可換格子ゲージ理論の
測定型量子シミュレーションと
anomaly inflow

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助野 裕紀氏(Stony Brook U.)との共著論文 arXiv:2210.10908, SciPost Phys. 14, 129 (2023) および助野氏・Aswin Parayil Mana氏との共同研究

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Plan of the talk

- 動機と背景
- 測定型量子シミュレーション

- リソース状態とanomaly inflow
- リソース状態と古典分配関数の双対性・SymTFT

Motivations and background

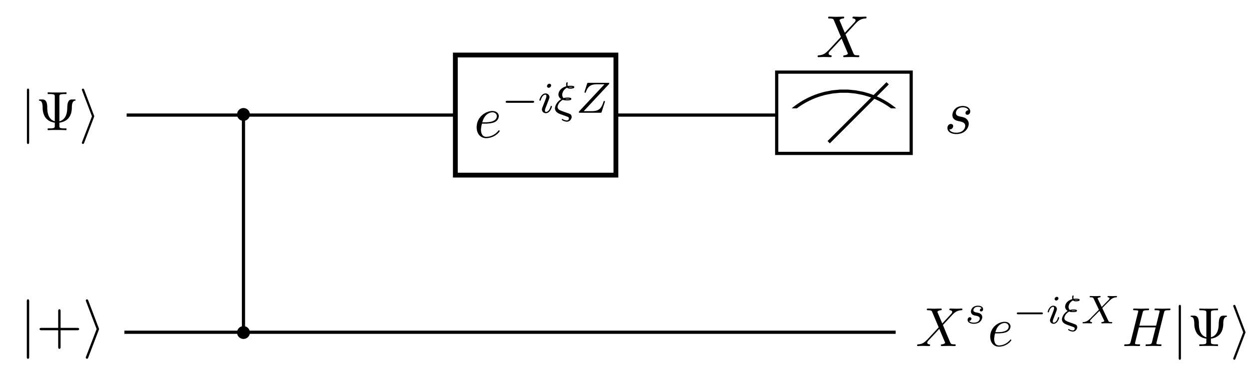
- Quantum simulation of lattice gauge theories is expected to become a major application of quantum computers.
- It's still too early to decide which simulation schemes will be the most efficient, and different schemes should be investigated.
- Simulation schemes can be roughly divided into digital and analog quantum simulations. I focus on digital schemes.

- Digital simulation uses the Suzuki-Trotter approximation to realize discrete time evolution.
- So far, most efforts have focused on circuit-based methods.
- In quantum computation, there are alternative quantum computation (QC) schemes: measurement-based QC, adiabatic QC, etc. We want to apply the idea of measurement-based QC for simulation.
- Does a quantum simulation scheme reflect intrinsic properties of the simulated field theory?

Review: measurement-based quantum computation (MBQC)

- Introduced by Raussendorf and Briegel (2001).
- Also called one-way quantum computation.
- An alternative computational scheme that replaces circuit-based computation.
- Uses quantum teleportation and adaptive measurements on a resource (cluster) state.

Gate teleportation



- X-eigenstate $X|\pm\rangle = \pm|\pm\rangle$
- $|\Psi\rangle$ is an arbitrary 1-qubit state
- Entangle $|\Psi\rangle$ and $|+\rangle$ by a controlled-Z gate CZ.
- Measure the first qubit in bases $\{e^{i\xi Z}|\pm\rangle\}$. The measurement outcome is $s = 0,1$ corresponding to $\pm 1 = (-1)^s$.
- The state on the second qubit becomes

$$X^s e^{-i\xi X} H|\Psi\rangle.$$

Up to X^s and H, the state and the unitary transformation $e^{-i\xi X}$ are teleported. X^s is an example of a **byproduct operator**.

Adaptive measurement

- Suppose that an earlier measurement in a bigger circuit had produced the state $|\Psi\rangle = X^t H |\Phi\rangle$, where $t = 0,1$ is the **known** measurement outcome. Suppose also that we wish to obtain $e^{-i\alpha X}|\Phi\rangle$.

- Substituting this to the teleportation formula $X^s e^{-i\xi X} H|\Psi\rangle$, we get $X^s e^{-i\xi X} H X^t H |\Phi\rangle = X^s Z^t e^{-i(-1)^t \xi X}|\Phi\rangle$.

- To get the desired state $e^{-i\alpha X}|\Phi\rangle$ (up to byproducts), we need to set ξ to $\xi = (-1)^t \alpha$. $\Rightarrow$ We need to adjust the measurement angle ξ adaptively according to earlier measurement outcomes. In this way we can achieve a deterministic computation.

Resource state

- Measurement based quantum computation is performed by adaptive one-qubit measurements on a **resource state**.

