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行列模型の手法によるJT重力研究の進展

酒井 一博
(明治学院大学)

奥山和美さん（信州大学）との共同研究

arXiv:1911.01659, 2004.07555, 2108.03876 （他6編）

姉川尊徳さん（大阪大学）、飯塚則裕さん（大阪大学）、奥山和美さんとの共同研究

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1. Introduction

- JT gravity is a simple model of 2d dilaton gravity (Jackiw '85, Teitelboim '83)

$$I = -\underbrace{\frac{S_0}{2\pi}\left[\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}R + \int_{\partial\mathcal{M}}\sqrt{h}K\right]}_{\substack{\text{topological term} \\ = S_0\chi(\mathcal{M})}} - \left[\underbrace{\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}\phi(R+2)}_{\text{sets } R=-2} + \underbrace{\int_{\partial\mathcal{M}}\sqrt{h}\phi(K-1)}_{\substack{\text{gives action} \\ \text{for boundary}}}\right]$$

(Throughout this talk we consider Euclidean JT gravity.)

(We follow the notation of Saad-Shenker-Stanford '19)

- It describes the low-energy dynamics of any near-extremal black hole.

- It has revived as a model for the $NAdS_2/NCFT_1$ correspondence

(Almheiri-Polchinski '14) (Maldacena-Stanford-Yang '16) (Jensen '16) (Engelsöy-Mertens-Verlinde '16)

low energy dynamics of the SYK model	=	1d Schwarzian theory	=	boundary description of bulk 2d JT gravity

◉ Saad-Shenker-Stanford showed that the partition functions of JT gravity correspond to the genus expansion of a double-scaled matrix integral.

(Saad-Shenker-Stanford '19)

1. Introduction (continued)

- 2d quantum gravity has been extensively studied since the 1980's.
  - Double scaled matrix model — counting of triangulations of surfaces

$$\mathcal{Z} = \int dH e^{-N\,\mathrm{Tr}\,V(H)}$$

(Brezin-Kazakov '90) (Douglas-Shenker '90) (Gross-Migdal '90)

  - Topological gravity — intersection theory on the moduli space of Riemann surfaces (Witten '90) (Witten '91)

- Witten conjecture (proved by Kontsevich) (Witten '91) (Kontsevich '92)

The above two theories are in fact equivalent.

- The generating function for the intersection numbers obeys the KdV equations and the string equation.

Q: How is the matrix integral of Saad-Shenker-Stanford understood in the context of traditional matrix models/topological gravity?

1. Introduction (continued)

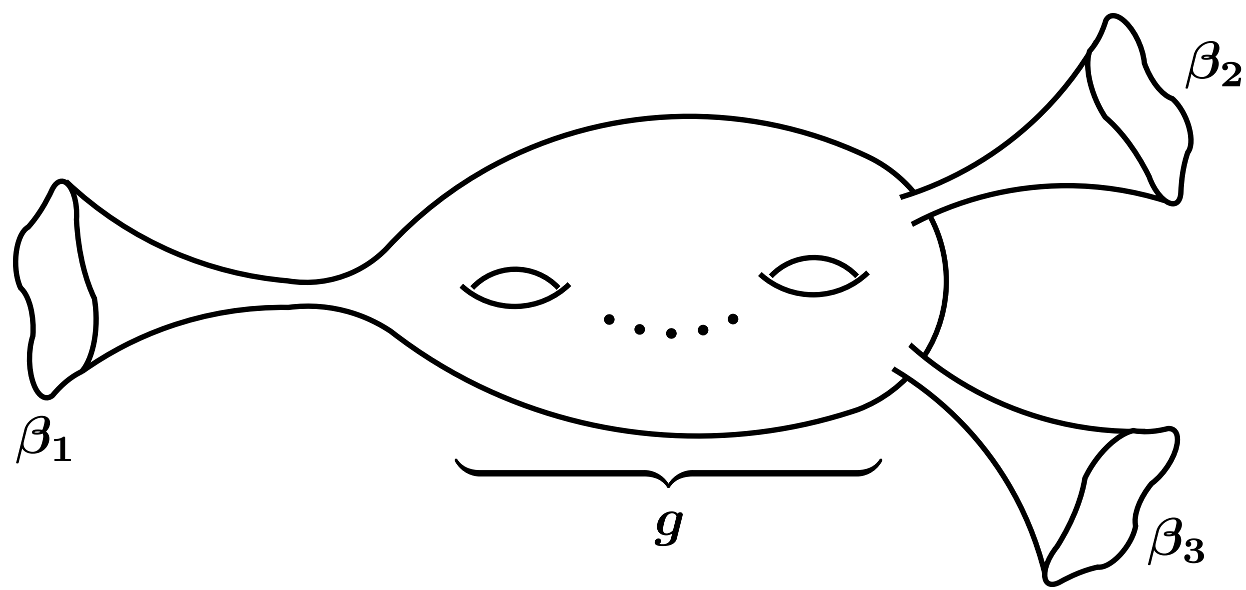
Main results

- JT gravity is a special case of 2d topological gravity

$$t_0 = t_1 = 0, \quad t_k = \frac{(-1)^k}{(k-1)!} \quad (k \geq 2)$$

- Multi-boundary correlators of 2d topological gravity are computed by simply solving the KdV equation

