Krylov complexity of free and interacting scalar QFTs with bounded power spectrum

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[arXiv:2212.14702]

with Hugo A. Camargo, Viktor Jahnke, Keun-Young Kim

Krylov complexity is a measure of operator growth.

Krylov complexity $K_{\mathcal{O}}(t) := \sum_n n|\varphi_n(t)|^2$

[D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, E. Altman, 2018]

Lanczos algorithm is a mathematical method for Krylov basis $\mathcal{O}_n$ for $\mathcal{O}(t)$.

$$\mathcal{O}(t) = \sum_{n=0} i^n \varphi_n(t) \mathcal{O}_n$$

Lanczos algorithm can determine Krylov basis $\mathcal{O}_n$ Lanczos coefficients b_n , and wave functions $\varphi_n(t)$.

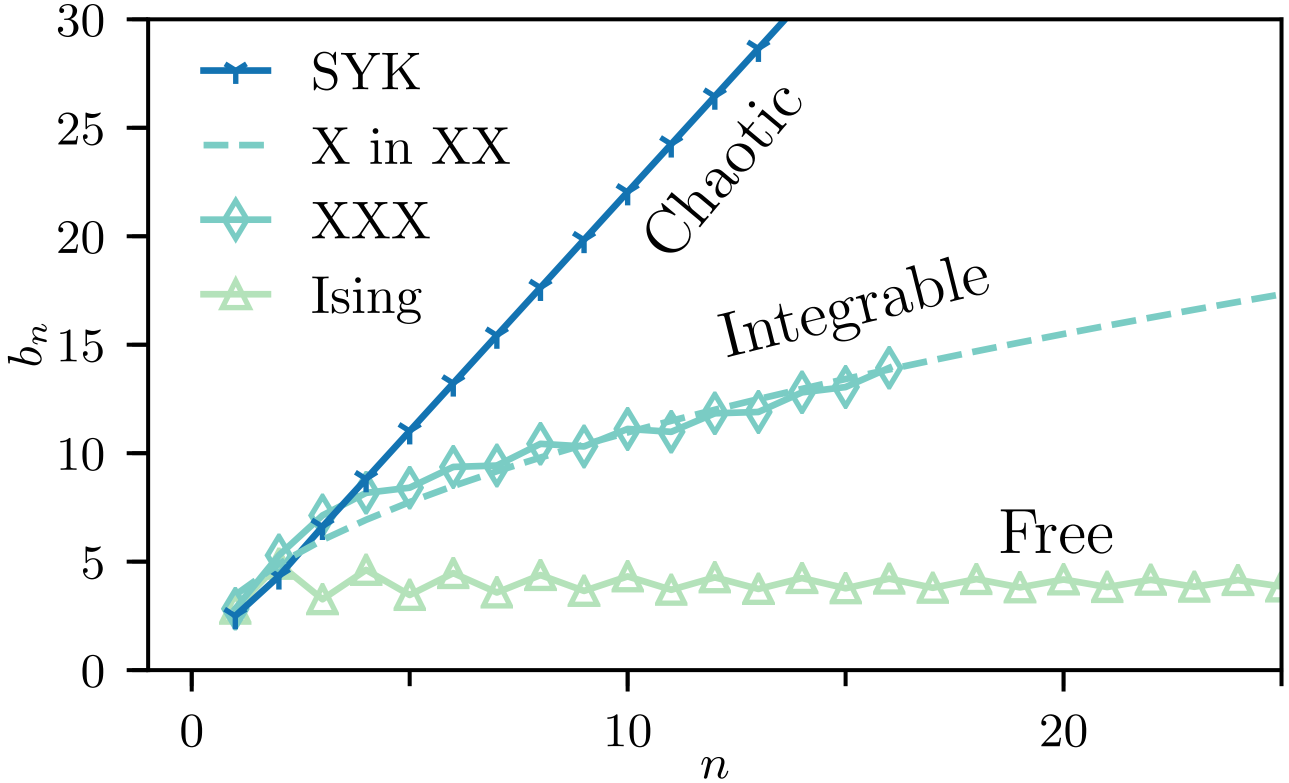
Universal operator growth hypothesis

Lanczos coefficient b_n of non-integrable systems in the thermodynamic limit grows linearly.

$$b_n \sim \alpha n + \gamma \quad \text{at large} \quad n$$

[D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, E. Altman, 2018]

$$|\mathcal{O}_{-1}) := 0,\ |\mathcal{O}_0) := |\mathcal{O}),\ \mathcal{L}|\mathcal{O}_n) = a_n|\mathcal{O}_n) + b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1})$$



- If Hilbert space is infinite, b_n can increase forever.
- If Hilbert space is finite, b_n should decrease to zero.

[R. Watanabe's talk]

λ_K bounds λ_L

Smooth linear behavior $b_n \sim \alpha n + \gamma$ implies the exponential growth behavior $K_{\mathcal{O}}(t) \sim e^{2\alpha t} = e^{\lambda_K t}$

[J.L.F. Barbon, E. Rabinovici, R. Shir, R. Sinha, 2019]

An exact example

$$C(t) = \frac{1}{(\cosh(\alpha t))^\eta}, \quad b_n = \alpha\sqrt{n(n-1+\eta)}, \quad K_{\mathcal{O}}(t) = \eta \sinh^2(\alpha t)$$

Generalized chaos bound

$$\lambda_L \le \lambda_K = 2\alpha$$

proved $(T = \infty)$

[D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, E. Altman, 2018]

$$\lambda_L \le \lambda_K \le 2\pi T$$

conjecture (finite T)

[A. Avdoshkin, A. Dymarsky, 2019], [Y. Gu, A. Kitaev, P. Zhang, 2021]

