Upper bound on the Atiyah–Singer index from tadpole cancellation

Hajime Otsuka (Kyushu University)

Reference :

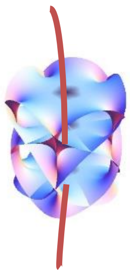
T. Kai, K. Ishiguro (KEK), S. Nishimura, H.O., M. Takeuchi (Kobe U), arXiv:2308.XXXXX

場の理論と弦理論2023, August 10th, 2023

Introduction — Atiyah–Singer index

- Huge number of 4D stable vacua (string landscape)

– $O(10^{500})$ Type IIB flux vacua *Ashok-Douglas ('04)*
– $O(10^{662})$ MSSM-like models in Heterotic on CYs *Constantin-He-Lukas ('18)*



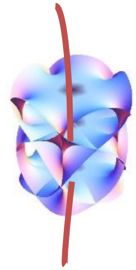
- 6D compactification → Degrees of freedom
  --- fluxes (VEVs of gauge fields)
  --- branes (wrapping sub-manifolds)

Question :

Generation number of quarks/leptons in the string landscape?

--- Counted by the Atiyah-Singer index : $\chi(M, V) = \int_M \mathrm{ch}(V) \wedge \mathrm{Td}(TM)$

--- Atiyah-Singer index : $\chi(M,V)=\int_M \mathrm{ch}(V)\wedge \mathrm{Td}(TM)$



Background fluxes on the internal manifold

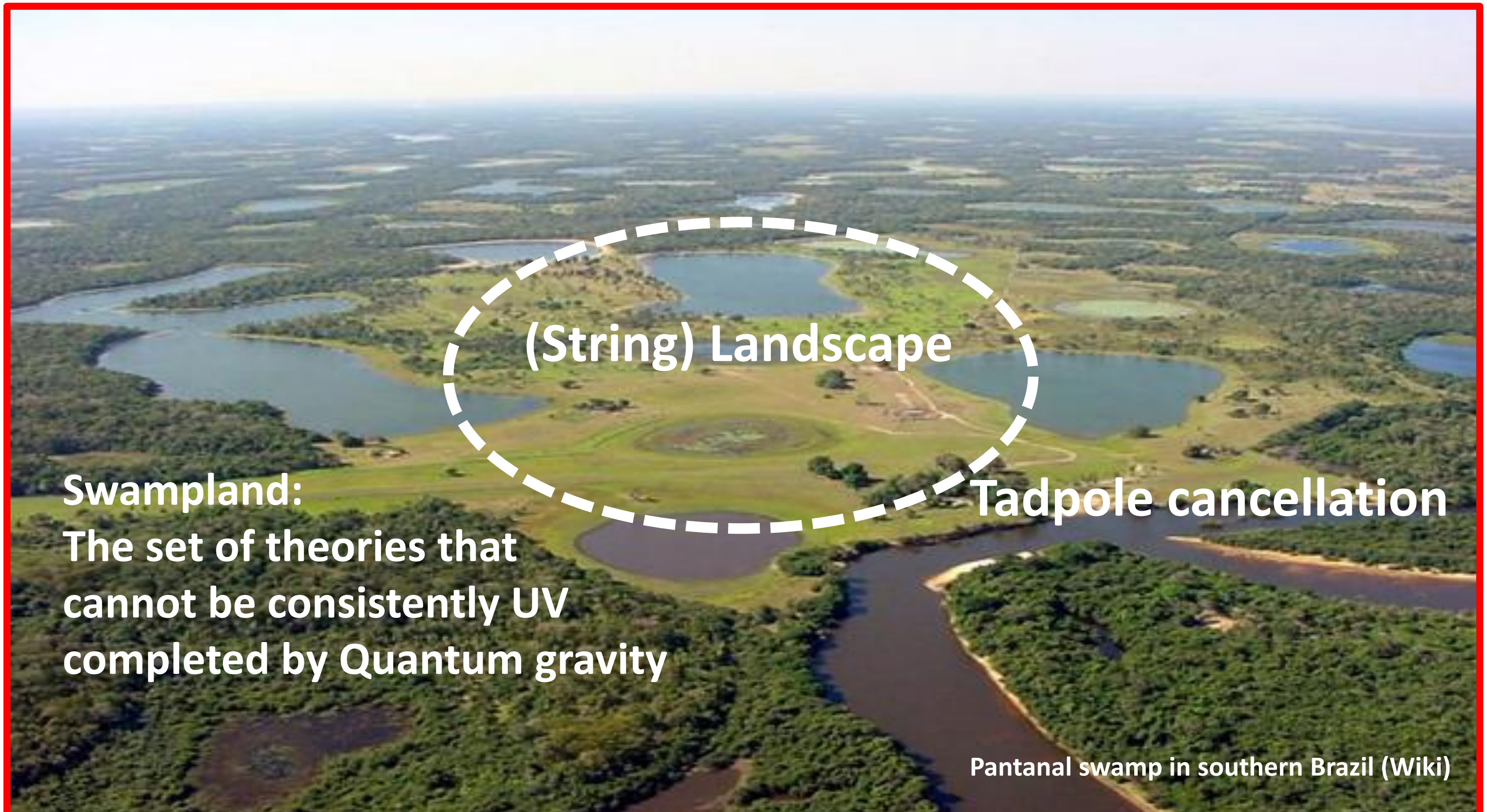
Since such background fluxes induce brane charges,
an arbitrary value of background fluxes is not allowed in compact spaces

-- constrained by the *tadpole cancellation condition*

Short summary

An arbitrary value of background fluxes is not allowed (constrained by *tadpole cancellation condition*)

→ Upper bound on the Atiyah-Singer Index $\chi(M,V) = \int_M \mathrm{ch}(V) \wedge \mathrm{Td}(TM)$



Outline

✓ **Introduction/Short summary**

- **Heterotic string theory with line bundles**

  — E_6 GUT in $E_8 \times E_8$ heterotic string

  — SU(5) GUT in $E_8 \times E_8$ heterotic string

  — Pati-Salam in $SO(32)$ heterotic string

  — Direct flux breaking (hypercharge flux) in $SO(32)$ heterotic string

- **Conclusions and Discussions**

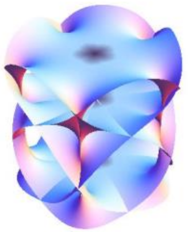
  — Type II string/ F-theory construction

Heterotic string models

- 10D $E_8 \times E_8'$ or $SO(32)$ heterotic string on Calabi-Yau threefolds

— CY compactifications → 4D N=1 SUSY

— Holomorphic vector bundle V on CY (with group $H \subset E_8$ or $SO(32)$)



Heterotic string models

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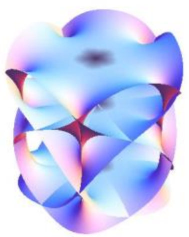

- 4D gauge symmetry

  — $E_8 \rightarrow G \times H$

Candelas-Horowitz-Strominger-Witten ('85)
L.Anderson, Y.H.He, A. Lukas ('08),...
Blumenhagen-Hocecker-Weigand ('05),...