$$Z_3(\beta_1, \beta_2, \beta_3) = \sum_{g=0}^{\infty}$$



Plan of the talk

1. Introduction

2. Path integral in JT gravity (review)

3. JT gravity as a special case of topological gravity

4. Genus expansion of multi-boundary correlators

5. Other expansions, FZZT branes and applications

6. Conclusions and outlook

2. Path integral in JT gravity

(Saad-Shenker-Stanford '19)

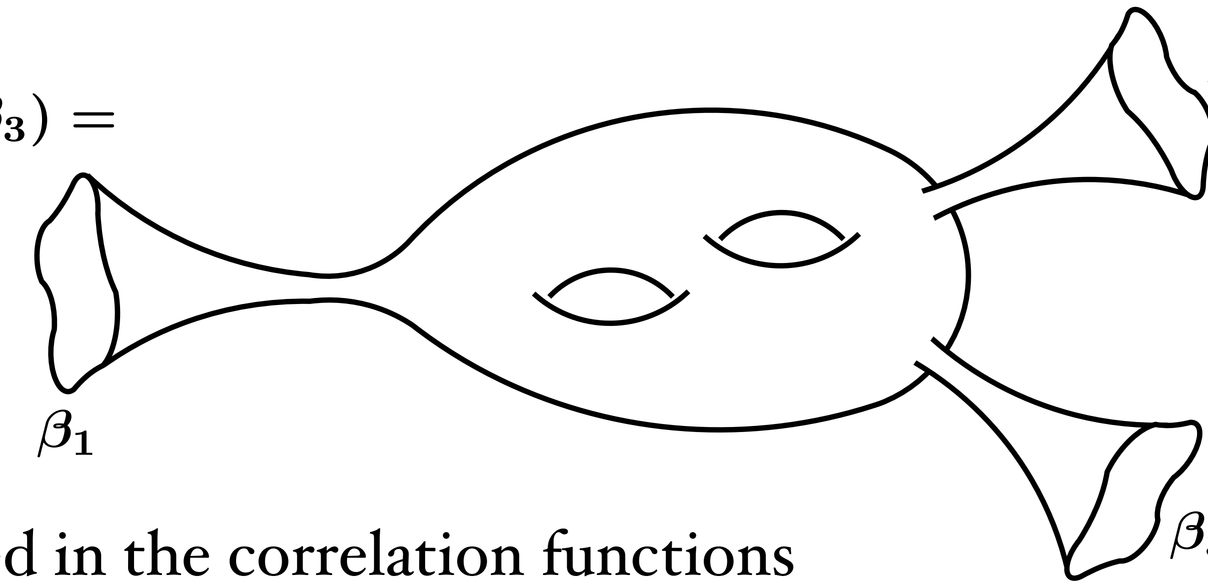
- JT gravity is a 2d dilaton gravity given by the action

$$I = \underbrace{-\frac{S_0}{2\pi}\left[\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}R + \int_{\partial\mathcal{M}}\sqrt{h}K\right]}_{\substack{\text{topological term} \\ = S_0\chi(\mathcal{M})}} - \left[\underbrace{\frac{1}{2}\int_{\mathcal{M}}\sqrt{g}\phi(R+2)}_{\text{sets } R=-2} + \underbrace{\int_{\partial\mathcal{M}}\sqrt{h}\phi(K-1)}_{\substack{\text{gives action} \\ \text{for boundary}}}\right]$$

- $\mathcal{M}$ has n boundaries of lengths $\beta_1/\epsilon, \ldots, \beta_n/\epsilon$, where $\phi = \gamma/\epsilon$ $(\epsilon \to 0)$

(We will set $\gamma = 1/2\pi^2$)

$Z_{g=2,n=3}(\beta_1, \beta_2, \beta_3) =$



- We are interested in the correlation functions

$$\langle Z(\beta_1)\cdots Z(\beta_n)\rangle_c = Z_n(\beta_1,\ldots,\beta_n) = \sum_{g=0}^{\infty}\frac{Z_{g,n}(\beta_1,\ldots,\beta_n)}{(e^{S_0})^{2g+n-2}}$$

$Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$

thermal partition function in the boundary theory interpretation

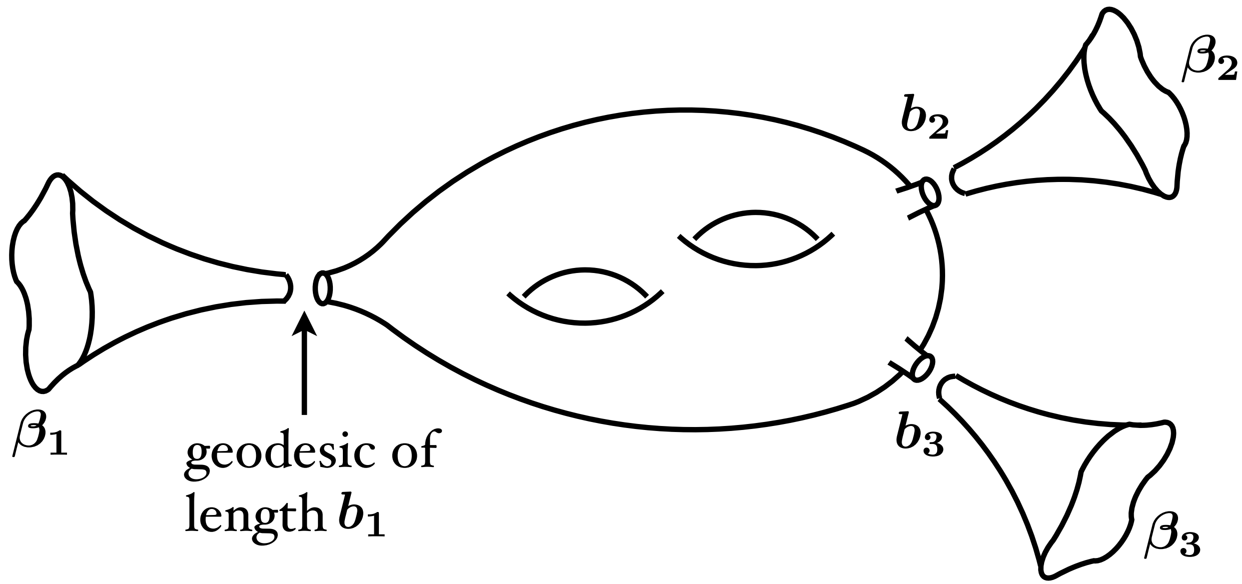
genus counting parameter

$$\begin{pmatrix} e^{-S_0} \\ \sim g_s \\ \sim \hbar \end{pmatrix}$$

2. Path integral in JT gravity (continued)

(Saad-Shenker-Stanford '19)

- The path integral can be evaluated as follows:



boundary action

$$Z_{g,n}(\beta_1,\ldots,\beta_n) = \int d(\text{bulk moduli}) \int \mathcal{D}(\text{boundary wiggles}) e^{\int_{\partial\mathcal{M}} \sqrt{h}\phi(K-1)}$$

$$= \int b_1 db_1 \cdots b_n db_n V_{g,n}(b_1,\ldots,b_n) \prod_{i=1}^{n} Z_{\text{Sch}}^{\text{trumpet}}(\beta_i, b_i)$$

$$Z_{\text{Sch}}^{\text{trumpet}}(\beta_i, b_i) = \sqrt{\frac{\gamma}{2\pi\beta_i}} \exp\left[-\frac{\gamma b_i^2}{2\beta_i}\right]$$

Weil-Petersson volume of the moduli space of hyperbolic Riemann surfaces with g handles and n geodesic boundaries of length $b_1,\ldots,b_n$

JT gravity as a matrix integral

(Saad-Shenker-Stanford '19)

Mirzakhani's recursion relation for Weil-Petersson volumes $\Longleftrightarrow$ Eynard-Orantin "topological recursion" formulation

(Mirzakhani '07)

|| (Eynard-Orantin '07)

loop equation for the matrix integral

- Saad-Shenker-Stanford showed that the JT gravity correlation functions are consistent with the recursion relation of the matrix integral with the input

$$\rho_0(E) = \frac{\gamma}{2\pi^2}\sinh(2\pi\sqrt{2\gamma E}) \quad \Longleftrightarrow \quad y(z) = \frac{\gamma}{2\pi}\sin(2\pi\sqrt{2\gamma}z)$$

(leading density of eigenvalues) (spectral curve)

- The input is determined from the JT path integral $Z_{0,1}(\beta)$ for a disk by

$$Z_{0,1}(\beta) = \int_0^\infty dE\rho_0(E)e^{-\beta E}$$

- This is a "double-scaled" matrix integral as $\rho_0(E)$ is not normalizable.