- As a resource state, one usually considers a **cluster state**

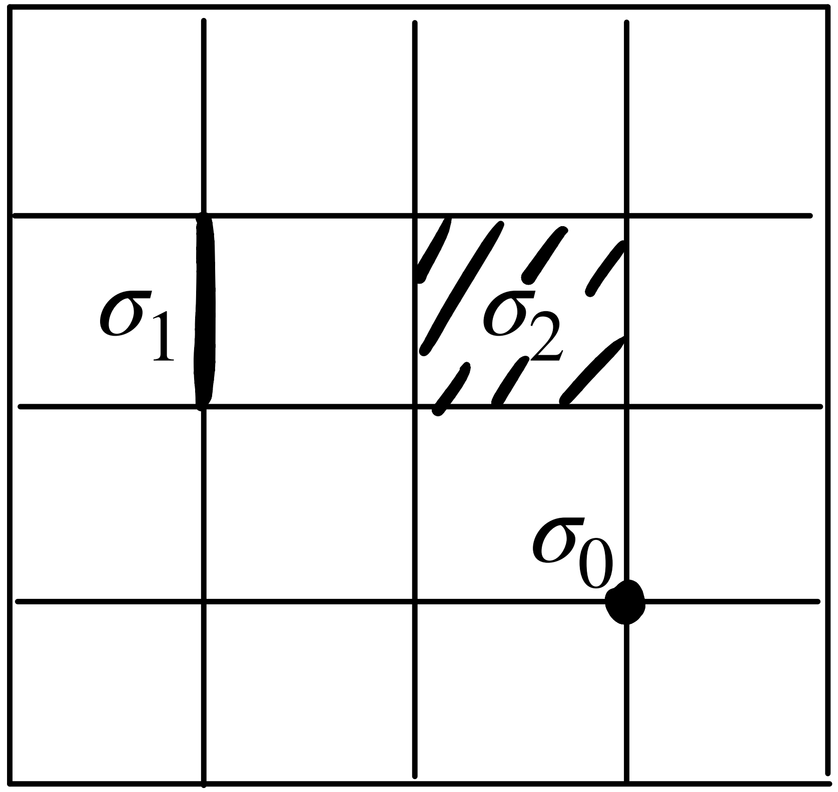
$$\bigotimes_{\text{pair}} CZ_{\text{pair}} |+\rangle^{\otimes \text{qubits}}.$$

- Cluster states can be constructed by a finite-depth circuit and can be characterized by stabilizer operators.

- Measurement-based quantum computation on a 2-dimensional cluster state is universal: it can reproduce any computation of a circuit-based quantum computation.
- There exist versions of MBQC and cluster states with discrete and continuous-variable qudits.
- Large-scale $\mathcal{O}(10^4)$ (continuous-variable) optical cluster states have been experimentally generated.

Review: Hamiltonian lattice gauge theory in 2+1 dimensions

- Cell complex for a square lattice.
  - 0-cells $\sigma_0 \in \Delta_0$
  - 1-cells $\sigma_1 \in \Delta_1$
  - 2-cells $\sigma_2 \in \Delta_2$
- Degrees of freedom (qubits) are on 1-cells (edges) $\sigma_1 \in \Delta_1$.



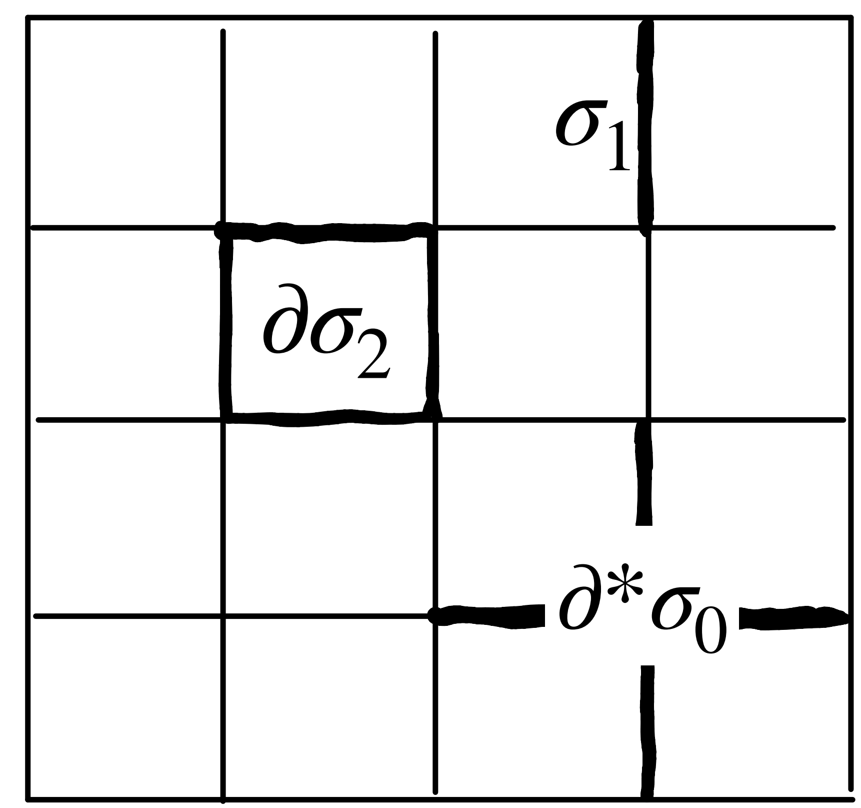
- Hamiltonian: $H = -\sum_{\sigma_1 \in \Delta_1} X(\sigma_1) - \lambda \sum_{\sigma_2 \in \Delta_2} Z(\partial \sigma_2)$ with $Z(\partial \sigma_2) = \prod_{\sigma_1 \subset \partial \sigma_2} Z(\sigma_1)$.

- Gauss law constraint: for any $\sigma_0 \in \Delta_0$,

$$X(\partial^* \sigma_0) | \psi_{\text{phys}} \rangle = | \psi_{\text{phys}} \rangle.$$

- The $\lambda \to \infty$ limit is Kitaev's toric code.

- Generalization: $\mathbb{Z}_2$ gauge theory in 2+1 dimensions = $M_{(3,2)}$ $\Rightarrow$ **Wegner's model** $M_{(d,n)}$: higher-form gauge theory in d dimensions. The $n = 1$ case is the Ising model.



