Integrable vortices on compact Riemann surfaces of genus one (~~and two~~)

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Introduction: Vortices

Vortices are ...

- 2-dimensional **topological solitons**
  It possesses a topological invariant, the **vortex number**
- Static solutions of 2+1d Abelian Higgs model

$$L = \int \left[\frac{-1}{2} F_{\mu\nu} F^{\mu\nu} + 2C \overline{D_\mu \phi} D^\mu \phi - (-C_0 + C|\phi|^2) \right] \Omega_0 dx dy$$

We would like to focus on the static energy (potential term) of this theory

Static energy functional of Abelian Higgs model

$$E = \int_M \left[\frac{1}{\Omega_0^2} F_{xy}^2 - \frac{2C}{\Omega_0} |D_i \phi|^2 + (-C_0 + C|\phi|^2) \right] \Omega_0 dx dy$$

- M is 2-dimensional space
- Ω_0 is a conformal factor of M i.e. $ds_M^2 = \Omega_0(dx^2 + dy^2)$
- It equals to the **Ginzburg-Landau theory** with a critical coupling constant

One can apply Bogomol'nyi completion to the energy E

The energy functional can be transformed into

$$E = \int_M \left[\left(\frac{F_{xy}}{\Omega_0} + C_0 - C|\phi|^2 \right)^2 - \frac{2C}{\Omega_0} |D_x \phi + i D_y \phi|^2 \right] \Omega_0 dxdy$$
$$- 2C_0 \int_M F_{xy} dxdy$$

- The last term is an integer (Vortex number)
- Bogomol'nyi equations (**Vortex eq.**) are derived from the formula: $D_x \phi + i D_y \phi = 0, \quad F_{xy} = \Omega_0(-C_0 + C|\phi|^2)$
- Solutions of the Bogomol'nyi eq.s minimize E

Jackiw-Pi vortex

When $(C_0, C) = (0, 1)$, the vortex eq. is called **Jackiw-Pi vortex eq.** It can be transformed into **Liouville's eq.** (solvable)

[Manton(2016)]

$$D_x\phi + iD_y\phi = 0,\ F_{xy} = \Omega_0|\phi|^2 \quad \Rightarrow \quad \nabla^2 \log|\phi| = \Omega_0|\phi|^2$$

The general solution of Liouville's eq. (Jackiw-Pi vortex):

$$\phi(z, \bar{z}) = \frac{f'(z)}{1 + |f(z)|^2} \quad f : \mathbb{R}^2 \to S^2$$

f is a meromorphic function

Jackiw-Pi eq. can also be derived from non-relativistic 2+1d **CS matter theory**. In this context, the JP vortex relates to the Hall effect.

[Horvathy-Zhang(2009)]

$$L_{CSM} = -\bar{\phi} D_t \phi + \frac{|\vec{D}\phi|^2}{2M} - \frac{\lambda}{2}|\phi|^4 - \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma}$$

The EOMs are

$$i\partial_t \phi = \left(-\frac{\vec{D}\cdot\vec{D}}{2M} - eA_t - \lambda|\phi|^2 \right)\phi, \qquad \frac{\kappa}{2}\epsilon^{\mu\alpha\beta} F_{\alpha\beta} = eJ^\mu$$

The Hamiltonian of the theory takes the form

$$H = \int \left(\frac{|D_{\pm}\phi|^2}{2M} + \frac{\lambda|\kappa|M - e^2}{2|\kappa|M} |\phi|^2 \right) dxdy$$

Assuming $D_+\phi = 0, \partial_t\phi = 0$ and setting $\lambda = e^2/(|\kappa|M)$, EOM of gauge field takes the form

$$\kappa F_{xy} = e|\phi|^2 \quad \Rightarrow \quad \nabla^2 \log|\phi| = 2e^2|\phi|^2$$