3. JT gravity as a special case of topological gravity (Okuyama-KS '19)

- Mirzakhani's ($\Leftrightarrow$ topological) recursion — a slow algorithm

  to compute $V_{g,1}(b)$ we need to know all the data of $V_{g',n}$ with $g'+n \leq g+1 \quad (n \geq 1)$

- Zograf proposed an efficient algorithm for computing the WP volume by solving the KdV equation. (Zograf '08)

  - KdV eq. must help us to compute the partition function of JT gravity. But how?

- KdV equation arises in the study of old matrix models of 2d gravity.

  - How is the matrix integral of Saad-Shenker-Stanford understood in terms of old matrix models?

    - SSS's proposal: $p \to \infty$ limit of the (2,p) minimal string theory
    - We propose another (perhaps more natural) understanding.

General 2d topological gravity

(Witten '90) (Witten '91)

Σ : a closed Riemann surface of genus g with n marked points $p_1,\ldots,p_n$

$\mathcal{M}_{g,n}$: the moduli space of Σ

- Intersection numbers (= correlation functions of 2d topological gravity)

$$\langle \kappa^m \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \kappa^m \psi_1^{d_1} \cdots \psi_n^{d_n}, \qquad m, d_1, \ldots, d_n \in \mathbb{Z}_{\geq 0}$$

κ : the first Miller-Morita-Mumford class $\propto$ the Weil-Petersson symplectic form

ψ_i : the first Chern class of the complex line bundle whose fiber is the cotangent space to p_i

- Generating functions

$(\tau_d = \psi^d)$

$$G(s, \{t_k\}) := \sum_{g=0}^{\infty} g_s^{2g-2} \left\langle e^{s\kappa + \sum_{d=0}^{\infty} t_d \tau_d} \right\rangle_g, \quad F(\{t_k\}) := \sum_{g=0}^{\infty} g_s^{2g-2} \left\langle e^{\sum_{d=0}^{\infty} t_d \tau_d} \right\rangle_g$$

- G and F are related as

(Mulase-Safnuk '06) (Dijkgraaf-Witten '18)

$$G(s, \{t_k\}) = F(\{t_k + \gamma_k s^{k-1}\})$$

with

$$\gamma_0 = \gamma_1 = 0, \quad \gamma_k = \frac{(-1)^k}{(k-1)!} \quad (k \geq 2)$$

JT gravity as a special case of 2d topological gravity

- Let us consider the one-boundary partition function of JT gravity

$$\langle Z(\beta)\rangle = e^{S_0} Z^{\text{disk}}_{\text{Sch}} + \sum_{g=1}^{\infty} e^{(1-2g)S_0} \int_0^{\infty} b db Z^{\text{trumpet}}_{\text{Sch}}(\beta, b) \underline{V_{g,1}(b)}$$

$$\| $$

where WP volume $V_{g,1}(b)$ is expressed as $\left\langle e^{2\pi^2\kappa + \frac{b^2}{2}\psi_1} \right\rangle_{g,1}$ (Mirzakhani '07)

- By using the selection rule

$$\langle \kappa^k \psi_1^l \rangle_{g,1} = 0 \quad \text{unless} \quad k + l = 3g - 2$$

one can evaluate the above integral as

$$\langle Z(\beta)\rangle = \frac{g_s}{\sqrt{2\pi}\beta^{3/2}} \left(g_s^{-2} e^{\beta^{-1}} + \sum_{d=0}^{\infty} \beta^{d+2} \underline{\sum_{g=1}^{\infty} g_s^{2g-2} \langle e^{\kappa} \psi_1^d \rangle_{g,1}} \right)$$

$$\| $$

$$\partial_d G^{g\geq 1}(s=1, \{t_k = 0\}) \qquad \left(\partial_d := \frac{\partial}{\partial t_d}\right)$$

$$\| $$

$$\partial_d F^{g\geq 1}(\{t_k = \gamma_k\})$$

- We have thus shown that the partition function of JT gravity is expressed entirely in terms of the general topological gravity with couplings turned on with the specific value $t_k = \gamma_k$. (Okuyama-KS '19)

4. Multi-boundary correlators in topological gravity

- The n-boundary correlator of topological gravity is given by

$$Z_n(\{\beta_i\},\{t_k\}) \simeq B(\beta_1)\cdots B(\beta_n)F(\{t_k\})$$

(The symbol $\simeq$ means that the equality holds up to an additional non-universal part when $3g$-3+n<0.)