Trotterization

- Ideally we want to implement the continuous time evolution e^{-iHt} for any t. Decompose $H = H_1 + H_2$. $H_1 = -\sum_{\sigma_1 \in \Delta_1} X(\sigma_1)$ and $H_2 = -\lambda \sum_{\sigma_2 \in \Delta_2} Z(\partial\sigma_2)$ do not commute.
- In digital quantum simulation (such as by quantum circuits), we implement $e^{-iH_1\delta t}$ and $e^{-iH_2\delta t}$ separately.
- Suzuki-Trotter approximation: $e^{-iHt} \simeq \left(e^{-iH_1 t/n} e^{-iH_2 t/n}\right)^n$.
- We want to realize $e^{-iH_1\delta t} = \prod_{\sigma_1 \in \Delta_1} e^{i\delta t X(\sigma_1)}$ and $e^{-iH_2\delta t} = \prod_{\sigma_2 \in \Delta_2} e^{i\lambda\delta t Z(\partial\sigma_2)}$.

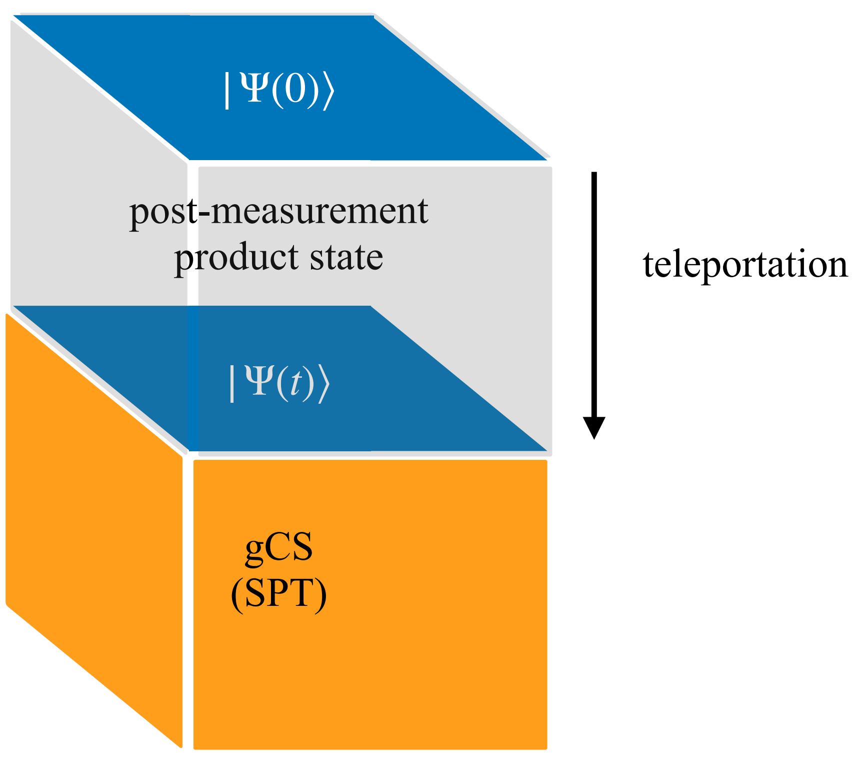
Proposal: measurement-based quantum simulation of abelian lattice gauge theories

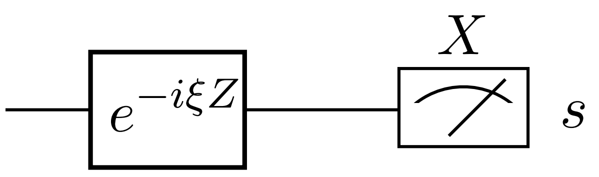
- Claim: we can implement the Trotterized time evolution $\left(e^{-iH_1 t/n} e^{-iH_2 t/n}\right)^n$ by

  1. preparing a generalized cluster state that reflects the spacetime structure of the gauge theory

  and then by

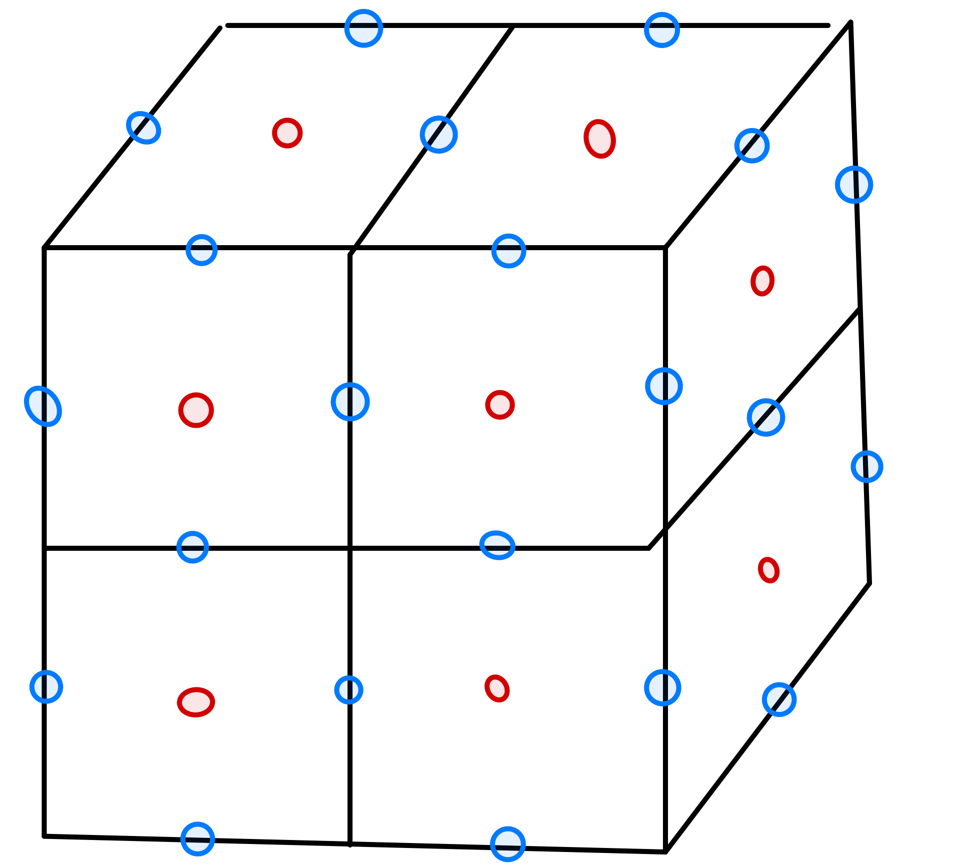
  2. performing adaptive single-qubit measurements adaptively in a prescribed **pattern**.





