Wave Function Renormalization and Flow of Couplings in Asymptotically Safe Quantum Gravity

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1 Introduction

We would like to understand how to formulate Quantum gravity (QG).

We want to consider a formulation that can deal with such phenomena.

⇒ Quantum gravity within the framework of local field theory.

- **The Einstein theory is non-renormalizable perturbatively.**
- **Higher-derivative (curvature) terms always appear in QG, e.g. quantized Einstein theory and (low-energy effective theory of) superstring theories!**
- **In 4D, quadratic (higher derivative) theory is renormalizable!** [K. S. Stelle, Phys. Rev. D16 (1977) 953.] ⇒ Possible UV completion? **But it is non-unitary!**

HDG

$$S_{HDG} = \int d^4x\sqrt{-g}\Big[\mathcal{V} - Z_N R + \frac{1}{2\lambda}{C_{\mu\nu\rho\lambda}}^2 + \frac{1}{\xi}R^2 - \frac{1}{\rho}E\Big],$$

$${C_{\mu\nu\rho\lambda}}^2 = R^2_{\mu\nu\alpha\beta} - 2R^2_{\mu\nu} + \frac{1}{3}R^2, \qquad E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2,$$

$$Z_N = \frac{1}{16\pi G_N}\ , \quad \mathcal{V} = 2\Lambda Z_N\ ,$$

To fully understand the theory, we need nonperturbative method.
$\Rightarrow$ (Functional or Exact) Renormalization Group! $\Rightarrow$ Asymptotic Safety

2 Asymptotic Safety in a nutshell

We consider effective "average" action obtained by integrating out all fluctuations of the fields with momenta larger than k.

$$e^{W_k(J)} = \int [D\phi] e^{-(S[\phi]+\Delta S_k[\phi])+\int J\phi} \text{ where } \Delta S_k[\phi] = \frac{1}{2}\int d^d q \phi(-q) R_k(q^2)\phi(q)$$

$R_A(q)$: a cutoff which gives suppression of IR modes

Its role is to remove the IR mode from the action, so that the path integral is carried out only over UV modes $\Rightarrow$ Legendre transf. $\Rightarrow$ $\Gamma_k[\phi]$

This is still divergent! But by introducing the cutoff function R_k

$$k\partial_k \Gamma_k(\Phi) = \frac{1}{2}\mathbf{tr}\left[\left(\frac{\partial^2 \Gamma_k}{\partial \Phi^A \partial \Phi^B} + R_k\right)^{-1} k\partial_k R_k\right] \quad \Leftarrow \quad \text{there is no divergence!}$$

because $k\partial_k R_k$ has contribution from modes only around $\sim k$

Functional (Exact) renormalization group equation (FRGE)!

Important fact

We look at the dependence of the effective average action on k, which gives the RG flow, free from any divergence and can be used to define quantum theory.

How?: FRGE gives flow of the effective action in the theory (coupling) space defined by suitable bases $\mathcal{O}_i$.

$$\Gamma_k = \sum_i g_i(k)\mathcal{O}_i \quad \Rightarrow \quad \frac{d\Gamma_k}{dt} = \sum_i \beta_i \mathcal{O}_i, \quad \beta_i = \frac{dg_i}{dt}$$

$$t \equiv \ln k$$

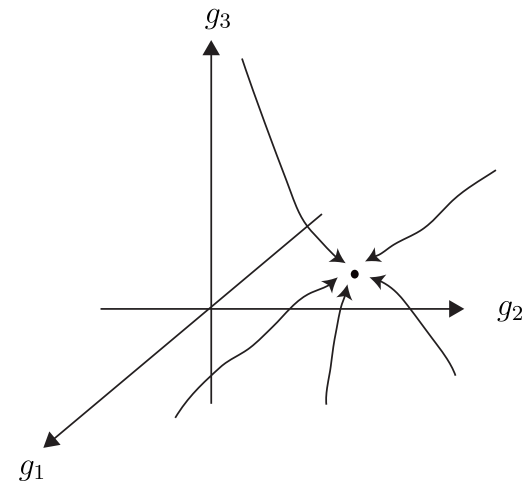


Figure 1: RG flow

We set initial conditions at some point and then flow to $k \to \infty$.

The flows may stop at FPs where $\beta = 0$.