My motivation

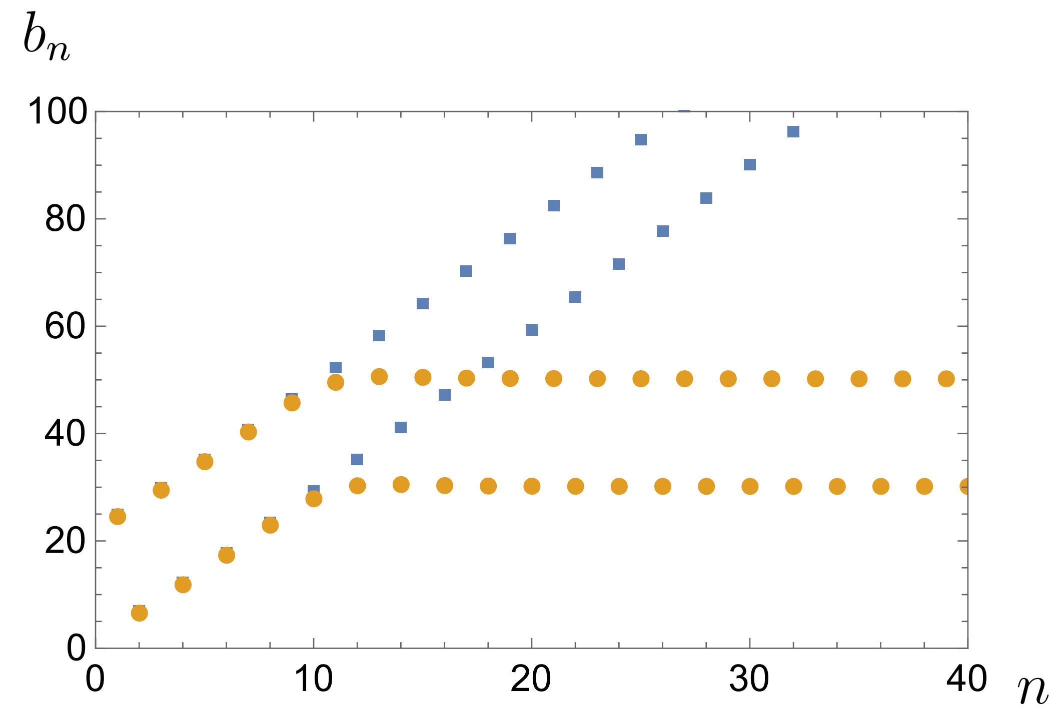
- Understand the meaning of CFT results in [A. Dymarsky, M. Smolkin, 2021]

- Understand how much difference of Krylov complexity in lattice and continuum theories

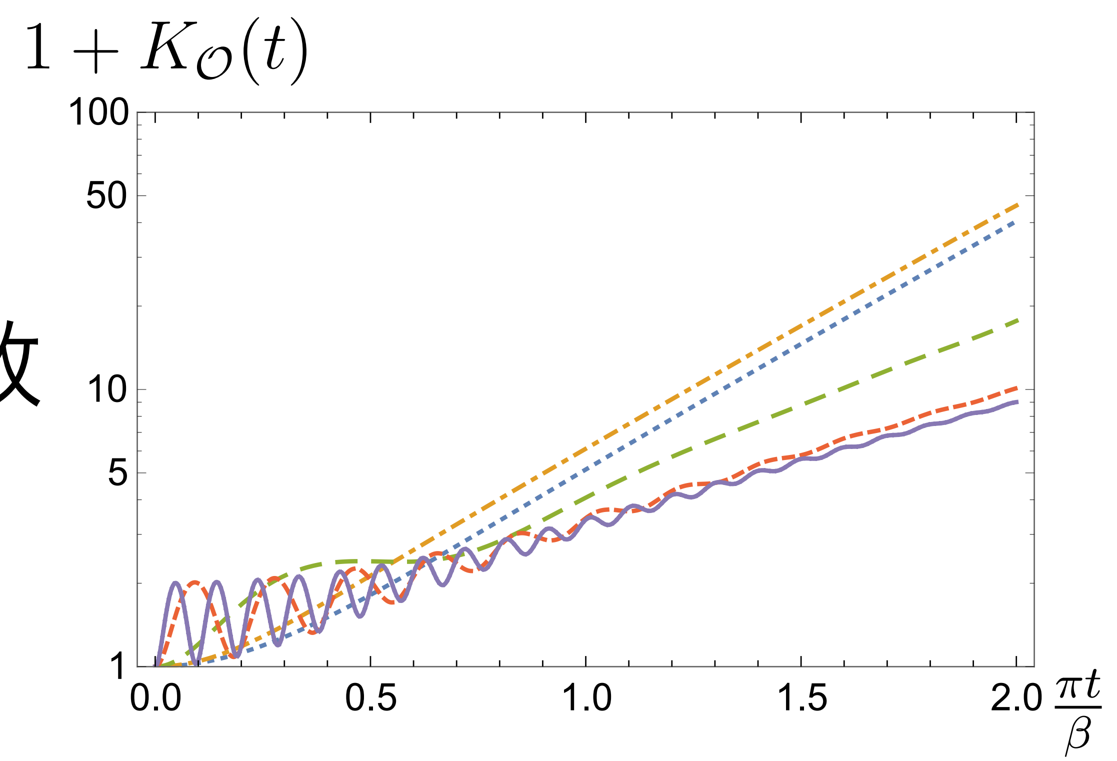
- Compute Krylov complexity of familiar and simple theories in QFT's textbooks

我々がやったこと

- 自由massive scalar場の理論のLanczos係数とKrylov complexityを調べた



- Mass gap とUV cutoffの効果を調べた



- 4d ϕ^3 と4d ϕ^4 理論のLanczos係数を摂動的に調べた

Outline

1. Moment method

2. Lanczos coefficients and Krylov complexity in scalar QFTs

3. Summary

Expansion of $\mathcal{O}(t)$ and inner product

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt} = \sum_{n=0} \frac{(it)^n}{n!}\mathcal{L}^n\mathcal{O} \qquad \mathcal{L}\mathcal{O} := [H, \mathcal{O}]$$

We want to construct an orthonormal basis for $\{\mathcal{L}^n\mathcal{O}\}$ choosing an inner product.