Resource state for $\mathbb{Z}_2$ lattice gauge theory in 2+1 dimensions

- Place one qubit on each 1-cell $\boldsymbol{\sigma}_1 \in \boldsymbol{\Delta}_1$ and 2-cell $\boldsymbol{\sigma}_2 \in \boldsymbol{\Delta}_2$ on a 3d cubic lattice.



- Entangle the neighboring 1-cells and 2-cells by controlled-Z gates.

$$|\mathrm{gCS}\rangle = \prod_{\boldsymbol{\sigma}_1 \subset \partial \boldsymbol{\sigma}_2} CZ_{\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2} |+\rangle^{\otimes \boldsymbol{\Delta}_1 \cup \boldsymbol{\Delta}_2}$$

- A version of three-dimensional cluster state.

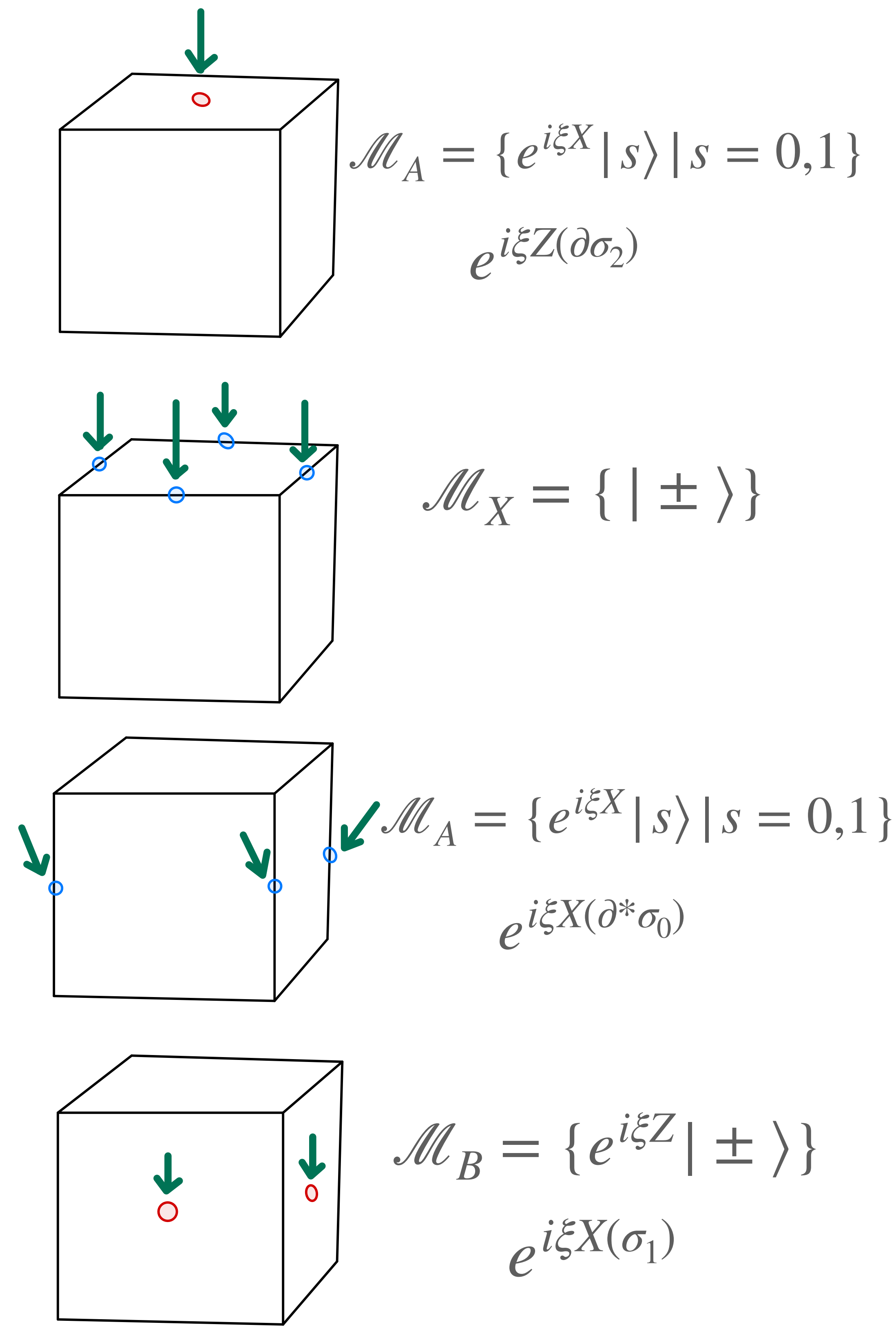
- Stabilizers $K(\boldsymbol{\sigma}_2) = X(\boldsymbol{\sigma}_2) Z(\partial \boldsymbol{\sigma}_2)$ and $K(\boldsymbol{\sigma}_1) = X(\boldsymbol{\sigma}_1) Z(\partial^* \boldsymbol{\sigma}_1)$.