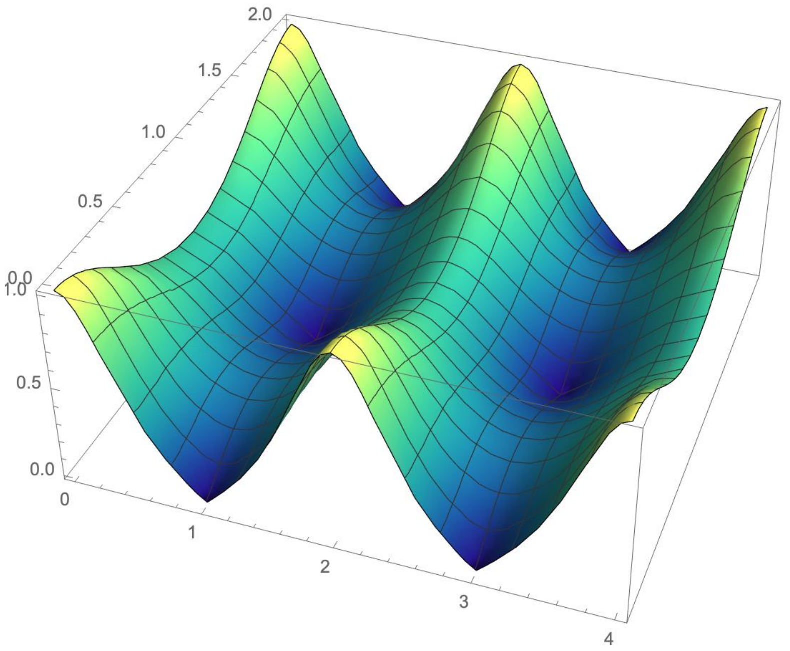
And solutions minimize H

Jackiw-Pi vortex on Torus

J.P. vortices are defined on the Torus if f is the doubly periodic function (**elliptic function**)

■**Example: Jacobi sn**

$$\phi_{\mathrm{sn}}(z,\bar{z};k)=\frac{\mathrm{cn}(z;k)\mathrm{dn}(z;k)}{1+|\mathrm{sn}(z;k)|^2}$$

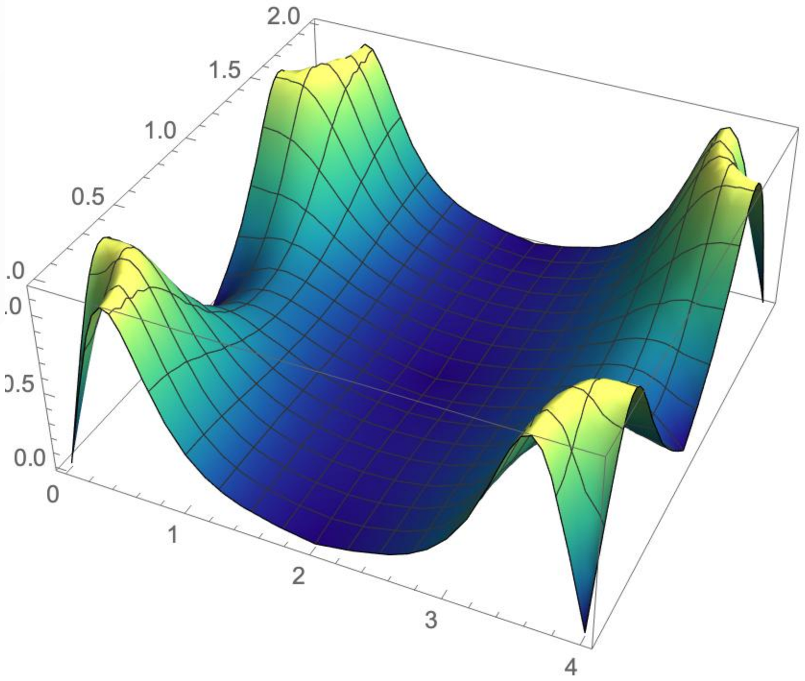


- is defined on $T^2=\mathbb{R}^2/\Lambda$,
  $\Lambda=4K(k)\mathbb{Z}+2iK'(k)\mathbb{Z}$
- Vortex number $N=4$

■Example: Weierstrass $\wp$

$$\phi_\wp(z,\bar{z}) = \frac{\wp'(z;\omega_1,\omega_2)}{1+|\wp(z;\omega_1,\omega_2)|^2}$$

- is defined on $T^2 = \mathbb{R}^2/\Lambda$, $\Lambda = 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}$
- Vortex number $N = 4$



Jackiw-Pi vortices on the torus are classified completely

[Akerblom-Cornelissen-Stavenga-Holten(2011)]

In previous examples, the vortex number N is given by

$$N = \frac{1}{2\pi} \int_M F_{xy} dx dy$$

- Naively the integral always vanishes if M is a compact surface (e.g. torus)
- However numerical integration gives $N \neq 0$

  [ACSH(2011)], [Olesen(1991)].