Choices of inner products

$$(A|B) := \mathrm{Tr}[A^\dagger B]/\mathrm{Tr}[1]$$ Infinte temperature

$$(A|B)^S_\beta := \frac{1}{2Z}\mathrm{Tr}[e^{-\beta H}(A^\dagger B + BA^\dagger)]$$ Standard inner product

$$(A|B)^W_\beta := \frac{1}{Z}\mathrm{Tr}[e^{-\beta H/2}A^\dagger e^{-\beta H/2}B]$$ Wightman inner product

Lanczos algorithm

Algorithm to construct a basis
for tridiagonalization of a Hermitian matrix

$$|\mathcal{O}_{-1}) := 0,\ |\mathcal{O}_0) := |\mathcal{O}),\ \mathcal{L}|\mathcal{O}_n) = a_n|\mathcal{O}_n) + b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1})$$

Krylov subspace

$$\mathrm{Span}\{\mathcal{L}^n\mathcal{O}\}$$

Krylov basis

$$|\mathcal{O}_n)$$

$$(\mathcal{O}_m|\mathcal{L}|\mathcal{O}_n) = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_2 & 0 & \cdots \\ 0 & b_2 & a_2 & b_3 & \cdots \\ 0 & 0 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

a_n, b_n : Lanczos coefficients

$a_n = 0$ for a Hermitian operator $\mathcal{O}$.

Lanczos coefficients can be determined from a 2pt function.

2pt function $C(t) := (\mathcal{O}|\mathcal{O}(-t)) = \sum_{n=0} M_n \frac{(-it)^n}{n!}$

Moments $M_n := \frac{1}{(-i)^n} \frac{d^n C(t)}{dt^n}\Big|_{t=0} = (\mathcal{O}_0|\mathcal{L}^n|\mathcal{O}_0)$

Moments determine Lanczos coefficients.

$$
\begin{aligned}
M_1 &= (\mathcal{O}_0|\mathcal{L}|\mathcal{O}_0) = a_0, \\
M_2 &= (\mathcal{O}_0|\mathcal{L}^2|\mathcal{O}_0) = a_0^2 + b_1^2, \\
M_3 &= (\mathcal{O}_0|\mathcal{L}^3|\mathcal{O}_0) = a_0^3 + 2a_0 b_1^2 + a_1 b_1^2 \,, \\
M_4 &= (\mathcal{O}_0|\mathcal{L}^4|\mathcal{O}_0) = (a_0 + a_1)^2 b_1^2 + (a_0^2 + b_1^2)^2 + b_1^2 b_2^2.
\end{aligned}
$$

Time evolution of $\varphi_n(t)$

$$|\mathcal{O}(t)) = \sum_{n=0} i^n \varphi_n(t)|\mathcal{O}_n), \quad \varphi_n(t) := i^{-n}(\mathcal{O}_n|\mathcal{O}(t))$$

$$|\mathcal{O}_{-1}) := 0,\ |\mathcal{O}_0) := |\mathcal{O}),\ \mathcal{L}|\mathcal{O}_n) = a_n|\mathcal{O}_n) + b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1})$$

$$\varphi_{-1}(t) := 0, \varphi_0(t) = C(-t), \frac{d\varphi_n(t)}{dt} = ia_n\varphi_n(t) + b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

From $C(t)$, we can determine a_n, b_n and solve $\varphi_n(t)$ recursively.

Then, we can compute $K_{\mathcal{O}}(t) := \sum_n n|\varphi_n(t)|^2$.

Outline

1. Moment method

2. Lanczos coefficients and Krylov complexity
   in scalar QFTs

3. Summary

How to compute b_n and $K_{\mathcal{O}}(t) := \sum_n n|\varphi_n(t)|^2$

2pt function

$$C(t) = \langle \phi(t - i\beta/2, \mathbf{0})\phi(0, \mathbf{0})\rangle_\beta$$

Wightman power spectrum

$$f^W(\omega) := \int \mathrm{d}t\, C(t) e^{i\omega t}$$

$$= \frac{1}{\sinh[\beta\omega/2]} \int \frac{\mathrm{d}^{d-1}\mathbf{k}}{(2\pi)^{d-1}} \rho(\omega, \mathbf{k})$$