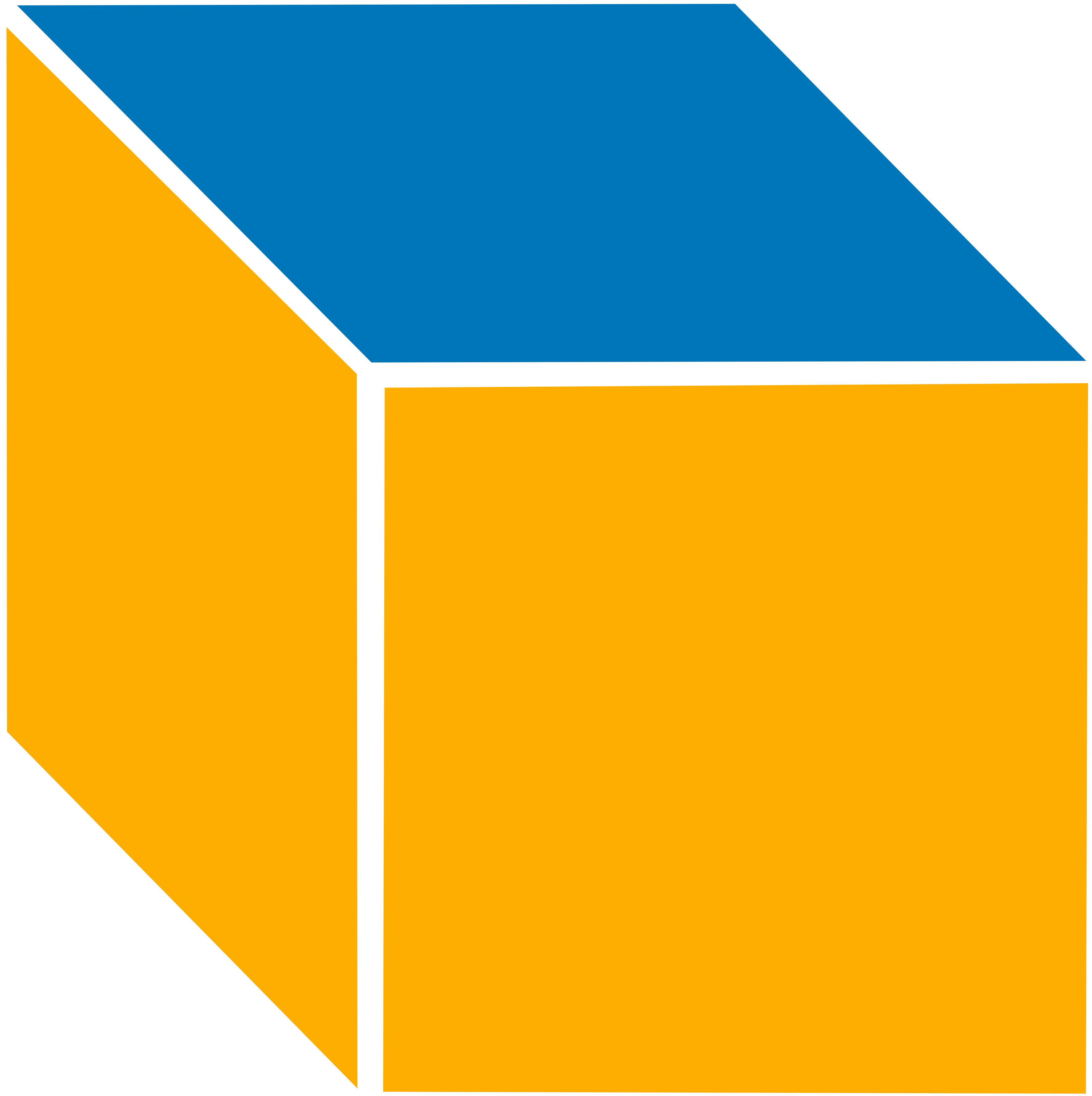
$$K(\boldsymbol{\sigma}_1)|\mathrm{gCS}\rangle = K(\boldsymbol{\sigma}_2)|\mathrm{gCS}\rangle = |\mathrm{gCS}\rangle$$