H : Non-abelian (Standard embedding, Monad construction,..
$H = SU(n), n = 3,4,5$ leads to $G = E_6, SO(10), SU(5)$

H : Abelian (line bundle models)
$H = U(1)^n, n = 2,3,4$ leads to $G = E_6, SO(10), SU(5)$

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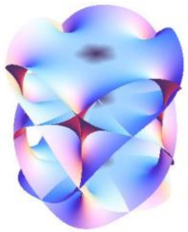

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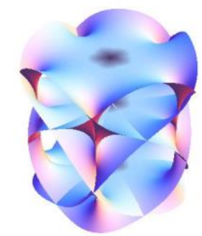
Conditions in heterotic string models

1. Tadpole cancellation condition (in compact spaces)

$$c_2(TM) - c_2(V) = [5\text{–brane}] \geq 0$$

from Bianchi identity of the three-form H
$$0 = dH = tr(F^2) - tr(R^2)$$



2. "K-theory condition"

$$c_1(V) = 0$$

3. SUSY condition

$$F_{ab} = F_{\bar{a}\bar{b}} = 0, \qquad g^{a\bar{b}} F_{a\bar{b}} = 0$$

4. Finite 4D gauge coupling

Upper bound on CY volume: $\mathcal{V} = g_s^2 \alpha_{\mathrm{GUT}}^{-1} \leq 25$

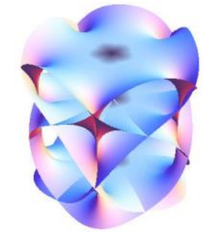
g_s : string coupling

4D SUSY E_6 GUT from Heterotic string on 6D CY

Candelas-Horowitz-Strominger-Witten ('85)

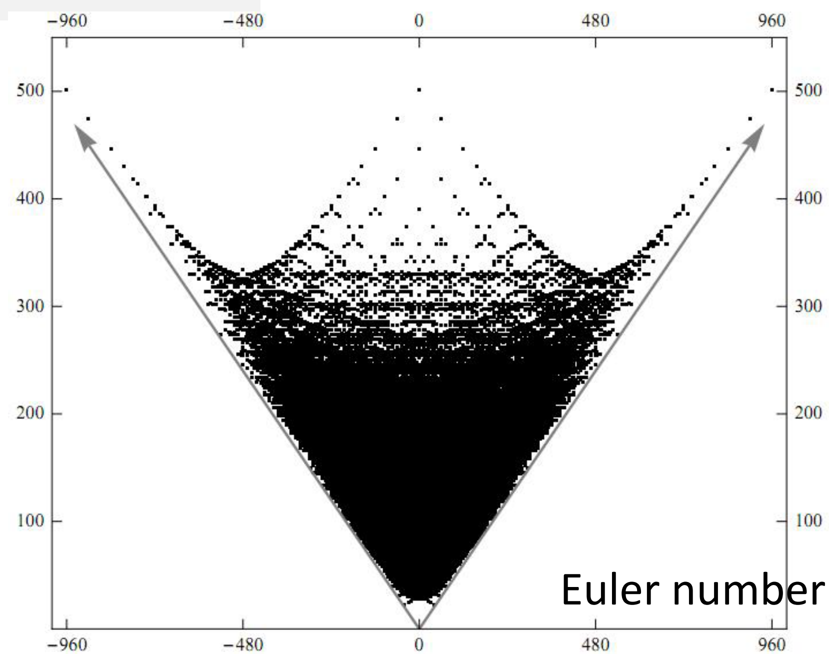
- 4D gauge symmetry :

$$E_8 \times E_8^{(\text{hidden})} \to E_6 \times SU(3) \times E_8^{(\text{hidden})}$$



- Index of 27 or $\overline{27}$ matter multiplets

$$N_{\text{gen}} = \frac{\text{Euler number}}{2}$$



CY3 in the Kreuzer-Skarke database

Kreuzer-Skarke ('00)

Largest Euler number = 960

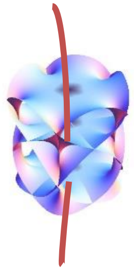
$$N_{\text{gen}} \leq 480$$

$E_8 \times E_8$ Heterotic Line Bundle Models on CY threefolds

Blumenhagen-Honecker-Weigand ('05), Anderson-Gray-Lukas-Palti ('11),….

- Multiple line bundles $V = \oplus_{a=1}^{4} L_a$ lead to semi-realistic SM spectra:

$$c_1(L_a) = \sum_{i=1}^{h^{1,1}} m_a^i w_i \qquad \frac{1}{2\pi}\int_{\Sigma_i} F_a = m_a^i \in \mathbb{Z}$$



-- Gauge group

$$E_8 \to SU(5) \times \Pi_{a=1}^{4} U(1)_a$$

-- Chiral zero-modes

$$248 \to \oplus_p (R_p\,, C_p)$$

-- Index of SU(5) matter multiplets

$$N_{\text{gen}} = \frac{1}{2}\int_M c_3(V) = \sum_{a,i} \frac{d_{ijk} m_a^i m_a^j m_a^k}{6}$$

due to $c_1(V) = 0$ ("K-theory condition")

$E_8 \times E_8$ Heterotic Line Bundle Models on CY threefolds

- Upper bound on the Atiyah-Singer index:

$$\left|N_{\text{gen}}\right| = \left|\sum_{a,i} \frac{d_{ijk} m_a^i m_a^j m_a^k}{6}\right| \leq \frac{|m_{max}|\ ||c_2(V)||}{3} \leq \frac{|m_{max}|\ ||c_2(TX)||}{3}$$

$\left|m_a^i\right| \leq |m_{max}|$

Tadpole cancellation
$c_2(V) \leq c_2(TX)$

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- Upper bound on flux quanta m_a^i :

Constantin-Lukas-Mishra ('15)

$$0 < \sum_a m_a^i G_{ij} m_a^j = \frac{1}{\mathcal{V}} t^i c_{2i}(V) \leq \frac{1}{\mathcal{V}} ||\boldsymbol{t}|| \times ||c_2(TX)||$$