Moments

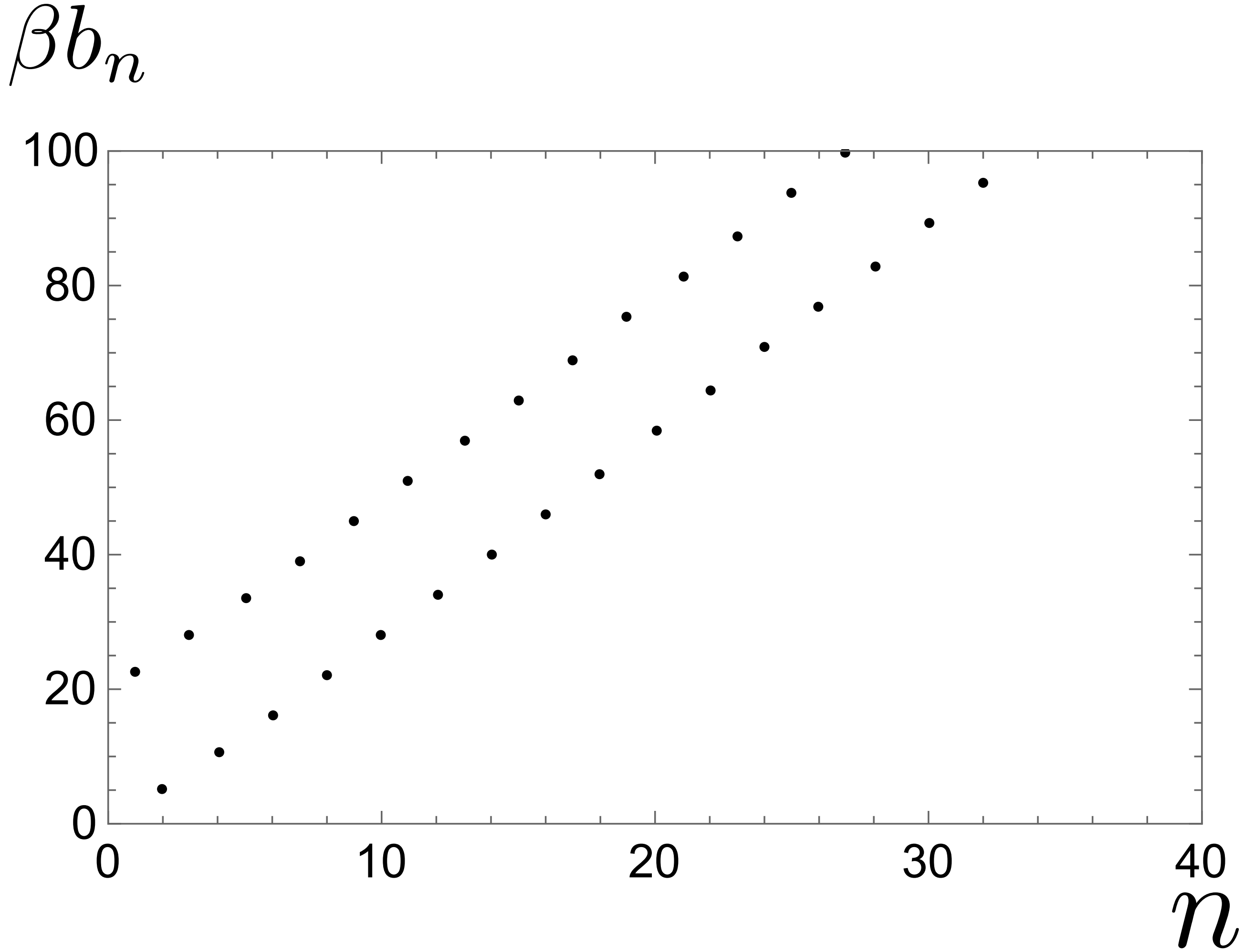
$$M_{2n} := \frac{1}{(-i)^{2n}} \frac{d^{2n}C(t)}{dt^{2n}}\Big|_{t=0}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, \omega^{2n} f^W(\omega)$$

From a given spectral function $\rho(\omega, \mathbf{k})$,

we can compute $C(t), M_{2n}, b_n, K_{\mathcal{O}}(t)$

b_n of free massive scalar theory



$$b_n \sim \frac{\pi}{\beta} n + \gamma_{\rm odd} \quad (\text{odd } n)$$

$$b_n \sim \frac{\pi}{\beta} n + \gamma_{\rm even} \quad (\text{even } n)$$

$$\gamma_{\rm odd} - \gamma_{\rm even} \sim m$$

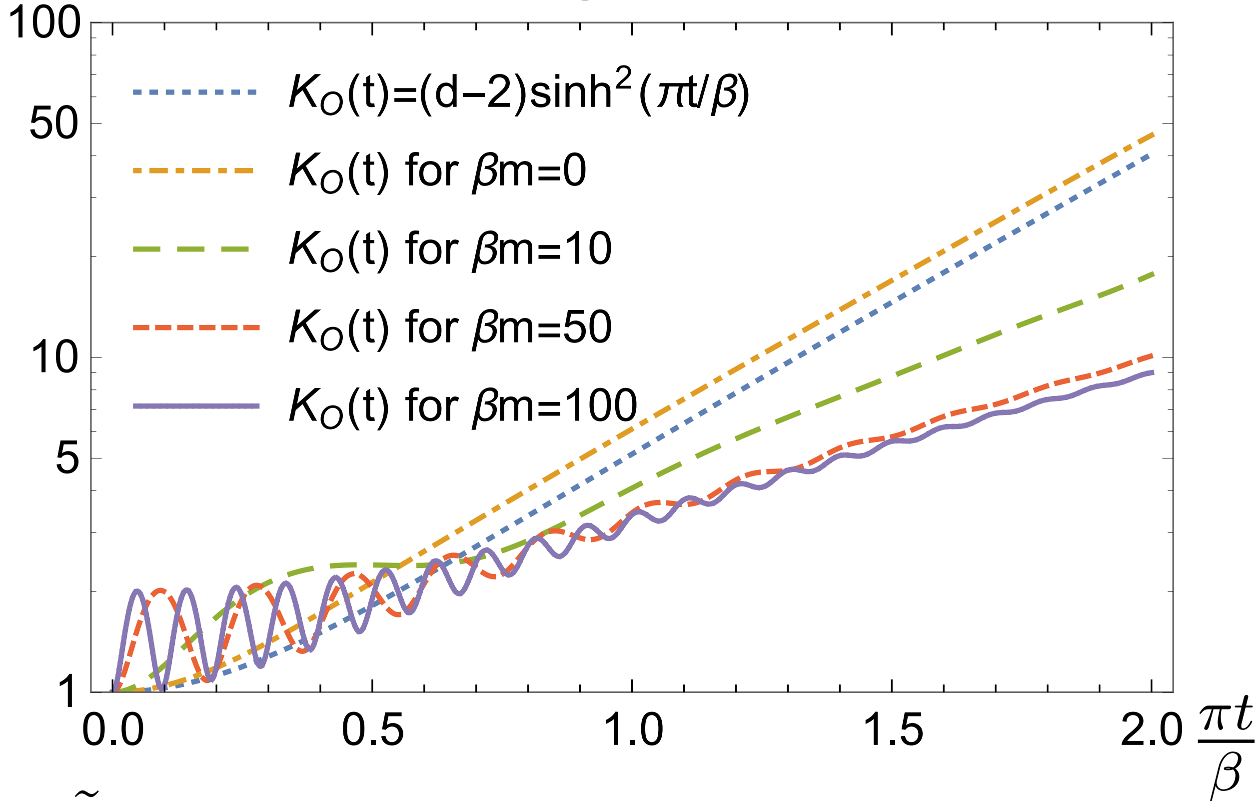
$$d = 5, \beta = 1, m = 20$$

Mass m causes the difference between $b_{\rm odd}$ and $b_{\rm even}$.