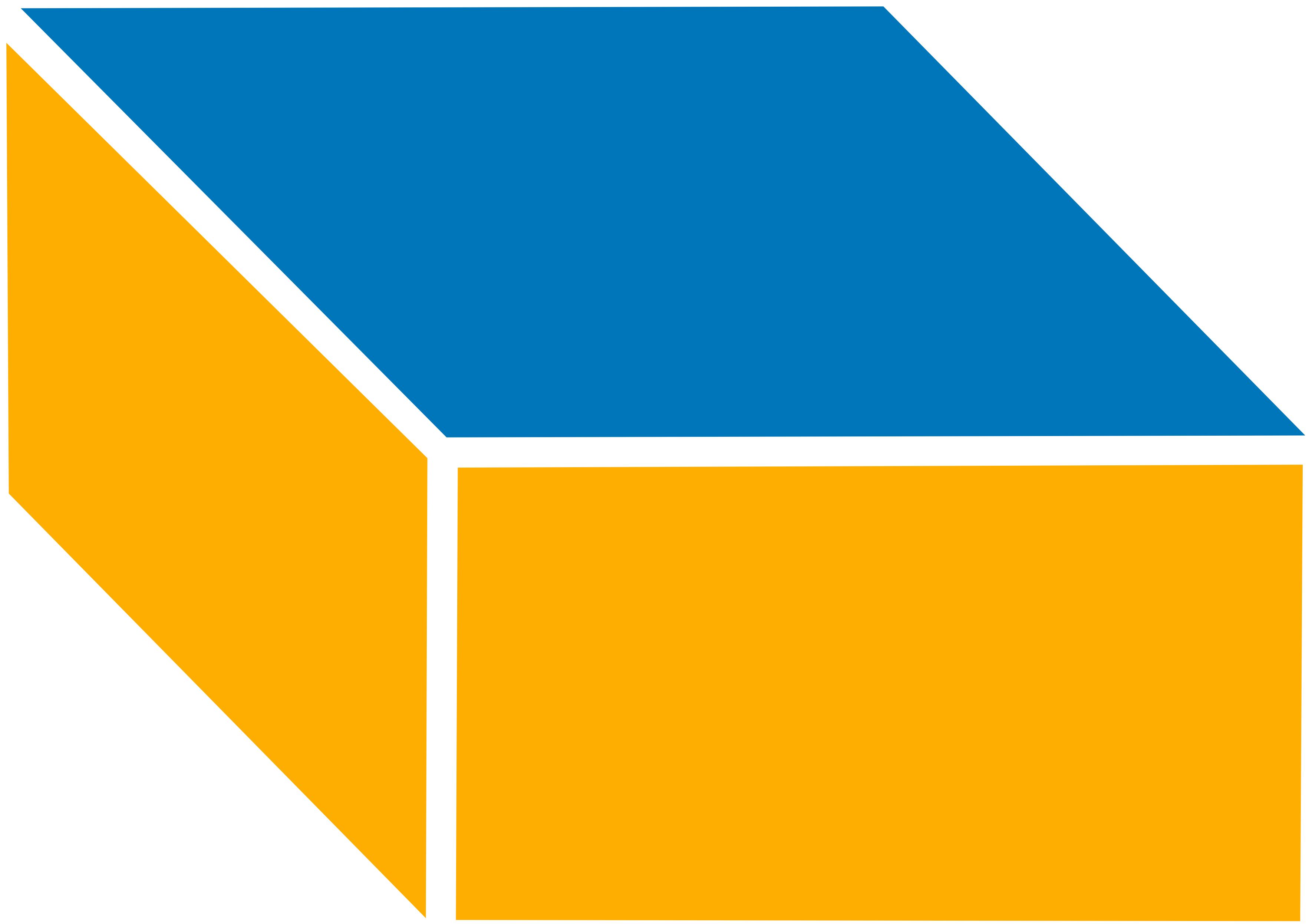
Measurement pattern = simulation protocol

- Trotterized time evolution is deterministically implemented by the measurement pattern and adaptive choices of the measurement angles ξ to absorb minus signs $(-1)^s$.

- Main result of the paper. The resource state reflects the spacetime structure of the simulated gauge theory.







Toward experimental realization

- The measurement-based approach requires only simple interactions (such as Ising interactions) between qubits because interactions are only used to create the resource state.
- Since the resource state includes the time direction, the measurement-based approach requires more qubits than the circuit-based approach.
- Possible experimental platforms:
    - Lattices formed by cold atoms
    - Continuous-variable cluster states created optically

Comparison with circuit-based simulation

- Simulation time is linear in the number of Trotter steps in both schemes.
  - $T_{\mathrm{MB}} \sim (\#\text{Trotter steps}) \times T_{\text{meas}}$
  - $T_{\mathrm{CB}} \sim (\#\text{Trotter steps}) \times T_{CZ}$
- In the measurement-based scheme, the resource state is created by a finite-depth circuit consisting of CZ. The number of necessary qubits grows linearly in the number of Trotter steps.

Comparison with classical simulation

- Exact diagonalization is only possible for up to tens of sites.
- Using tensor network methods, low-entanglement states are accessible for up to thousands of sites.
- In MBQS, the number of required qubits scales linearly with the number of Trotter steps.
- MBQS may have an advantage for problems with high-entanglement states if there are sufficiently many $\mathcal{O}(10^4)$ qubits of good quality.

Other aspects and generalizations

- Generalizations to $\mathbb{Z}_N$ gauge groups and the Kitaev Majorana chain are given in the paper. (Other generalizations in progress.)
- Non-compact U(1) ($\mathbb{R}$) gauge group discussed in the paper. Compact U(1) case to be explored.
- Correction of Gauss law violation discussed in the paper.
- Scheme for imaginary time evolution given in the paper.

SPT order of the resource state

- Claim: the natural resource state (qubits on n- and $(n-1)$-cells) for simulating Wegner's model $M_{(d,n)}$ is protected by global $\mathbb{Z}_2$ $(n-1)$ - and $\mathbb{Z}_2$ $(d-n)$-form symmetries. (For $d=3$, $n=2$, shown by Yoshida.)