(Moore-Seiberg-Staudacher '91)

where

$$B(\beta) = g_{\rm s}\sqrt{\frac{\beta}{2\pi}}\sum_{d=0}^{\infty}\beta^d\frac{\partial}{\partial t_d}$$

"boundary creation operator"

Witten conjecture (Kontsevich theorem)

(Witten '90, '91) (Kontsevich '92)

(1) $u := g_s^2 \partial_0^2 F$ obeys the KdV equations ($k = 1$: traditional KdV)

$$\partial_k u = \partial_0 \mathcal{R}_{k+1} \qquad \left(\partial_k := \frac{\partial}{\partial t_k}\right)$$

$\mathcal{R}_k$ are the Gelfand-Dikii differential polynomials of u

$$\mathcal{R}_0 = 1, \quad \mathcal{R}_1 = u, \quad \mathcal{R}_2 = \frac{u^2}{2} + \frac{D_0^2 u}{12}, \quad \mathcal{R}_3 = \frac{u^3}{6} + \frac{u D_0^2 u}{12} + \frac{(D_0 u)^2}{24} + \frac{D_0^4 u}{240}, \quad \cdots .$$

$$(D_k := g_s \partial_k)$$

(2) F obeys the string equation

$$\partial_0 F = \frac{t_0^2}{2 g_s^2} + \sum_{k=0}^{\infty} t_{k+1} \partial_k F$$

These equations uniquely determine F.

Izykson-Zuber variables and polynomial structure (Itzykson-Zuber '92)

- Izykson-Zuber introduced variables

$$I_n = I_n(u_0, \{t_k\}) = \sum_{\ell=0}^{\infty} t_{n+\ell} \frac{u_0^\ell}{\ell!} \quad (n \geq 0)$$

$$(u_0 := \partial_0^2 F_0)$$

in which genus expansion of F is neatly formulated:

$$F = \sum_{g=0}^{\infty} g_s^{2g-2} F_g \quad \text{with} \quad \begin{aligned} F_0 &= \frac{1}{2}\int_0^{u_0} dv (I_0(v, \{t_k\}) - v)^2 \quad (\Leftrightarrow \ u_0 = I_0) \\ F_1 &= -\frac{1}{24}\log(1 - I_1) \\ F_2 &= \frac{1}{1152}\frac{I_4}{(1-I_1)^3} + \frac{29}{5760}\frac{I_2 I_3}{(1-I_1)^4} + \frac{7}{1440}\frac{I_2^3}{(1-I_1)^5} \end{aligned}$$

(genus zero string equation)

> F_g $(g \geq 2)$ are polynomials in I_n $(n \geq 2)$ and $(1 - I_1)^{-1}$

(Itzykson-Zuber '92) (Eguchi-Yamada-Yang '95) (Zhang-Zhou '19)

- In the JT gravity case, I_n reduce to numerical values

$$I_0 = I_1 = 0, \quad I_n = \frac{(-1)^n}{(n-1)!} \ (n \geq 2)$$

Change of variables

(Zograf '08)

- Using the polynomial structure, we have only to solve the traditional KdV equation to determine F_g.

$$\partial_1 u = u\partial_0 u + \frac{g_s^2}{12}\partial_0^3 u \qquad (u = g_s^2 \partial_0^2 F)$$

- To solve it, it is enough to treat t_0 and t_1 as independent variables and regard the rest as parameters.

- Instead of t_0 and t_1 let us take

$$u_0 := \partial_0^2 F_0 \quad \text{and} \quad t := (\partial_0 u_0)^{-1} = 1 - I_1$$

as independent variables. In terms of these new variables we have

$$\partial_0 = \frac{1}{t}(\partial_{u_0} - I_2 \partial_t), \qquad \partial_1 = u_0 \partial_0 - \partial_t.$$

This change of variables (first introduced by Zograf), combined with the property $\partial_{u_0} I_n = I_{n+1}$ $(n \geq 2)$, enables us to solve the KdV equation recursively and determine F_g very efficiently.

KdV equation for multi-boundary correlators

- Let us introduce the notation

$$\hbar := \frac{g_s}{\sqrt{2}}, \quad x := \frac{t_0}{\hbar} \quad \tau := \frac{t_1}{\hbar}, \quad {}' := \partial_x = \hbar\partial_0, \quad \dot{} := \partial_\tau = \hbar\partial_1$$

$$W_n := Z_n', \qquad W_0 := F', \qquad u = 2W_0' = 2F''$$

- Integrating the KdV equation $\dot{u} = uu' + \frac{1}{6}u'''$ once in t_0 we have

$$\dot{W}_0 = (W_0')^2 + \frac{1}{6}W_0''' \qquad \cdots \ (*)$$

- Applying $B(\beta)$ on both sides of this equation we obtain

$$\dot{W}_1 = uW_1' + \frac{1}{6}W_1'''$$

- Similarly, applying $B(\beta_1)\cdots B(\beta_n)$ on $(*)$ we obtain

$$\dot{W}_n(\beta_1,\ldots,\beta_n) = \sum_{I\subset N} W'_{|I|}W'_{|N-I|} + \frac{1}{6}W_n'''(\beta_1,\ldots,\beta_n)$$

$$N = \{1,2,\ldots,n\}, \quad I = \{i_1,i_2,\ldots,i_{|I|}\}, \quad W'_{|I|} = W'_{|I|}(\beta_{i_1},\ldots,\beta_{i_{|I|}})$$