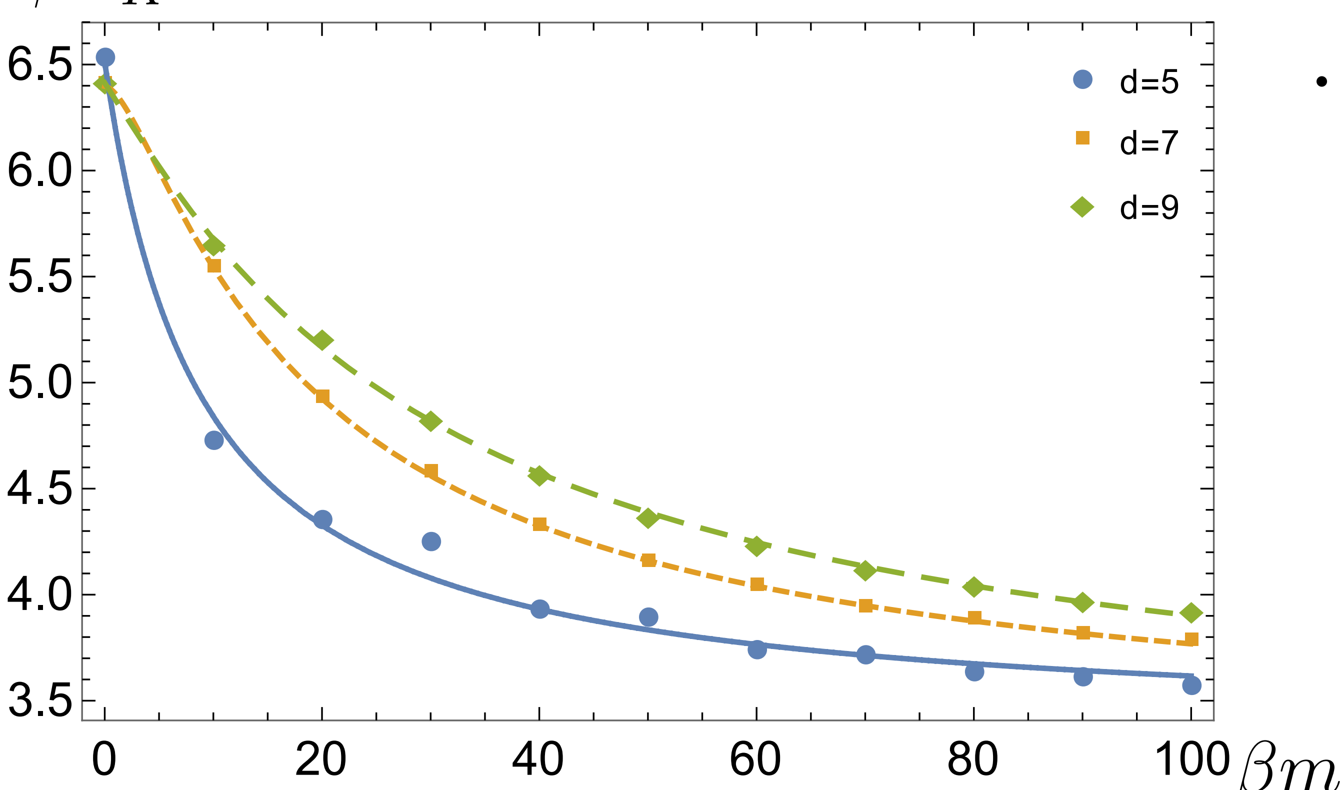
b_n is not smooth with respect to n due to mass.

$K_{\mathcal{O}}(t)$ of free massive scalar theory

$1+K_{\mathcal{O}}(t)$ Log plot $d=5$



$\beta\tilde{\lambda}_K$



$$K_{\mathcal{O}}(t) \sim e^{\tilde{\lambda}_K t} \quad \left(1.5 \leq \frac{\pi t}{\beta} \leq 2.0\right)$$