- Vector bandle argument explains it: N is given by the transition function on the intersections of patches

  [Manton-Sutcliffe(2004)]

Calculation of Vortex Number

[Miyamoto-Nakamula(2023)] (in preparation)

Instead of the bandle argument, we would like to give an **analytical method**: We regularize singular points and calculate N directly

- Liouville's equation $\nabla^2 \log|\phi| = 2|\phi|^2$ is not defined at zeros of ϕ (Singular Points)
- Then we treat $\widetilde{T}^2 = T^2 \backslash \mathrm{S.P.}$ which has boundary around zeros
- Then we apply Green's theorem to $\frac{1}{2\pi}\int F_{xy} dx dy$ on $\widetilde{T}^2$

Using $F_{xy} = |\phi|^2$, one obtains

$$\frac{1}{2\pi}\int_{\widetilde{T}^2} F_{xy} dxdy = \frac{1}{4\pi}\int_{\widetilde{T}^2} \partial_x^2 \log|\phi|^2 + \partial_y^2 \log|\phi|^2 dxdy$$
$$= \frac{-i}{4\pi}\oint_{\partial\widetilde{T}^2} \partial \log|\phi|^2 dz - \overline{\partial} \log|\phi|^2 d\bar{z}$$

where $\partial\widetilde{T}^2$ are infinitesimal circles C_{ξ_i} around zeros $\{\xi_i\}$
$\partial := (\partial_x - i\partial_y)/2,\ \overline{\partial} := (\partial_x + i\partial_y)/2$ are Wirtinger derivatives

- ϕ is not holomorphic nor anti-holomorphic. Then one can apply ∂, $\overline{\partial}$ to the function of ϕ

Considering ϕ as a two-variable function, one can expand it

Let ξ_0 be a simple zero of ϕ
Then around ξ_0, ϕ can be expanded as follows

$$\phi(z,\bar{z}) \sim \partial\phi|_{z=\xi_0}(z-\xi_0) + \overline{\partial}\phi|_{z=\xi_0}(\overline{z-\xi_0}) + \cdots$$

- We write these coeff.s c^0, c^1 for short

Substituting it into $\partial \log |\phi|^2 dz$, one obtains

$$\partial \log |\phi|^2 dz = \left(\frac{\partial \phi}{\phi} + \frac{\partial \overline{\phi}}{\overline{\phi}} \right) dz$$

$$\sim \left(\frac{c^0 + \cdots}{c^0 (z - \xi_0) + c^1 \overline{(z - \xi_0)} + \cdots} + \frac{c^1 + \cdots}{c^0 \overline{(z - \xi_0)} + c^1 (z - \xi_0) + \cdots} \right) dz$$

Let $z = \xi_0 + \epsilon e^{i\theta}$ around ξ_0, then $dz = i\epsilon e^{i\theta} d\theta$

Hence, in $\epsilon \to 0$

$$\partial \log |\phi|^2 dz \sim \left(\frac{c^0}{c^0 e^{i\theta} + c^1 e^{-i\theta}} + \frac{c^1}{c^0 e^{-i\theta} + c^1 e^{i\theta}} \right) i e^{i\theta} d\theta$$

By the same calculation, one obtains

$$\overline{\partial} \log |\phi|^2 d\bar{z} \sim \left(\frac{c^1}{c^0 e^{i\theta} + c^1 e^{-i\theta}} + \frac{c^0}{c^0 e^{-i\theta} + c^1 e^{i\theta}} \right) (-i) e^{-i\theta} d\theta$$