Genus expansion of multi-boundary correlators

- The multi-boundary correlators at genus zero are known

$$Z_1^{g=0}(\beta) = \frac{1}{g_s}\sqrt{\frac{\beta}{2\pi}} \int_{-\infty}^{u_0} dv \, (I_0(v) - v) \, e^{\beta v}$$

$$Z_n^{g=0}(\{\beta_i\}) = \sqrt{\frac{\prod_{i=1}^n \beta_i}{(2\pi)^n}} \frac{(g_s\partial_0)^{n-2} e^{\sum_{i=1}^n \beta_i u_0}}{\sum_{i=1}^n \beta_i} \quad (n \geq 2)$$

(Ambjørn-Jurkiewicz-Makeenko '90) (Moore-Seiberg-Staudacher '91)

- By solving the KdV equation for W_n with the above initial condition we are able to compute higher genus corrections efficiently up to any order.

(Okuyama-KS '20)

- The results for JT gravity is recovered by simply setting

$$I_0 = I_1 = 0, \quad I_n = \frac{(-1)^n}{(n-1)!} \; (n \geq 2)$$

5. Other expansions, FZZT branes and applications

- So far we have considered the genus expansion: $\beta \sim 1$

$$\text{small } \hbar \text{ expansion}, \quad \beta : \text{finite}$$

- One can calculate some other expansions by solving the KdV equation.
  - ’t Hooft expansion (open string/WKB like) $\beta \sim \hbar^{-1}$ (Okuyama-KS ’19)

$$\text{small } \hbar \text{ expansion}, \quad \beta : \text{large}, \quad \lambda = \hbar\beta : \text{fixed}$$

  - τ-scaling limit (suitable for SFF) $\mathrm{Im}\,\beta \sim \hbar^{-1}$ (Saad-Stanford-Yang-Yao ’22) (Blommaert-Kruthoff-Yao ’22) (Weber-Haneder-Richter-Urbina ’22) (Okuyama-KS ’23) (Anegawa-Iizuka-Okuyama-KS ’23)

$$\text{small } \hbar \text{ expansion}, \quad \beta = \tilde{\beta} + it\,,$$
$$\tilde{\beta} : \text{finite}, \quad t : \text{large}, \quad \tau = t\hbar : \text{fixed}$$

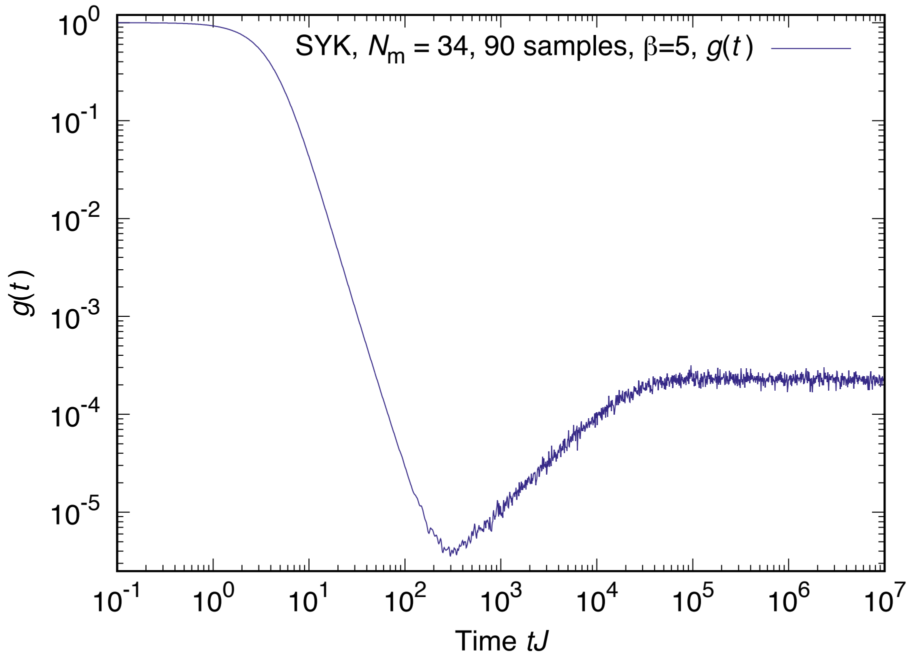
  - low temperature expansion (Airy like) $\beta \sim \hbar^{-2/3}$ (Okuyama-KS ’19)

$$\text{small } T = \beta^{-1} \text{ expansion}, \quad \hbar : \text{small}, \quad h = \hbar\beta^{3/2} : \text{fixed}$$