- For $\beta m = 0$, $K_{\mathcal{O}}(t) \sim e^{\frac{2\pi}{\beta}t}$

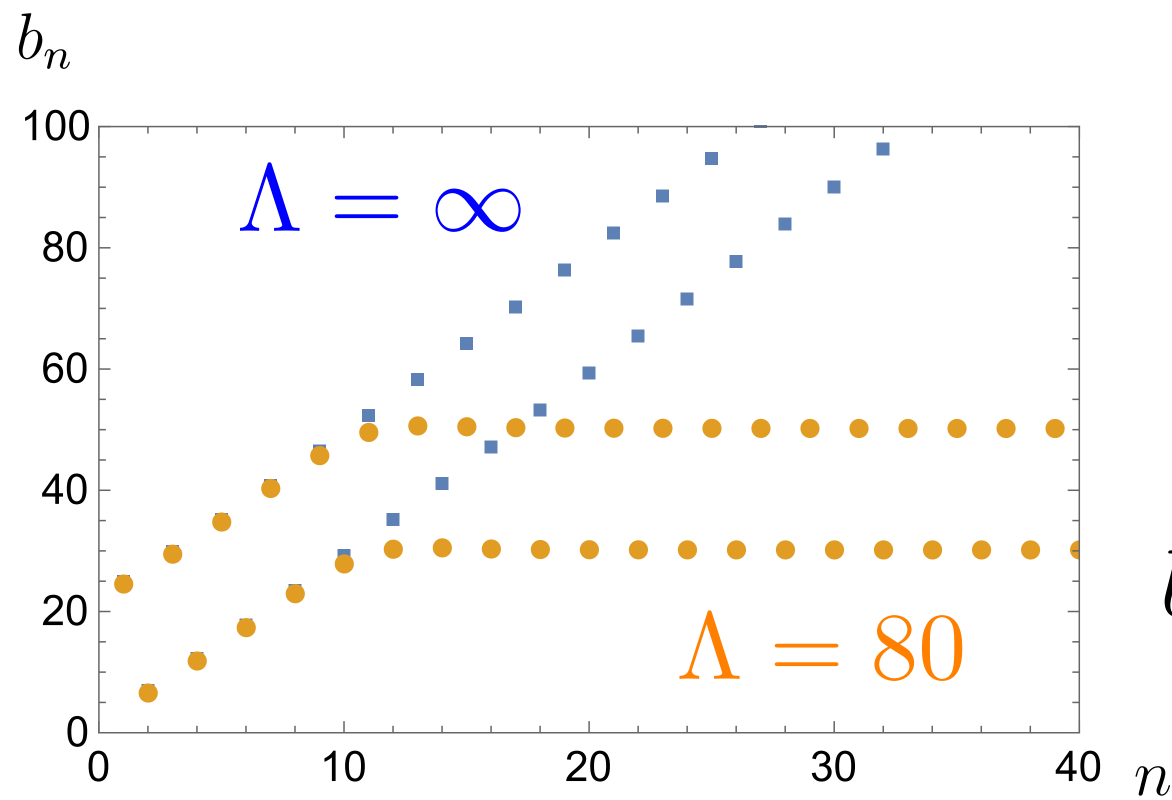
  [A. Dymarsky, M. Smolkin, 2021]

- For $\beta m \neq 0$, $\tilde{\lambda}_K$ decreases due to mass.

- Mass violates the smoothness of b_n for $K_{\mathcal{O}}(t) \sim e^{2\alpha t}$ from $b_n \sim \alpha n + \gamma$

b_n With finite UV cutoff Λ $\quad (d=5)$

$$f^W(\omega) \sim N(m,\beta,\Lambda)\,(\omega^2 - m^2)\,e^{-\frac{\beta|\omega|}{2}}\,\Theta\left(|\omega| - m\,,\; \Lambda - |\omega|\right)$$



$b_{\text{odd}} \sim (\Lambda + m)/2$

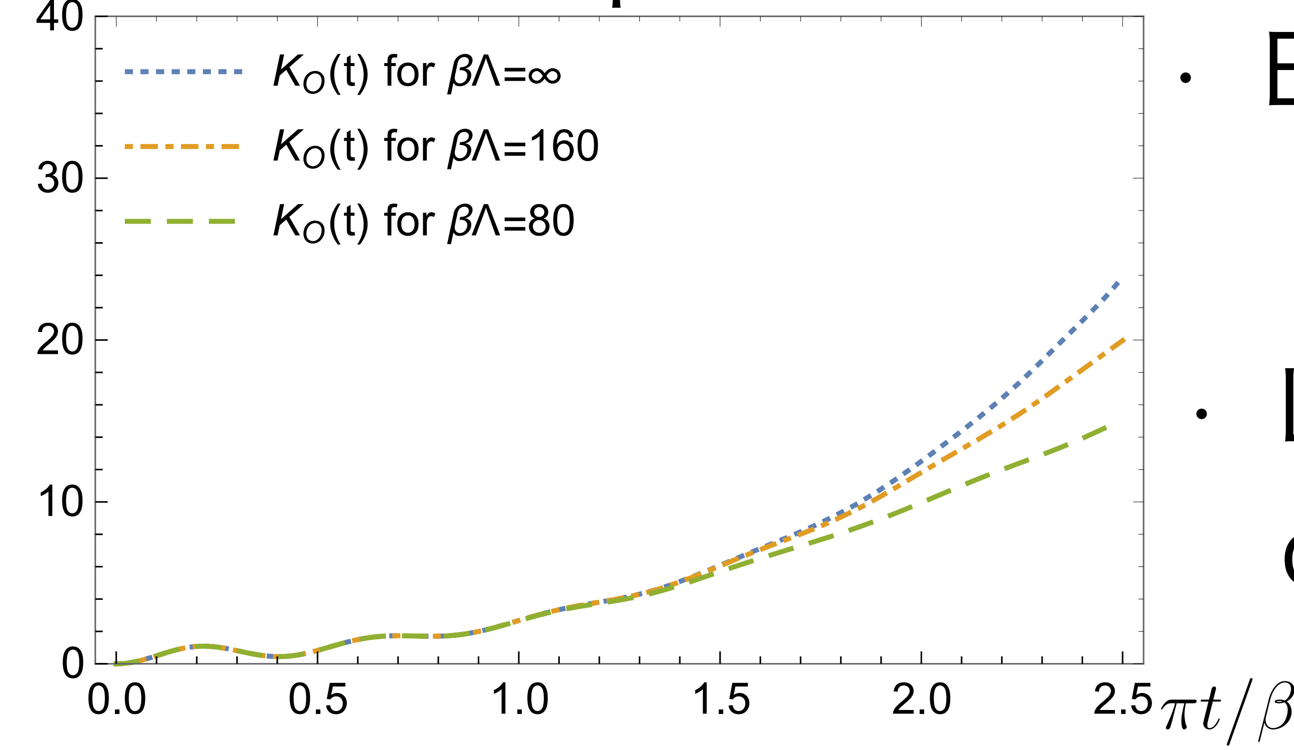
$b_{\text{even}} \sim (\Lambda - m)/2$

$$(d=5, m=20, \beta=1)$$

UV cutoff Λ causes the saturation of b_n .