- For the $\mathbb{Z}_2$ gauge theory in $2+1$ dimensions $M_{(3,2)}$, they are both one-form symmetries generated by membrane (surface) operators $\prod_{\boldsymbol{\sigma}_2 \subset z_2} X(\boldsymbol{\sigma}_2)$ with 2-cycle z_2 ($\partial z_2 = 0$) and $\prod_{\boldsymbol{\sigma}_1 \subset z_2^*} X(\boldsymbol{\sigma}_1)$ with dual 2-cycle z_2^* ($\partial^* z_2^* = 0$).

- The SPT order of the resource state for $M_{(d,n)}$ can be demonstrated by showing that "gauging" the symmetries of the resource state and the product state give rise to distinct topological orders. [Levin-Gu, Yoshida]

リソース状態と anomaly inflow

- Claim: the anomaly of the simulated boundary theory $M_{d,n}$ is canceled by the bulk resource state $|\mathrm{gCS}_{d,n}\rangle$.
- More precisely, the relevant anomalous symmetry is $\mathbb{Z}_2^{[d-n]} \times \mathbb{Z}_2^{[n-1]}$ present in a particular (toric code) limit of $M_{d,n}$.
- The resource state $|\mathrm{gCS}_{d,n}\rangle$ for the MBQS of $M_{d,n}$ is the cluster state on a d-dimensional hypercubic lattice with qubits on n- and $(n-1)$-cells. We believe that the continuum description is given by the classical action $S = \frac{i}{\pi}\int B_n \wedge B_{d-n+1}$.

- Quantum Wegner model $M_{d,n}$: $H = -\sum_{\sigma_{n-1}} X(\sigma_{n-1}) - \lambda \sum_{\sigma_n} Z(\partial \sigma_n)$

  with $X(\partial^* \sigma_{n-2}) \,|\, \text{phys}\rangle = |\, \text{phys}\rangle$.

- Generalized toric code $TC_{d,n}$:

  $H = -\sum_{\sigma_n} Z(\partial \sigma_n) - \sum_{\sigma_{n-2}} X(\partial^* \sigma_{n-2})$.

- The partition function is a functional of background gauge fields.

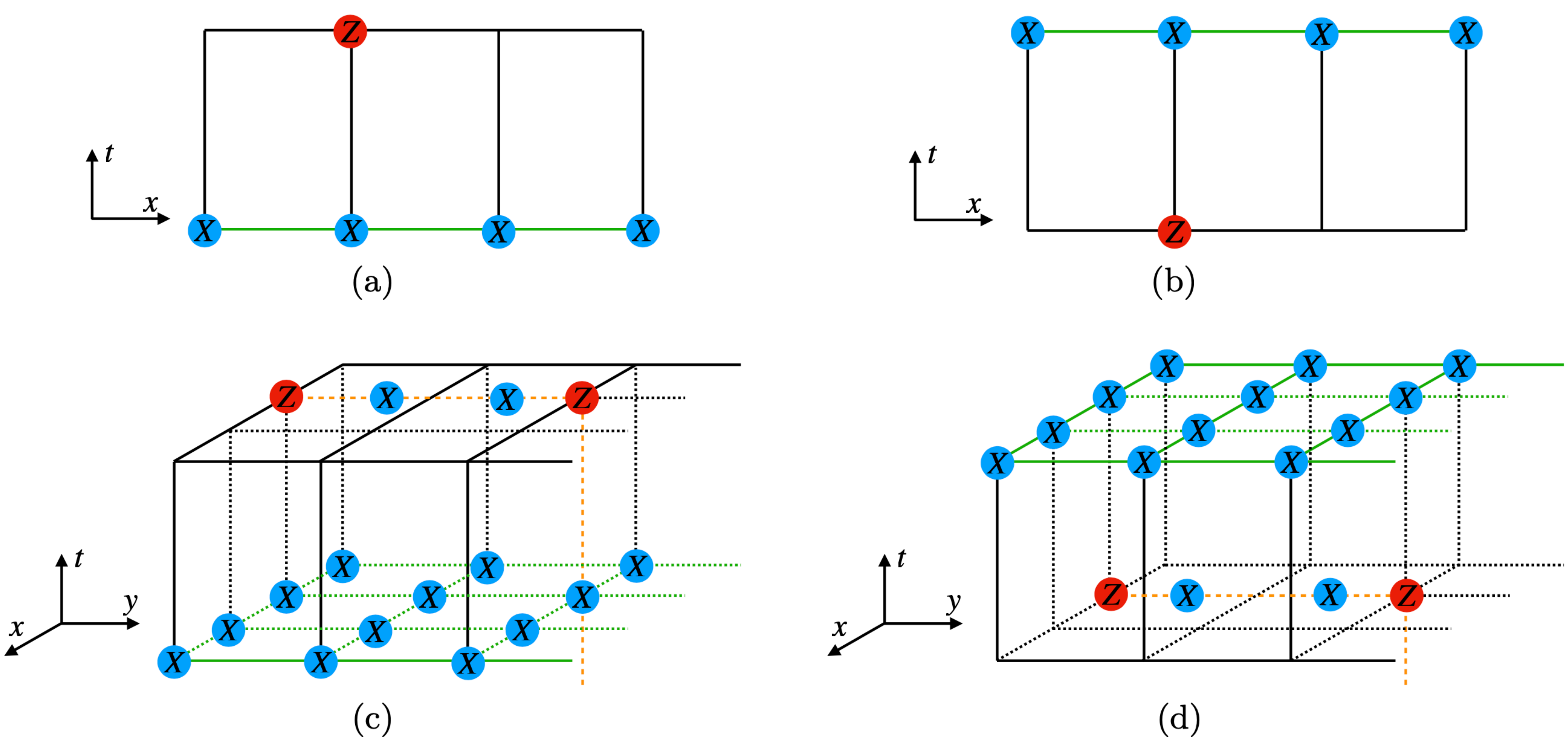
- Background gauge fields are Poincare-dual to the world-volume of symmetry defects.