G_{ij} : Moduli metric

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$$\tilde{G}_{ij} := \frac{\mathcal{V}}{||\boldsymbol{t}||} G_{ij}$$

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$$\Downarrow$$

$$\tilde{G}_{ij} := \frac{\mathcal{V}}{||\boldsymbol{t}||} G_{ij}$$

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$$0 < \sum_a m_a^i \tilde{G}_{ij} m_a^j \leq ||c_2(TX)||$$

$E_8 \times E_8$ Heterotic Line Bundle Models on CY threefolds

- Upper bound on flux quanta m_a^i :

$$0 < \sum_{a,i} m_a^i \tilde{G}_{ij} m_a^j \leq ||c_2(TX)||$$

$$\tilde{G}_{ij} \coloneqq \frac{\mathcal{V}}{||\boldsymbol{t}||} G_{ij}$$

Eigen$(\tilde{G}_{ij})|_{\min} = \lambda_{\min}$

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$$\sum_{a,i} \left(m_a^i\right)^2 \leq \frac{||c_2(TX)||}{\lambda_{\min}}$$

$$\mathrm{Eigen}(\tilde{G}_{ij})|_{\min} = \lambda_{\min}$$

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--- using the Cauchy-Schwarz inequality with $\sum_{a=1}^{n} m_a^i = 0$

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--- using the Cauchy-Schwarz inequality with $\sum_{a=1}^{n} m_a^i = 0$

$$\sum_{a=1,i} \left(m_a^i\right)^2 \geq \sum_{b=2,i} \left(m_b^i\right)^2 + \sum_i \left(m_1^i\right)^2$$

$$\geq \sum_i \frac{\left(m_2^i + \cdots + m_n^i\right)^2}{n-1} + \sum_i \left(m_1^i\right)^2 = \frac{n}{n-1} \sum_i \left(m_1^i\right)^2$$

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$$\sum_{i} \left(m_a^i\right)^2 \leq \frac{n-1}{n} \frac{||c_2(TX)||}{\lambda_{\min}}$$

n : # of U(1)

$E_8 \times E_8$ Heterotic Line Bundle Models on CY threefolds

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$$\sum_{i} \left(m_a^i\right)^2 \leq \frac{n-1}{n} \frac{||c_2(TX)||}{\lambda_{\min}}$$

n : # of U(1)

If $m_a^i \neq 0$ for $\forall a, i$

$$(m_{\max})^2 \leq \frac{n-1}{n} \frac{||c_2(TX)||}{\lambda_{\min}} - (h^{1,1} - 1)$$

$E_8 \times E_8$ Heterotic Line Bundle Models on CY threefolds

- Upper bound on the Atiyah-Singer index:

$$|N_{\text{gen}}| = \left| \sum_{a,i} \frac{d_{ijk} m_a^i m_a^j m_a^k}{6} \right| \leq \frac{|m_{max}|\, ||c_2(V)||}{3} \leq \frac{|m_{max}|\, ||c_2(TX)||}{3}$$

$|m_a^i| \leq |m_{max}|$

Tadpole cancellation
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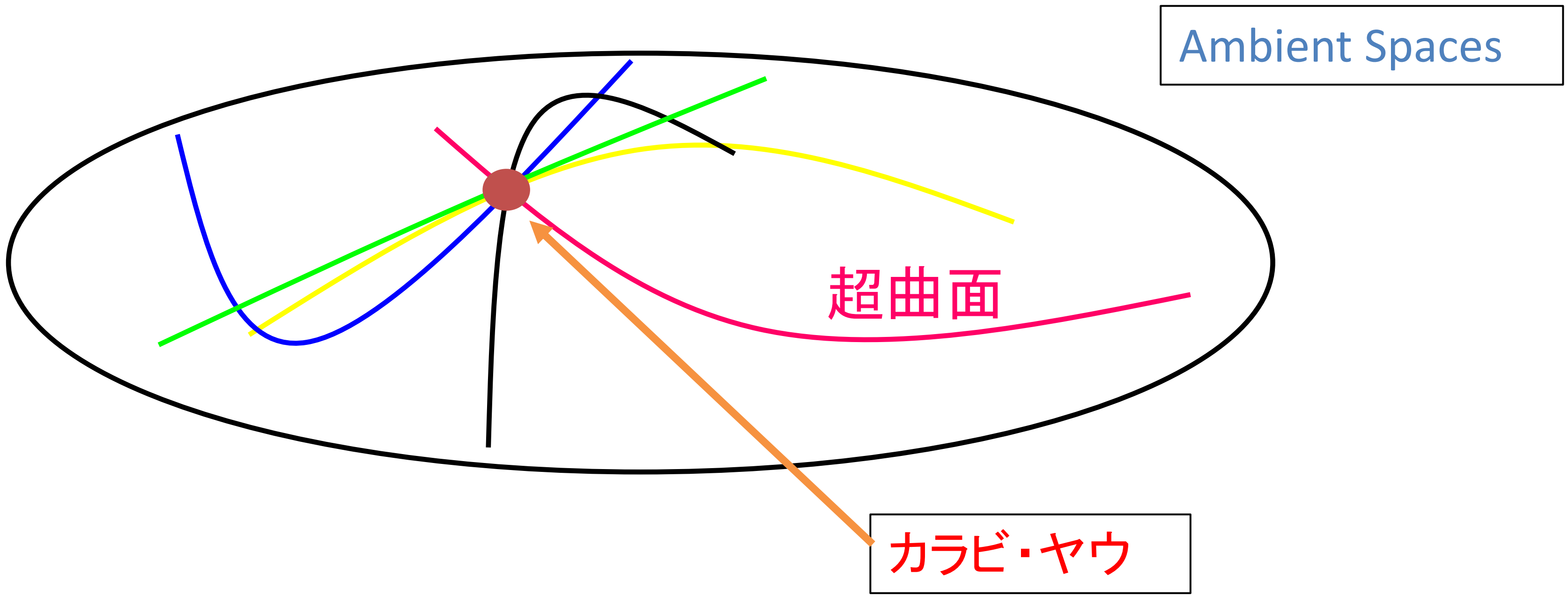
We estimate the bound on "favorable" complete intersection CYs (CICYs).