Spectral form factor (SFF)

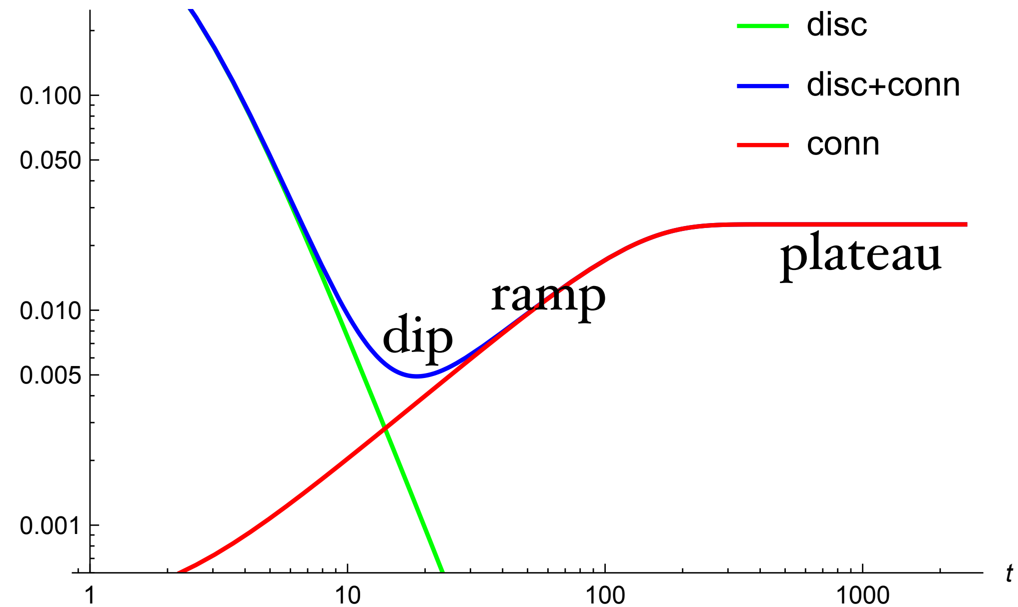
$$g(t) = \frac{\langle Z(\beta,t)Z^*(\beta,t)\rangle_J}{\langle Z(\beta)\rangle_J^2} \qquad Z(\beta,t) = \mathrm{Tr}(e^{-\beta H - iHt})$$
$$Z(\beta) = \mathrm{Tr}(e^{-\beta H})$$

SYK model



(Cotler, Gur-Ari, Hanada, Polchinski, Saad, Shenker, Stanford, Streicher and Tezuka JHEP05(2017)118 [arXiv:1611.04650] Fig.1)

Airy gravity (≈ JT gravity)



(Okuyama-KS '20)
(Anegawa-Iizuka-Okuyama-KS '23)

- It was thought that the plateau behavior is due to a doubly non-perturbative effect. Gravity interpretation was missing.
- It turns out that the plateau can be derived analytically.

(Okuyama-KS '20)
(Saad-Stanford-Yang-Yao '22)
(Blommaert-Kruthoff-Yao '22)
(Weber-Haneder-Richter-Urbina '22)

FZZT branes in JT gravity and topological gravity

- Adding FZZT brane = adding vector degrees of freedom

$$\det(\boldsymbol{\xi} + \boldsymbol{H}) = \int d\boldsymbol{\chi} d\bar{\boldsymbol{\chi}}\, e^{\bar{\boldsymbol{\chi}}(\boldsymbol{\xi}+\boldsymbol{H})\boldsymbol{\chi}}$$

$\chi, \bar{\chi}$: Grassmann-odd vector variables

- Anti FZZT brane

$$\det(\boldsymbol{\xi} + \boldsymbol{H})^{-1} = \int d\boldsymbol{\phi} d\bar{\boldsymbol{\phi}}\, e^{\bar{\boldsymbol{\phi}}(\boldsymbol{\xi}+\boldsymbol{H})\boldsymbol{\phi}}$$