If all couplings go to finite FPs at UV, physical quantities are well defined, giving the UV finite theory $\Rightarrow$ Asymptotic safety

+ There are finite number of the couplings $\Rightarrow$ Predictability

This does not mean that there is no "divergences" but the theory becomes scale invariant such that the divergences are under control!

This defines nonperturbative renormalizability.

When integrated to $k = 0$, we get the standard effective action $\Gamma_{k=0}[\phi]$.

An important consequence of the FRGE is that the gravitational couplings depend on the energy scale k.

Those operators whose couplings go to FPs in the infinite energy are called relevant operators, and repel irrelevant operators and others marginal.

An important problem in this approach is how many relevant operators are there.
These are similar to the renormalizable interactions in perturbative quantum field theory!

The theory must contain these interactions.

In our earlier work, we have studied this problem including operators up to quadratic curvature terms on arbitrary backgrounds!

Arbitrary background is important because there is Gauss-Bonnet theory in four dimensions.

We have found evidence that there are only 3 relevant operators (K. Falls, N.O., R. Percacci) for the beta functions containing $\frac{1}{G}$ in the first nontrivial order.

This cannot be connected to the low energy because we expect G goes to zero.

It is very difficult to get full beta functions involving all orders in $\frac{1}{G}$.

We got the full beta functions from B. Knorr who did the calculation by mathematica.

Unfortunately we did not find good nontrivial fixed point in high energy.

Not only that, it was difficult to find flow between high and low energies.

However, there is an important point to consider.

3 Wave function renormalization

It is well known that the wave function renormalization "constant" is unphysical parameter which does not affect any physical quantities.

This point has not been taken into account in most of the literature on the asymptotic safety until recently.

Here we improve this situation.

3.1 Einstein gravity

Consider the Einstein theory with the cosmological constant:

$$S = \int d^4x\sqrt{g}\left(2\Lambda - \frac{1}{16\pi G_N}R\right).$$

Under the wave function renormalization

$$g_{\mu\nu} \to g'_{\mu\nu} = Z g_{\mu\nu}$$

we have

$$\sqrt{g} \to Z^2\sqrt{g'}, \qquad \sqrt{g}R \to Z\sqrt{g}R'$$

The vacuum energy Λ and the Newton coupling changes as

$$\Lambda \to Z^2\Lambda', \qquad G_N \to Z^{-1}G'_N \quad \Rightarrow \quad \Lambda G_N^2 \text{ is invariant}$$

Namely

$$S_{eff}[Zg'_{\mu\nu}, \Lambda(t), G_N(t), t] = S_{eff}[g'_{\mu\nu}, Z^2\Lambda(t), Z^{-1}G_N(t), t].$$

It does not make sense to find FPs separately for Λ and Γ_N!
This modifies the FRGE as

$$\begin{cases} \dot{\tilde{\Lambda}} + 4\tilde{\Lambda} = \frac{1}{32\pi}(A_1 + A_2\eta_G) + 2\zeta\tilde{\Lambda}, \\ \dot{\tilde{G}} - 2\tilde{G} = (B_1 + B_2\eta_G) - \zeta\tilde{G}, \end{cases} \quad \Rightarrow \quad \tilde{\Lambda}\tilde{G}^2 = \text{ invariant}$$

Dimensionless couplings: $\Lambda = \tilde\Lambda k^4, \qquad G_N = \tilde G k^{-2}$

The usual optimized cutoff

$$R_k = (k^2 - \Delta)\theta(k^2 - \Delta), \quad (\Delta = -g^{\mu\nu}\nabla_\mu\nabla_\nu)$$

breaks the invariance under the wave function renormalization!

We choose

$$R_k = (\sqrt{\tilde\Lambda}\, k^2 - \Delta)\theta(\sqrt{\tilde\Lambda}\, k^2 - \Delta)$$

$$\Rightarrow \quad A_1 = \frac{(1+128\tilde G\sqrt{\tilde\Lambda}\,)(\dot{\tilde\Lambda}+4\tilde\Lambda)}{4\pi(1-32\pi\tilde G\sqrt{\tilde\Lambda}\,)}, \qquad A_2 = \frac{5\tilde\Lambda}{6\pi(1-32\pi\tilde G\sqrt{\tilde\Lambda}\,)},$$

$$B_1 = -\frac{(11-288\pi\tilde G\sqrt{\tilde\Lambda}+7(32\pi\tilde G\sqrt{\tilde\Lambda}\,)^2(\dot{\tilde\Lambda}+4\tilde\Lambda)}{12\pi(1-32\pi\tilde G\sqrt{\tilde\Lambda}\,)^2\sqrt{\tilde\Lambda}}, \quad B_2 = -\frac{160\pi\tilde G\sqrt{\tilde\Lambda}+1}{12\pi(1-32\pi\tilde G\sqrt{\tilde\Lambda}\,)}\sqrt{\tilde\Lambda},$$