Candelas-Dale-Lutken-Schimmrigk ('88)
Candelas-Lutken-Schimmrigk ('88)

Complete Intersection Calabi-Yau = 完全交叉カラビ・ヤウ



- Constructing CY manifolds as the intersection of hypersurfaces
- Ambient spaces are given by products of complex projective space

$$\mathbb{P}^n \equiv \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \mathbb{C}^*$$

$$(x_0, x_1, \ldots, x_n) \sim (\lambda x_0, \lambda x_1, \ldots, \lambda x_n)$$

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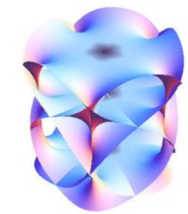


Quintic $x_i \quad \mathbb{P}^4 \; [5]$ $\qquad \sum_{i=1}^{5} x_i^5 = 0$

Tian-Yau
$$\begin{matrix} x_i & \mathbb{P}^3 \\ y_i & \mathbb{P}^3 \end{matrix} \begin{bmatrix} 1 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix} \qquad \sum_{i=0}^{3} x_i y_i = 0, \quad \sum_{i=0}^{3} (x_i)^3 = 0, \quad \sum_{i=0}^{3} (y_i)^3 = 0.$$

General CICY

Candelas-Dale-Lutken-Schimmrigk ('88)
Candelas-Lutken-Schimmrigk ('88)

$$\begin{matrix} \mathbb{P}^{n_1} \\ \mathbb{P}^{n_2} \\ \vdots \\ \mathbb{P}^{n_m} \end{matrix} \begin{bmatrix} q_1^1 & q_2^1 & \cdots & q_K^1 \\ q_1^2 & q_2^2 & \cdots & q_K^2 \\ \vdots & \vdots & \ddots & \vdots \\ q_1^m & q_2^m & \cdots & q_K^m \end{bmatrix}_{m \times K}$$

$$q_j^r \in \mathbb{Z}_{\neq 0}$$

$$\sum_{r=1}^{m} n_r - K = 3$$

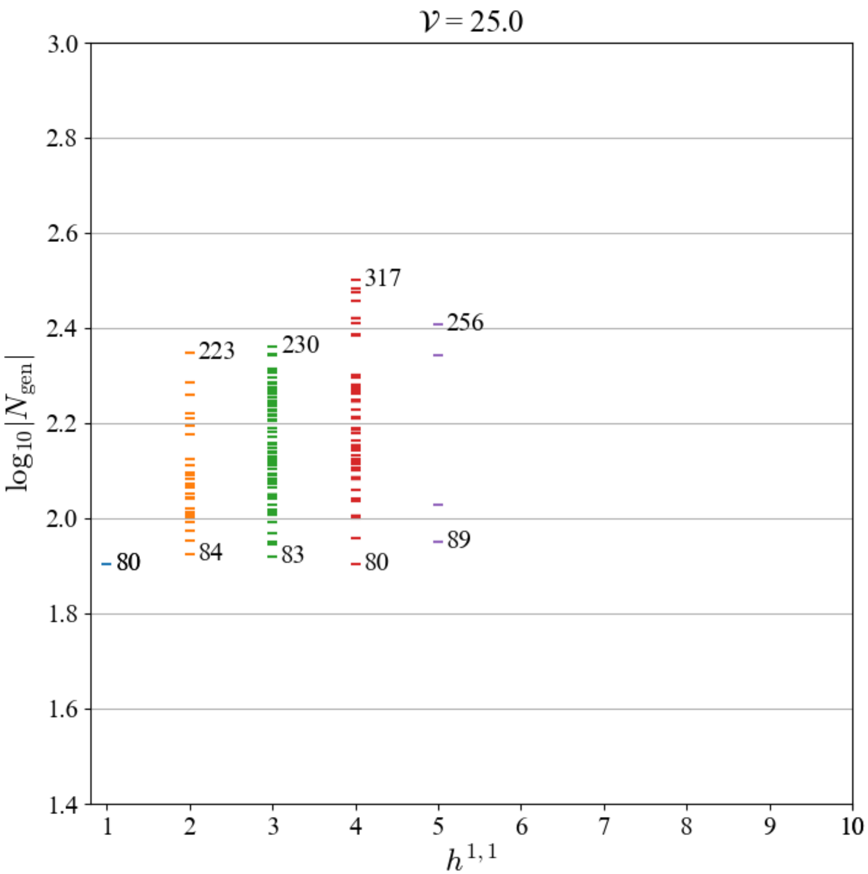
CY conditions

$$\sum_{j=1}^{K} q_j^r = n_r + 1$$

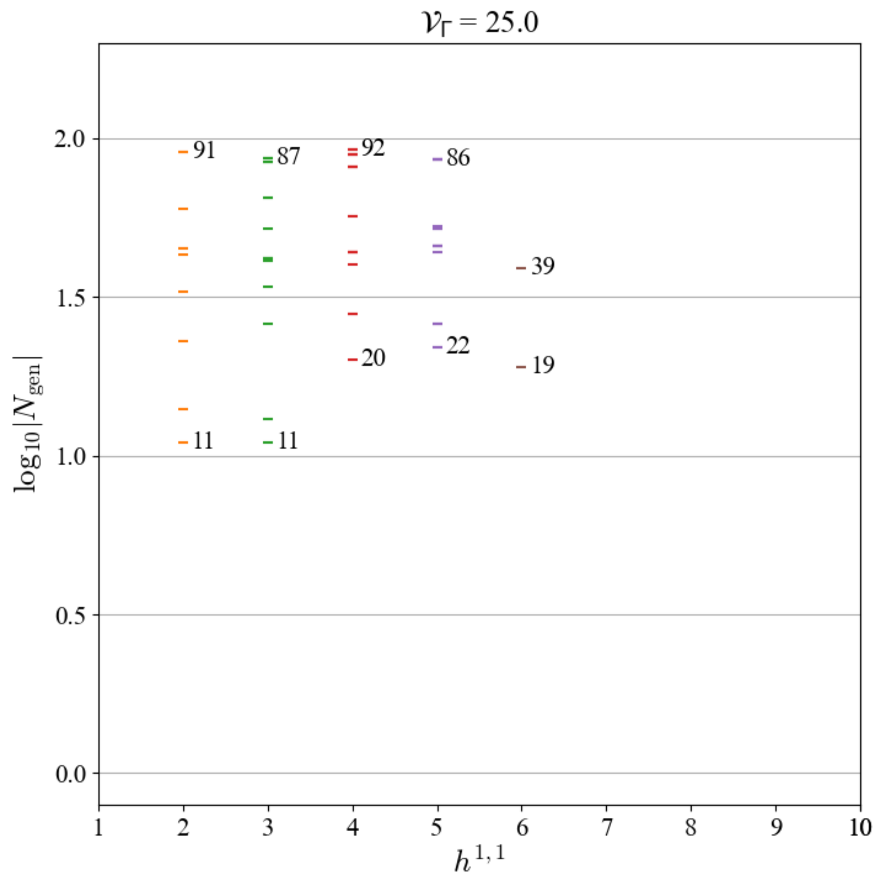
$$r = 1, \cdots, m$$

Upper bound on Atiyah-Singer Index in SU(5) GUT

CICYs ($\mathcal{M}$) with $\mathcal{V}$ = 25



Quotient CICYs ($\mathcal{M}/\Gamma$) with $\mathcal{V}_\Gamma$=25



Note that the CY volume is upper bounded by $\mathcal{V} = g_s^2 \alpha_{\rm GUT}^{-1} \leq 25$