Then the integrand becomes

$$\left(\frac{c^0 i e^{i\theta}}{c^0 e^{i\theta} + c^1 e^{-i\theta}} + \frac{c^1 i e^{i\theta}}{c^0 e^{-i\theta} + c^1 e^{i\theta}} \right) d\theta + \left(\frac{c^1 i e^{-i\theta}}{c^0 e^{i\theta} + c^1 e^{-i\theta}} + \frac{c^0 i e^{-i\theta}}{c^0 e^{-i\theta} + c^1 e^{i\theta}} \right) d\theta$$

$$= 2i d\theta$$

Hence for each simple zero one obtains 1 vortex number

■Example 1: ϕ_{sn}

$\phi_{\mathrm{sn}}(z,\bar{z};k) = \frac{\mathrm{cn}(z;k)\mathrm{dn}(z;k)}{1+|\mathrm{sn}(z;k)|^2}$

ϕ_{sn} has four simple zeros $\{K, K+iK', 3K, 3K+iK'\}$

One can calculate

$$c^0 = \left(\frac{-\mathrm{sn}\ \mathrm{dn}^2 - k^2\mathrm{sn}\ \mathrm{cn}^2}{(1+|\mathrm{sn}|^2)} + \frac{-\mathrm{cn}^2\ \mathrm{dn}^2\ \overline{\mathrm{sn}}}{(1+|\mathrm{sn}|^2)^2}\right)\Bigg|_{z=\xi_0},$$

$$c^1 = \frac{-\mathrm{sn}\ |\mathrm{cn}\ \mathrm{dn}|^2}{(1+|\mathrm{sn}|^2)^2}\Bigg|_{z=\xi_0}$$

	K	$K+iK'$	$3K$	$3K+iK'$
c^0	$\frac{-k'^2}{2}$	$\frac{k'^2}{k(1+\|1/k\|^2)}$	$\frac{k'^2}{2}$	$\frac{-k'^2}{k(1+\|1/k\|^2)}$
c^1	0	0	0	0

Then all integrand around zeros takes the form $\left(\frac{c^0 ie^{i\theta}}{c^0 e^{i\theta}} + \frac{c^0 ie^{-i\theta}}{c^0 e^{-i\theta}}\right) d\theta$
$= 2id\theta$
Hence the vortex number is

$$N = 4 \times \left(\frac{-i}{4\pi}\int_0^{2\pi} 2id\theta\right) = 4$$

■ Example 2: $\phi_\wp$

$$\phi_\wp(z,\bar{z}) = \frac{\wp'(z;\omega_1,\omega_2)}{1+|\wp(z;\omega_1,\omega_2)|^2}$$

$\phi_\wp$ has four simple zeros $\{0,\omega_1,\omega_2,\omega_1+\omega_2\}$

Then one obtains

$$c^0 = \left.\frac{\wp''(1+|\wp|^2) - \overline{\wp}\wp'^2}{(1+|\wp|^2)^2}\right|_{z=\xi_0}, \quad c^1 = \left.\frac{-\wp|\wp'|^2}{(1+|\wp|^2)^2}\right|_{z=\xi_0}$$

For $\{\omega_1, \omega_2, \omega_1 + \omega_2\}$, the calculation is same as ϕ_{sn} case:
$c^0 = \text{complex const.}, c^1 = 0$
Then the integrands for $\{\omega_1, \omega_2, \omega_1 + \omega_2\}$ are $2id\theta$

For $\{0\}$, $\wp$ and $\wp'$ have a pole
In neighborhood of the point 0 one can write $\wp \sim \frac{1}{z^2},\ \wp' \sim \frac{-2}{z^3}$
Then in $z \to 0$

$$c^0 \sim \frac{1}{(1+|z|^4)^2}\left(6\frac{|z|^8}{z^4} + 6\frac{|z|^4}{z^4} - \frac{g_2}{2}(|z|^8 + z|^4) - 4\frac{|z|^4}{z^4}\right) \to 2e^{-4i\theta}$$