Solve for $\dot{\tilde\Lambda}$ and $\dot{\tilde G}$:

$$\dot{\tilde\Lambda} = f_1(\tilde\Lambda, \tilde G, \zeta), \qquad \dot G = f_2(\tilde\Lambda, \tilde G, \zeta)$$

We use the freedom to fix the cosmological constant to a constant.

$$f_1(\tilde\Lambda_0, \tilde G, \zeta) = 0 \quad \Rightarrow \quad \zeta \quad \Rightarrow \quad \beta_G$$

The beta function is written solely in terms of invariant $\eta \equiv 32\pi\tilde{G}\sqrt{\tilde{\Lambda}}$

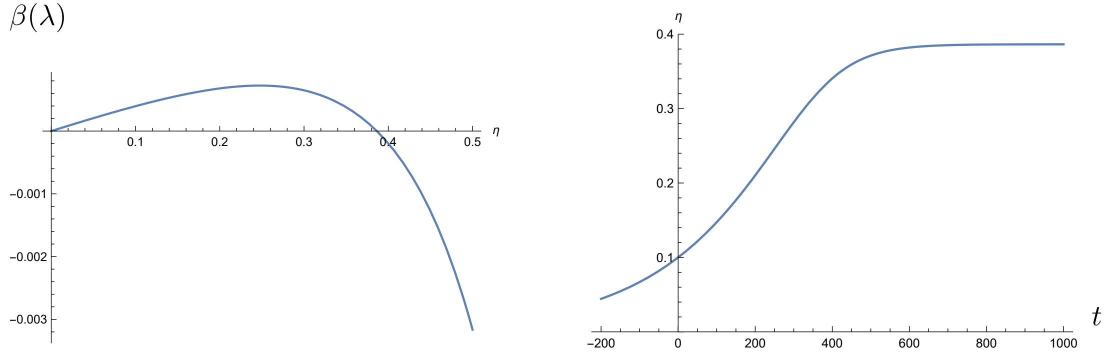
$$\dot{\eta} = \frac{2(8 - 19\eta + \eta^2 - 14\eta^3)\eta}{5 - 6\eta - 5\eta^2 + 384\pi^2(1-\eta)^2} \quad \Rightarrow \quad \textbf{Typical AS behavior! see next fig.}$$

What behaviors in the UV and IR limits:

$$\tilde{G} = G_N k^2 \to \ \textbf{finite}, \ (k \to \infty, \ \ \textbf{asymptotic safety}); \ \ \tilde{G} \to 0 \ (k \to 0).$$

We set the boundary condition $\eta = 0.1$ **at** $t = 0$**.**

The beta function and the behavior of η **is shown in (a) and (b):**



(a) (b)

Figure 2: (a) Beta function for η. (b) Flow of η for the boundary condition $\eta = 0.1$ at $t = 0$.

3.2 Quadratic gravity

We can do the same analysis with higher curvature terms for the full beta functions including all orders in $\frac{1}{G}$,

$$
\begin{aligned}
\dot{\tilde{\Lambda}} &= -4\tilde{\Lambda} + \tilde{\Lambda}\, f_1(\eta,\lambda,\xi) + 2\zeta\tilde{\Lambda}, \\
\dot{\tilde{G}} &= 2\tilde{G} + 16\pi\tilde{G}^2\sqrt{\tilde{\Lambda}}\, f_2(\eta,\lambda,\xi) - \zeta\tilde{G}, \\
\dot{\lambda} &= -2\lambda^2 f_3(\eta,\lambda,\xi), \\
\dot{\xi} &= -\xi^2 f_4(\eta,\lambda,\xi).
\end{aligned}
$$

Setting $\dot{\tilde{\Lambda}} = 0$ to solve for ζ again, and substituting it into other equations, we get beta functions for the invariants η, λ, ξ.