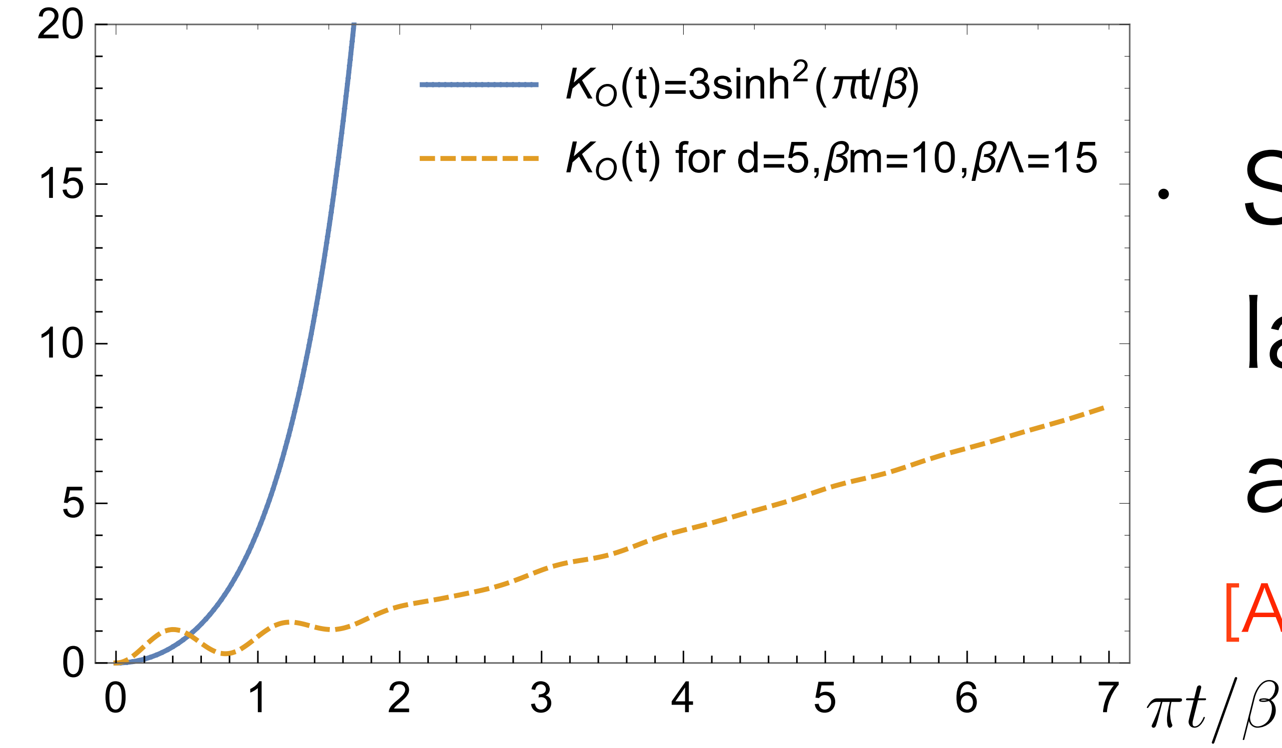
$K_{\mathcal{O}}(t)$ With finite UV cutoff Λ

$K_{\mathcal{O}}(t)$ Linear plot



- Early-time exponential growth, independent of Λ
- Late-time linear growth due to saturation of b_n

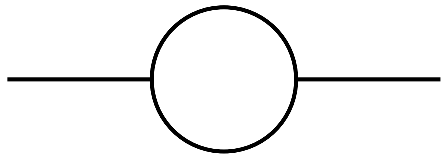
$K_{\mathcal{O}}(t)$ Linear plot

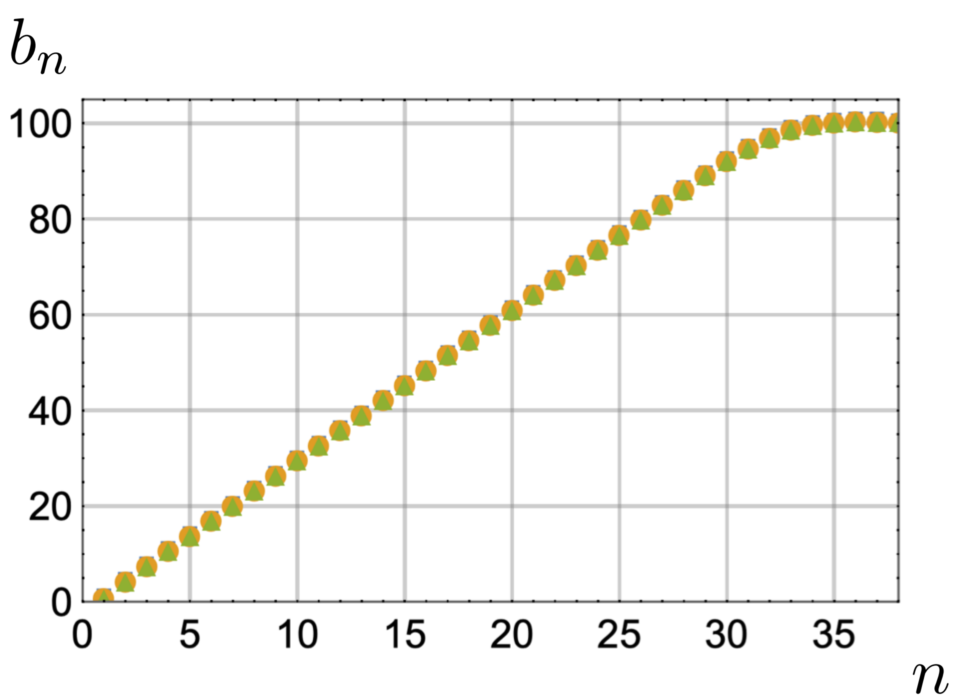


- Saturation of b_n and late-time linear growth of $K_{\mathcal{O}}(t)$ are consistent with free lattice.