($\sum_{i,j,k} d_{ijk} < \sum_{i,j,k} d_{ijk} t^i t^j t^k = 6\mathcal{V} \leq 150$)

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- **Conclusions and Discussions**

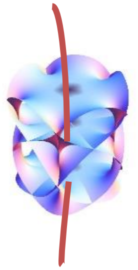
  — Type II string/ F-theory construction

$SO(32)$ Heterotic Line Bundle Models on CY threefolds

Otsuka ('18),....

- Multiple line bundles $V = \bigoplus_a L_a$ lead to semi-realistic SM spectra:

$$c_1(L_a) = \sum_{i=1}^{h^{1,1}} m_a^i w_i \qquad \frac{1}{2\pi}\int_{\Sigma_i} F_a = m_a^i \in \mathbb{Z}$$



-- Gauge group

$$SO(32) \to SU(4)_C \times SU(2)_L \times SU(2)_R \times \Pi_a U(1)_a \times SO(16)_{\rm hid}$$

-- Chiral zero-modes

$$496 \to \bigoplus_p (R_p\,, C_p)$$

-- Index of matter multiplets

$$N_{\rm gen} = \frac{1}{2}\int_M c_3(C_p) + \frac{1}{12} c_2(TM) c_1(C_p)$$

$(SU(4)_C \times SU(2)_L \times SU(2)_R \times SO(16))_{U(1)_1,U(1)_2,U(1)_4,U(1)_5}$	$(SU(3)_C \times SU(2)_L \times SO(16))_{U(1)_1,U(1)_2,U(1)_3,U(1)_4,U(1)_5}$	Matter	Index
Visible sector			
$(4,2,1,1)_{1,1,0,0}$	$(3,2,1)_{1,1,1,0,0}$ $(1,2,1)_{1,1,-3,0,0}$	Q_1 L_1	$\chi(\mathcal{M}, L_1 \otimes L_2)$
$(4,2,1,1)_{-1,1,0,0}$	$(3,2,1)_{-1,1,1,0,0}$ $(1,2,1)_{-1,1,-3,0,0}$	Q_2 L_2	$\chi(\mathcal{M}, L_1^{-1} \otimes L_2)$
$(15,1,1,1)_{0,0,0,0}$	$(\bar{3},1,1)_{0,0,-4,0,0}$	$u^c_{R_1}$	$\chi(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$
$(6,1,1,1)_{0,2,0,0}$	$(\bar{3},1,1)_{0,2,2,0,0}$ $(3,1,1)_{0,2,-2,0,0}$	$d^c_{R_1}$ $\bar{d}^c_{R_2}$	$\chi(\mathcal{M}, L_2^2)$
$(1,1,1,1)_{2,0,0,0}$	$(1,1,1)_{2,0,0,0,0}$	n_1	$\chi(\mathcal{M}, L_1^2)$
$(\bar{4},1,2,1)_{0,-1,-1,0}$	$(\bar{3},1,1)_{0,-1,-1,-1,0}$ $(1,1,1)_{0,-1,3,0,1}$ $(1,1,1)_{0,-1,3,-1,0}$ $(\bar{3},1,1)_{0,-1,-1,0,1}$	$u^{c\,4}_{R_2}$ $e^{c\,5}_{R_1}$ $n^{c\,4}_2$ $d^{c\,5}_{R_3}$	$\chi(\mathcal{M}, L_2^{-1} \otimes L_4^{-1})$
$(\bar{4},1,2,1)_{0,-1,1,0}$	$(\bar{3},1,1)_{0,-1,-1,1,0}$ $(1,1,1)_{0,-1,3,1,0}$ $(1,1,1)_{0,-1,3,0,-1}$ $(\bar{3},1,1)_{0,-1,-1,0,-1}$	$d^{c\,4}_{R_3}$ $e^{c\,4}_{R_1}$ $n^{c\,5}_2$ $u^{c\,5}_{R_2}$	$\chi(\mathcal{M}, L_2^{-1} \otimes L_4)$
$(1,2,2,1)_{1,0,-1,0}$	$(1,2,1)_{1,0,0,-1,0}$ $(1,2,1)_{1,0,0,0,1}$	L^4_3 $\bar{L}^5_4$	$\chi(\mathcal{M}, L_1 \otimes L_4^{-1})$
$(1,2,2,1)_{1,0,1,0}$	$(1,2,1)_{1,0,0,0,-1}$ $(1,2,1)_{1,0,0,1,0}$	L^5_3 $\bar{L}^4_4$	$\chi(\mathcal{M}, L_1 \otimes L_4)$
$(1,1,3,1)_{0,0,0,0}$	$(1,1,1)_{0,0,0,1,1}$	$e^{c\,45}_{R_2}$	$\chi(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$
$(1,1,1,1)_{0,0,2,0}$	$(1,1,1)_{0,0,0,1,-1}$	$n^{c\,45}_3$	$\chi(\mathcal{M}, L_4^2)$
Hidden sector			
$(4,1,1,16)_{0,-1,0,0}$	$(3,1,16)_{0,-1,-1,0,0}$ $(1,1,16)_{0,-1,3,0}$	– –	$\chi(\mathcal{M}, L_2^{-1})$
$(1,2,1,16)_{1,0,0,0}$	$(1,2,16)_{1,0,0,0,0}$	–	$\chi(\mathcal{M}, L_1)$
$(1,1,2,16)_{0,0,1,0}$	$(1,1,16)_{0,0,0,1,0}$ $(1,1,16)_{0,0,0,0,-1}$	– –	$\chi(\mathcal{M}, L_4)$
$(1,1,1,120)_{0,0,0,0}$	$(1,1,120)_{0,0,0,0,0}$	–	$\chi(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$