Unfortunately, we have not found any reasonable nontrivial fixed point!

In the case of asymptotically free FP $\lambda_* = \xi_* = 0$:
numerical analysis $\Rightarrow$ they go to zero keeping the ratio of these couplings finite with $\omega_* \equiv -\frac{3\lambda_*}{\xi_*} = -0.02286$. $\Rightarrow$ See the next figure.

This can be shown analytically.

For small couplings

$$
\begin{cases}
\beta_\lambda = -\frac{133}{160\pi^2}\lambda^2, \\
\beta_\xi = -\frac{5(72\lambda^2 - 36\lambda\xi + \xi^2)}{576\pi^2}
\end{cases}
\Rightarrow \frac{d\chi}{d\lambda} \propto (\chi-\alpha)(\chi-\beta), (\chi = \frac{\lambda}{\xi}) \quad \Rightarrow \quad
\begin{array}{l} \xi = \alpha\lambda \\ \xi = \beta\lambda \end{array}
\quad (\alpha = 131.2, \beta = 0.5487)
$$

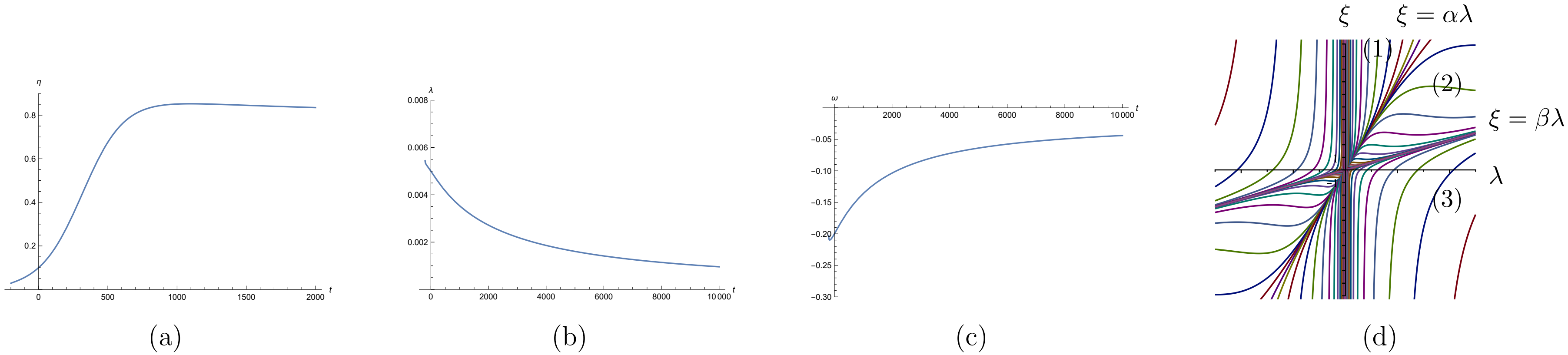


Figure 3: The flow of η (a), and the flows of (b) λ and (c) ω. (d) Small λ, ξ region. The flow direction is from right to left.

Solutions for this system have three regions: (see figure (d))

$$(1)\ \xi > \alpha\lambda, \qquad (2)\ \alpha\lambda > \xi > \beta\lambda, \qquad (3)\ \beta\lambda > \xi.$$

1. **The flow always occur in the direction of decreasing λ.**
2. **Those points in region (1) and (2) flows into the origin tending the line $\xi = \alpha\lambda \Rightarrow \omega_* = -\frac{3}{\alpha} = -0.02286$.**

The couplings go to asymptotically free FP as long as they are in the region $\xi > \beta\lambda > 0$, converging to the origin along the line $y = \alpha\xi$

$\Rightarrow$ both terms are relevant!

4 Summary

The redundant wave function renormalization should be taken into account, with higher curvature terms.

This affects the counting of the number of relevant operators.

We have shown that flow equations can be written solely in terms of the invariant η (and λ, ξ), and the nonperturbative FP in UV is smoothly connected to the perturbative gravity in IR.

We did not find nontrivial FPs for higher order couplings.

However we cannot exclude the possibility of the existence of these FPs, since it depends on the scheme.

Or this might be an artifact of truncation.

Future perspective

- **Search for nontrivial fixed points.**
- **Further higher curvature terms?**
- **Low-energy behavior.**