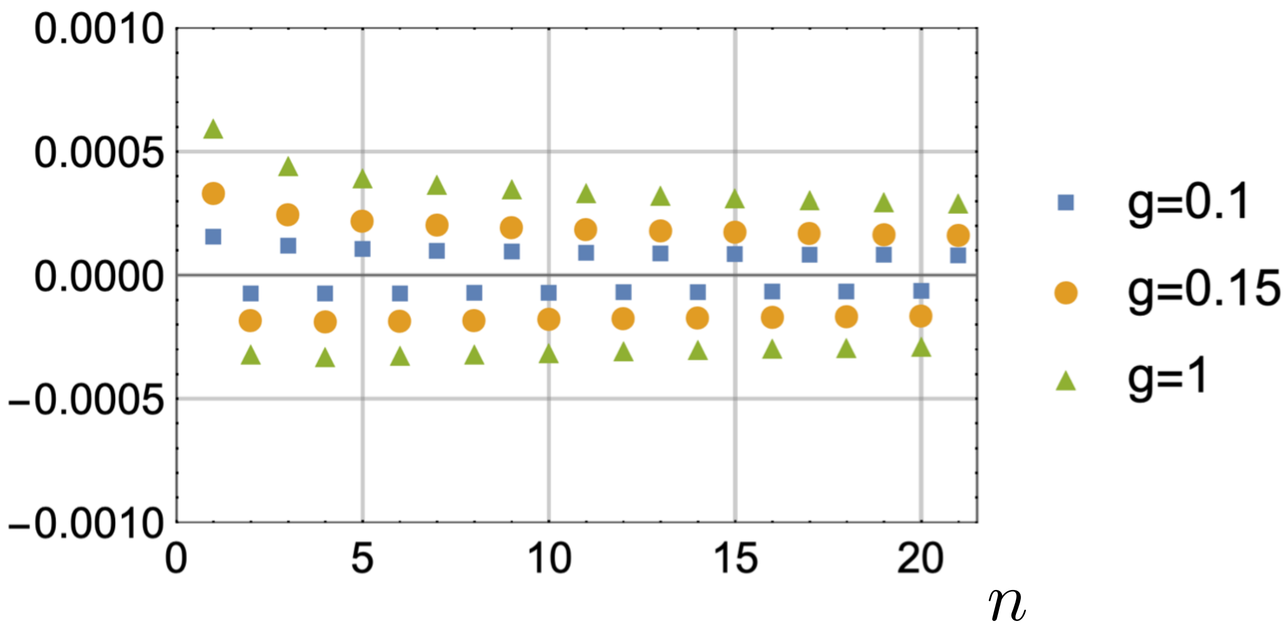
[A. Avdoshkin, A. Dymarsky, M. Smolkin, 2022]

b_n in 4d $g\phi^3/3!$ theory $(m=0, \beta=1, \Lambda=200)$

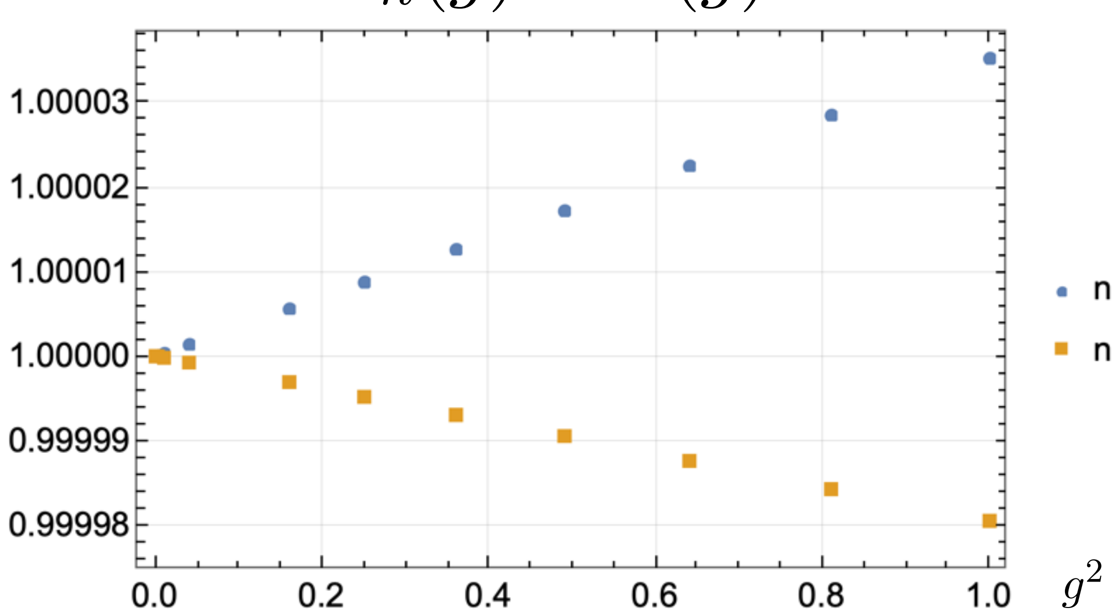
One-loop self energy $\Pi_E =$ 



$\delta b_n = b_n(g) - b_n(0)$



$b_n(g) \sim \alpha(g)n$



$g\phi^3/3!$ causes the difference between b_{odd} and b_{even}.

But, the difference is small because of the perturbation.

まとめ

- Lanczos係数 b_n とKrylov complexity $K_{\mathcal{O}}(t)$ は量子多体系のoperator growthの指標

- 自由 massive scalar場の理論のLanczos係数とKrylov complexityを調べた

- 場の理論のMass gap とUV cutoffが b_n $K_{\mathcal{O}}(t)$ に影響

- 4d ϕ^3 と4d ϕ^4 理論のLanczos係数を摂動的に調べた
ただし、摂動なので効果はすごく小さい

展望

- Mass gapがある場合のλ_Lの計算および λ_K との比較

$$\lambda_L \leq \lambda_K \leq \frac{2\pi}{\beta}$$

- 他の理論での解析

ϕ^4 matrix theory, TTbar deformed QFT

- Krylov complexity in AdS/CFT