$(SU(4)_C \times SU(2)_L \times SU(2)_R \times SO(16))_{U(1)_1,U(1)_2,U(1)_4,U(1)_5}$	$(SU(3)_C \times SU(2)_L \times SO(16))_{U(1)_1,U(1)_2,U(1)_3,U(1)_4,U(1)_5}$	Matter	Index
Visible sector			
$(4,2,1,1)_{1,1,0,0}$	$(3,2,1)_{1,1,1,0,0}$ $(1,2,1)_{1,1,-3,0,0}$	Q_1 L_1	$\chi(\mathcal{M}, L_1 \otimes L_2)$
$(4,2,1,1)_{-1,1,0,0}$	$(3,2,1)_{-1,1,1,0,0}$ $(1,2,1)_{-1,1,-3,0,0}$	Q_2 L_2	$\chi(\mathcal{M}, L_1^{-1} \otimes L_2)$
$(15,1,1,1)_{0,0,0,0}$	$(\bar{3},1,1)_{0,0,-4,0,0}$	$u^c_{R_1}$	$\chi(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$
$(6,1,1,1)_{0,2,0,0}$	$(\bar{3},1,1)_{0,2,2,0,0}$ $(3,1,1)_{0,2,-2,0,0}$	$d^c_{R_1}$ $\bar{d}^c_{R_2}$	$\chi(\mathcal{M}, L_2^2)$
$(1,1,1,1)_{2,0,0,0}$	$(1,1,1)_{2,0,0,0,0}$	n_1	$\chi(\mathcal{M}, L_1^2)$
$(\bar{4},1,2,1)_{0,-1,-1,0}$	$(\bar{3},1,1)_{0,-1,-1,-1,0}$ $(1,1,1)_{0,-1,3,0,1}$ $(1,1,1)_{0,-1,3,-1,0}$ $(\bar{3},1,1)_{0,-1,-1,0,1}$	$u^{c\,4}_{R_2}$ $e^{c\,5}_{R_1}$ $n^{c\,4}_2$ $d^{c\,5}_{R_3}$	$\chi(\mathcal{M}, L_2^{-1} \otimes L_4^{-1})$
$(\bar{4},1,2,1)_{0,-1,1,0}$	$(\bar{3},1,1)_{0,-1,-1,1,0}$ $(1,1,1)_{0,-1,3,1,0}$ $(1,1,1)_{0,-1,3,0,-1}$ $(\bar{3},1,1)_{0,-1,-1,0,-1}$	$d^{c\,4}_{R_3}$ $e^{c\,4}_{R_1}$ $n^{c\,5}_2$ $u^{c\,5}_{R_2}$	$\chi(\mathcal{M}, L_2^{-1} \otimes L_4)$
$(1,2,2,1)_{1,0,-1,0}$	$(1,2,1)_{1,0,0,-1,0}$ $(1,2,1)_{1,0,0,0,1}$	L_3^4 $\bar{L}_4^5$	$\chi(\mathcal{M}, L_1 \otimes L_4^{-1})$
$(1,2,2,1)_{1,0,1,0}$	$(1,2,1)_{1,0,0,0,-1}$ $(1,2,1)_{1,0,0,1,0}$	L_3^5 $\bar{L}_4^4$	$\chi(\mathcal{M}, L_1 \otimes L_4)$
$(1,1,3,1)_{0,0,0,0}$	$(1,1,1)_{0,0,0,1,1}$	$e^{c\,45}_{R_2}$	$\chi(\mathcal{M}, \mathcal{O}_{\mathcal{M}})$
$(1,1,1,1)_{0,0,2,0}$	$(1,1,1)_{0,0,0,1,-1}$	$n^{c\,45}_3$	$\chi(\mathcal{M}, L_4^2)$
Hidden sector			

Index of hidden sector = 0

$(SU(4)_C \times SU(2)_L \times SU(2)_R \times SO(16))_{U(1)_1,U(1)_2,U(1)_4,U(1)_5}$	$(SU(3)_C \times SU(2)_L \times SO(16))_{U(1)_1,U(1)_2,U(1)_3,U(1)_4,U(1)_5}$	Matter	Index
Visible sector			
$(6,1,1,1)_{0,2,0,0}$	$(\bar{3},1,1)_{0,2,2,0,0}$ $(3,1,1)_{0,2,-2,0,0}$	$d^c_{R_1}$ $\bar{d}^c_{R_2}$	$\chi(\mathcal{M}, L_2^2)$
Hidden sector			

Vector-like quarks :

$$|N_{\rm gen}^{\rm (vec)}| \leq \frac{|m_{\rm max}||c_2(V)|}{4} \leq \frac{|m_{\rm max}||c_2(TX)|}{4}$$

Tadpole cancellation

$c_2(V) \leq c_2(TX)$

Index of hidden sector = 0

$(SU(4)_C \times SU(2)_L \times SU(2)_R \times SO(16))_{U(1)_1,U(1)_2,U(1)_4,U(1)_5}$	$(SU(3)_C \times SU(2)_L \times SO(16))_{U(1)_1,U(1)_2,U(1)_3,U(1)_4,U(1)_5}$	Matter	Index
Visible sector			

Higgs + Quarks(Leptons) :

$$|N_{\text{gen}}(H) + 2N_{\text{gen}}^{(\text{quark})}| \leq \frac{|m_{\max}|\, ||c_2(V)||}{2} \leq \frac{|m_{\max}|\, ||c_2(TX)||}{2}$$