- The ’t Hooft anomaly is the non-invariance of the (boundary) partition function under the gauge transformations of the background gauge fields. Such transformations are equivalent to the deformations of symmetry defects.

- On the lattice, symmetry defects (both space-like and time-like) can be explicitly constructed. Gauge non-invariance is equivalent to the non-commutation of symmetry generators (logical operators).

- In the coupled boundary+bulk system, symmetry generators on the boundary get extended into the bulk. The total partition function is invariant under deformations of defects.

Example: $(d, n) = (2,1)$



The amplitudes for (a) and (b) have a relative minus sign.
When the boundary is coupled to the bulk, the minus sign is compensated an additional sign that arises from the symmetry generator acting on a bulk excitation.

リソース状態と古典分配関数の双対性

- Using the resource state $|\mathrm{gCS}_{d,n}\rangle$, define $|\Phi_{d,n}\rangle = (\bigotimes_{\sigma_{n-1}} \langle + |) \cdot |\mathrm{gCS}_{d,n}\rangle$.
- Up to the Hadamard transform, this is a state in the generalized toric code $TC_{d+1,n+1}$: $Z(\partial^* \sigma_{n-1})|\Phi_{d,n}\rangle = X(\partial \sigma_{n+1})|\Phi_{d,n}\rangle = |\Phi_{d,n}\rangle$.
- Consider the product state $\langle \Omega(J)| := \bigotimes_{\sigma_n} \langle 0 | e^{JX(\sigma_n)}$.
- The overlap $\langle \Omega(J) | \Phi_{d,n}\rangle$ (sometimes called the strange correlator) equals the classical partition function of $M_{d,n}$.

- Let $|\Phi^*_{d,d-n}\rangle$ be the state constructed in the same way as $|\Phi_{d,d-n}\rangle$ but on the dual lattice rather than the original lattice.

- Let H be the simultaneous Hadamard transform. Both $\mathrm{H}|\Phi_{d,n}\rangle$ and $|\Phi^*_{d,d-n}\rangle$ belong to the code subspace of $TC_{d+1,n+1}$ [Raussendorf, Bravyi, Harrington].

- They are related as

$$\mathrm{H}|\Phi_{d,n}\rangle = \frac{1}{|H_n(T^d,\mathbb{Z}_2)|}\left(\sum_{[\boldsymbol{z}_n]\in H_n(T^d,\mathbb{Z}_2)} Z(\boldsymbol{z}_n)\right)|\Phi^*_{d,d-n}\rangle\,,$$

$$|\Phi^*_{d,d-n}\rangle = \frac{1}{|H_{d-n}(T^d,\mathbb{Z}_2)|}\left(\sum_{[\boldsymbol{z}^*_{d-n}]\in H_{d-n}(T^d,\mathbb{Z}_2)} X(\boldsymbol{z}^*_{d-n})\right)\mathrm{H}|\Phi_{d,n}\rangle\,,$$

- We also have $\langle \Omega(J) \,|\, \mathtt{H} \propto \langle \Omega(J^*) \,|$ with $J^* = -\dfrac{1}{2} \log \tanh J$.
- From $\langle \Omega(J) \,|\, \Phi_{d,n} \rangle = \langle \Omega(J) \,|\, \mathtt{H} \cdot \mathtt{H} \,|\, \Phi_{d,n} \rangle$, we get an equality between the partition functions of two lattice models on the torus, showing the precise duality (cf. van den Nest, Dür, Briegel '06)

$$M_{d,n} \simeq M_{d,d-n} / \mathbb{Z}_2^{[d-1-n]}.$$

- $\langle \Omega(J) \,|\, \Phi_{d,n} \rangle$ is the partition function of $M_{d,n}$. $\langle \Omega(J) \,|\, \mathtt{H} \,|\, \Phi^*_{d,d-n} \rangle$ is the partition function of $M_{d,n} / \mathbb{Z}_2^{[n-1]}$. $|\, \Phi_{d,n} \rangle$ and $\mathtt{H} \,|\, \Phi^*_{d,d-n} \rangle$ are stabilized by different logical operators (Wilson loop-like operators) and define different topological boundary conditions.
- The topological field theory underlying $TC_{d+1,n+1}$ (BF theory) plays the role of the so-called symmetry topological field theory (SymTFT). [Gaiotto-Kulp, Apruzzi et al., Freed et al., Kaidi-Ohmori-Zheng, …]

Other aspects

- The relation between the (generalized) toric code and the resource state is a special case of the so-called “foliation” construction of a cluster state from a CSS code.
- The entangler $\prod CZ$ that appears in the cluster state can be used to implement the Kramers-Wannier duality as an operator acting on the Hilbert space [Tantivasadakarn et al.] One can exhibit non-invertible symmetry and compute the fusion rule.
- Generalization to fracton models in progress.

Future directions

- More general gauge theories: non-abelian gauge groups, fracton models.
- More general fermions.
- Relate SPT order to computational power.
- Experimental realizations.
- Quantum simulation on cloud quantum computers with (adaptive) mid-circuit measurement capabilities.