1								
e_2			1		1					
e_3				1						
e_4	1									1
e_5	1								1	
$\bar{e}_1$									1	1
$\bar{e}_2$						1				
$\bar{e}_3$					1		1			
$\bar{e}_4$								1		
$\bar{e}_5$			1				1			

伊原ゼータ関数の辺表示　　Bass 1992

$$\zeta_G(q) = \det(1 - qW)^{-1}$$