Tadpole cancellation
$c_2(V) \leq c_2(TX)$

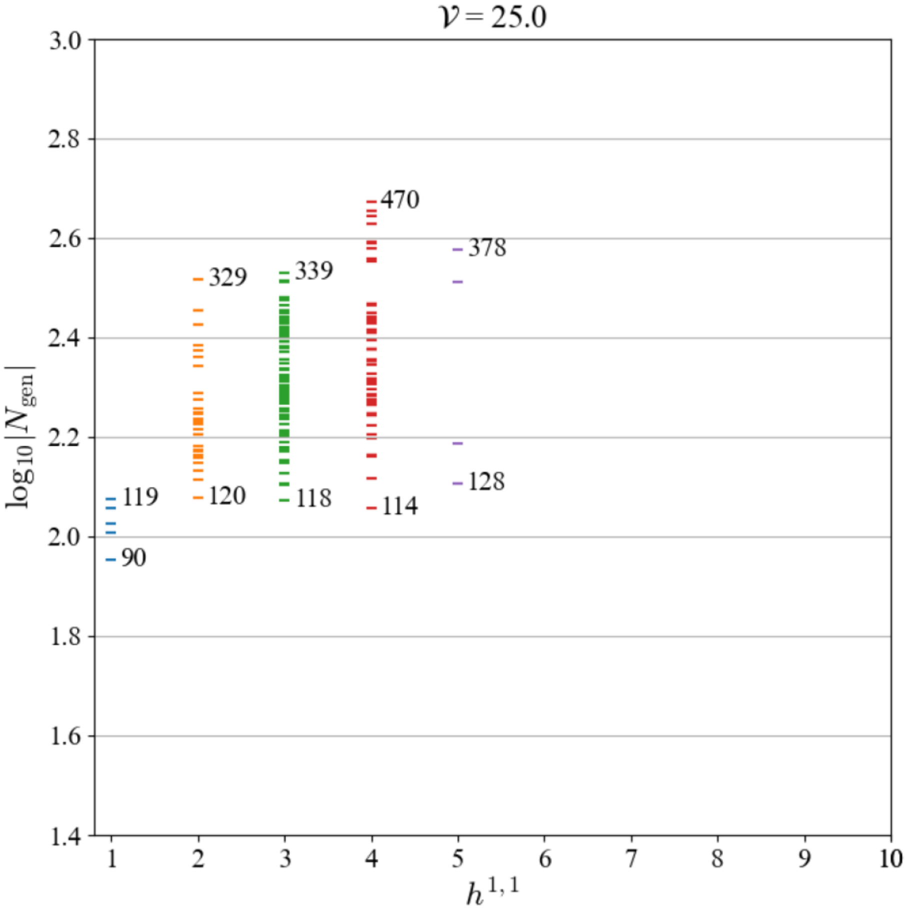
Higgs :

If $N_{\text{gen}}^{(\text{quark})} = -3$

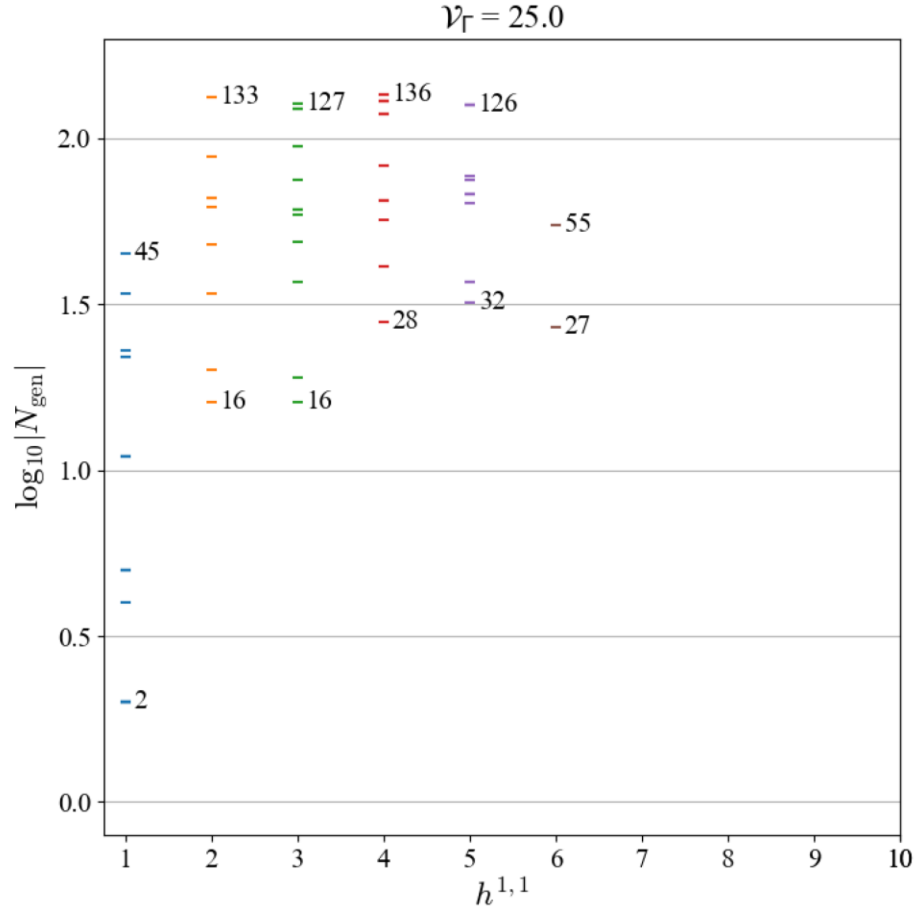
$$|N_{\text{gen}}(H)| \leq \frac{|m_{\max}||c_2(TX)|}{2} - 6$$

Upper bound on #Higgs in Pati-Salam

CICYs ($\mathcal{M}$) with $\mathcal{V}$ = 25



Quotient CICYs ($\mathcal{M}/\Gamma$) with $\mathcal{V}_\Gamma$=25



Note that the CY volume is upper bounded by $\mathcal{V} = g_s^2 \alpha_{\rm GUT}^{-1} \leq 25$
($\sum_{i,j,k} d_{ijk} < \sum_{i,j,k} d_{ijk} t^i t^j t^k = 6\mathcal{V} \leq 150$)

Outline

✓ **Introduction/Short summary**

- **Heterotic string theory with line bundles**

  ✓ E_6 GUT in $E_8 \times E_8$ heterotic string

  ✓ SU(5) GUT in $E_8 \times E_8$ heterotic string

  ✓ Pati-Salam in $SO(32)$ heterotic string

  — Direct flux breaking (hypercharge flux) in $SO(32)$ heterotic string

- **Conclusions and Discussions**

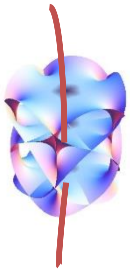
  — Type II string/ F-theory construction

$SO(32)$ Heterotic Line Bundle Models on CY threefolds

Otsuka ('18),….

- Multiple line bundles $V=\oplus_a L_a$ lead to semi-realistic SM spectra:

$$c_1(L_a)=\sum_{i=1}^{h^{1,1}} m_a^i w_i$$



-- Gauge group

$$SO(32)\to SU(3)_C\times SU(2)_L\times U(1)_Y\times \Pi_a U(1)_a\times SO(16)_{\rm hid}$$

-- Chiral zero-modes

$$496\ \to \oplus_p (R_p\,,C_p)$$

-- Index of quarks/leptons

$$|N_{\rm gen}|\leq \frac{|m_{\rm max}||c_2(V)|}{6}\leq \frac{|m_{\rm max}||c_2(TX)|}{6}$$

Tadpole cancellation
$c_2(V)\leq c_2(TX)$

Other CY threefolds ?