$$c^1 \sim \frac{1}{(1+|z|^4)^2}\frac{-4|z|^2}{z^2} \to -4e^{-2i\theta}$$

Then the integrand around 0 takes form

$$\left(\frac{2ie^{-3i\theta} - 4ie^{-3i\theta}}{2e^{-3i\theta} - 4e^{-3i\theta}} + \frac{2ie^{-5i\theta} - 4ie^{-i\theta}}{2e^{-5i\theta} - 4e^{-i\theta}}\right)d\theta$$

$$= 2id\theta$$

Integration of this is $2i\pi$, which corresponds to 1 vortex number

Hence the vortex number of $\phi_{\wp}$ equals 4 as we expected

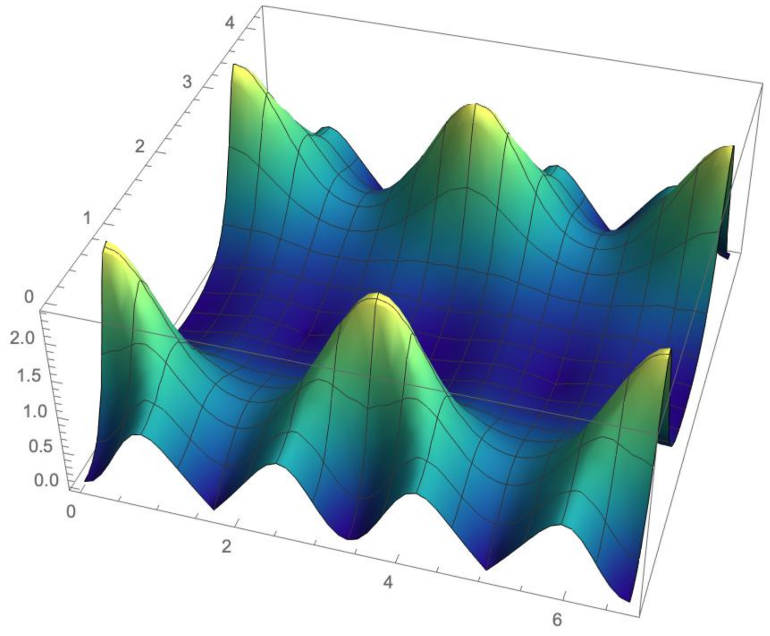
■Zeros of order $n > 1$ case

There is the solution that has zeros of order $n > 1$

$$\phi \sim c^{00}(z-\xi_0)^2 + c^{01}(z-\xi_0)(\overline{z-\xi_0}) + c^{10}(z-\xi_0)(\overline{z-\xi_0}) + c^{11}(\overline{z-\xi_0})^2 + \cdots$$

Calculating $c^{00}, c^{01}, \ldots$, one can obtain the integrand which does not vanish in $\epsilon \to 0$

then one can calculate the vortex number



ϕ made of $f = \mathrm{sn}^3(z)$

This has zeros of order 2

Conclusion

- Vortices are 2-dimensional topological soliton
- Jackiw-Pi vortex eq. is one of the integrable vortex eq. and can be defined on the torus
- Analytical calculation method of the vortex number of Jackiw-Pi vortices on the torus is given

■Future works

- Calculation method for the vortices on the higher genus surface

Buckup

There exist Five integrable vortex eq.s

These Eq.s have the geometrical interpretation

[Baptista(2014)], [Manton(2016)]

(C_0, C)	**Name**	**Eq.**
$(0, 1)$	Jackiw-Pi	$\nabla^2 \log \lvert\phi\rvert = \Omega_0 \lvert\phi\rvert^2$
$(1, 1)$	Popov	$\nabla^2 \log \lvert\phi\rvert = \Omega_0(-1 + \lvert\phi\rvert^2)$
$(-1, 1)$	Ambjørn-Olesen	$\nabla^2 \log \lvert\phi\rvert = \Omega_0(1 + \lvert\phi\rvert^2)$
$(-1, 0)$	Bradlow	$\nabla^2 \log \lvert\phi\rvert = \Omega_0$
$(-1, -1)$	Taubes	$\nabla^2 \log \lvert\phi\rvert = \Omega_0(1 - \lvert\phi\rvert^2)$