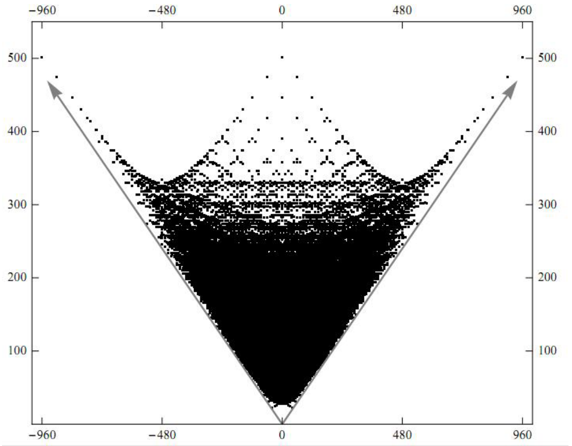
- For the CY threefolds in the Kreuzer-Skarke database

Kreuzer-Skarke ('00)
Demirtas-Long-McAllister-Stillman ('18),...

For $h^{1,1} \geq 25$

CY volume : $\mathcal{V} \geq (h^{1,1})^{-\frac{1}{2}} ||\boldsymbol{t}||^3$

Moduli metric : $G_{ij} \simeq (h^{1,1})^{-\frac{1}{2}} ||\boldsymbol{t}||$



$$||c_2(TX)|| \geq \sum_{a,i} m_a^i \tilde{G}_{ij} m_a^j \geq \alpha (h^{1,1})^{\frac{1}{2}}$$

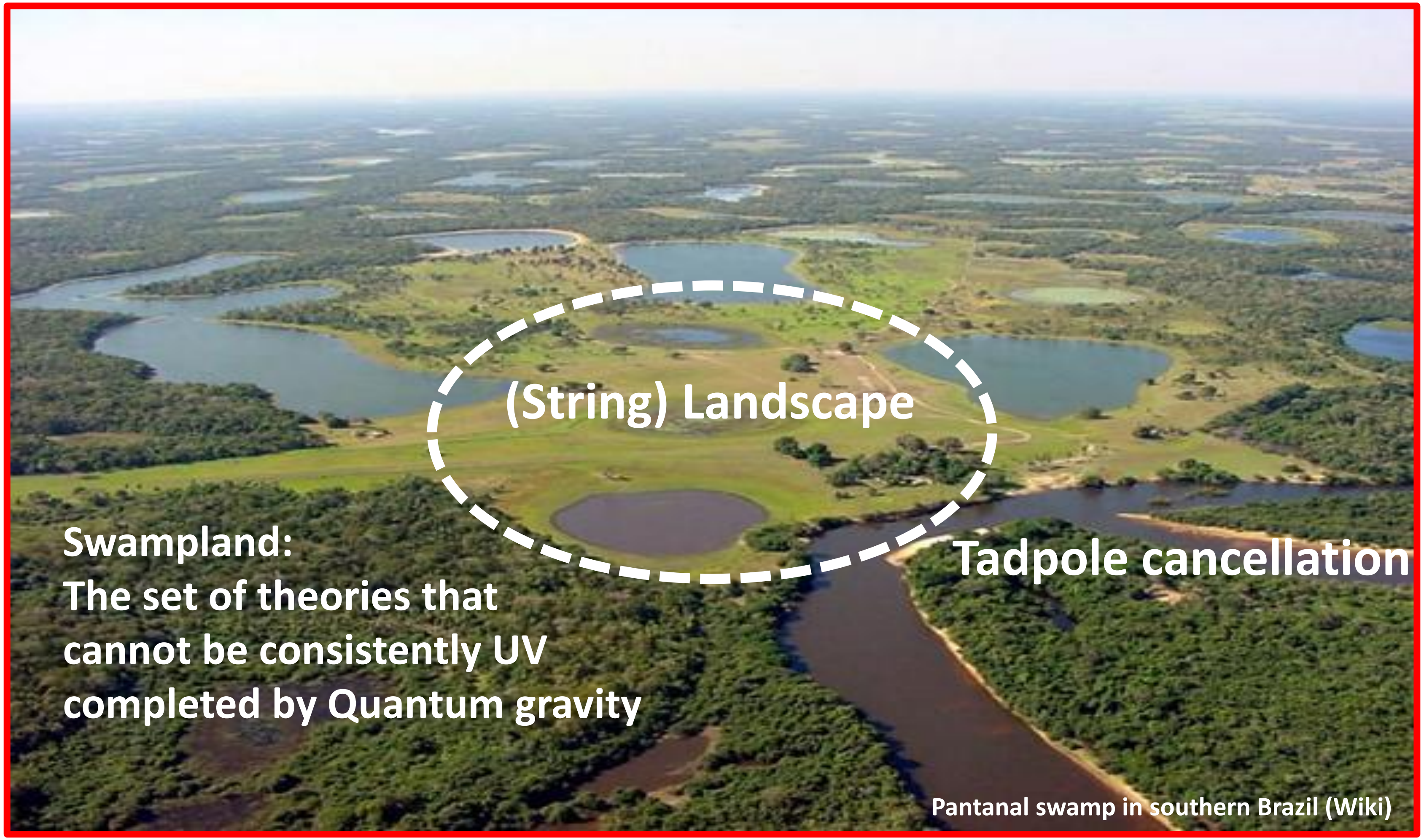
α : O(1) coefficients

$$\tilde{G}_{ij} := \frac{\mathcal{V}}{||\boldsymbol{t}||} G_{ij}$$

Tadpole cancellation
$c_2(TX) \geq c_2(V)$

Conclusion and Discussions

- We propose an upper bound on the Atiyah-Singer index in heterotic string theories on CY with line bundles



Conclusion and Discussions

- We propose an upper bound on the Atiyah-Singer index in heterotic string theories on CY with line bundles

- Generation number of quarks/leptons, Higgs are constrained by the tadpole cancellation :

$$|N_{\rm gen}| \leq \frac{|m_{\rm max}||c_2(TX)|}{4}$$

$$|m_{\rm max}| \leq \frac{n-1}{n}\frac{||c_2(TX)||}{\lambda_{\rm min}} - (h^{1,1}-1)$$

n : # of U(1)

- For $h^{1,1} \geq 25$

$$||c_2(TX)|| \geq \alpha (h^{1,1})^{\frac{1}{2}}$$

α : O(1) coefficients

- Similar upper bound in Type IIB string theory