$\phi, \bar{\phi}$: Grassmann-even (bosonic) vector variables

Effect of adding FZZT brane in JT gravity

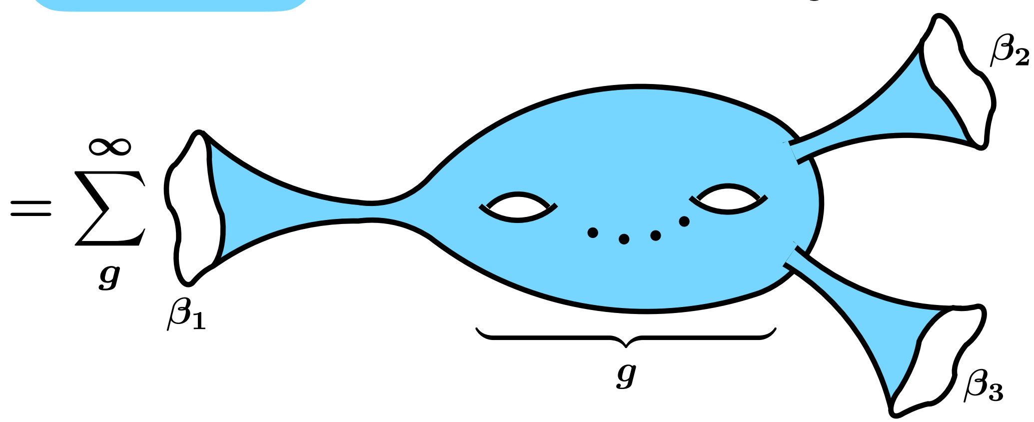
- We show that (when $\mathrm{Re}\,(\xi=\frac{1}{2}z^2)>0$) (Okuyama-KS '21)

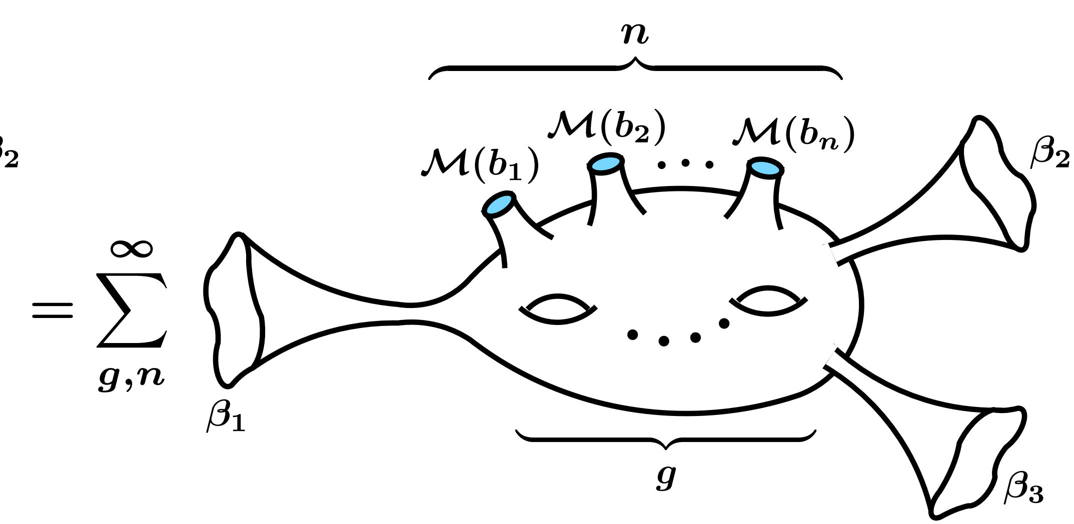
$$\left\langle \det(\xi+H)\prod_{i=1}^{m}Z(\beta_i)\right\rangle_{\mathrm{c}}$$

$$=\sum_{g,n=0}^{\infty}\frac{g_{\mathrm{s}}^{2g-2+n+m}}{n!}\prod_{j=1}^{n}\int_0^\infty db'_j\mathcal{M}(b'_j)\prod_{i=1}^{m}\int_0^\infty b_i db_i Z_{\mathrm{trumpet}}(\beta_i,b_i)V_{g,n+m}(b',b)$$

Insertion of an FZZT brane
$=$ Sum over topologies with extra boundaries with factor $\mathcal{M}(b)=-e^{-zb}$

$\langle\det(\xi+H)Z(\beta_1)Z(\beta_2)Z(\beta_3)\rangle_{\mathrm{c}}$





FZZT brane amplitudes in general topological gravity

- For finite N, the correlators of determinant operators are well known

(Morozov '94) (Brezin-Hikami '00)

$$\left\langle \prod_{i=1}^{k} \det(\xi + H) \right\rangle_{N\times N} = \frac{1}{\Delta(\xi)} \det\Big(P_{N+i-1}(\xi_j)\Big)_{i,j=1,\ldots,k}$$

$$\Delta(\xi) = \prod_{i<j}(\xi_i - \xi_j), \qquad \int d\lambda e^{-V(\lambda)} P_n(\lambda) P_m(\lambda) = h_n \delta_{n,m}$$

- In the double scaling limit (i.e. for general topological gravity) we have

$$\left\langle \prod_{i} \det(\xi_i + H) \right\rangle_{\mathrm{c}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left[-\sum_{k=0}^{\infty}\sum_{i} g_{\mathrm{s}}(2k-1)!!\, z_i^{-2k-1}\partial_k\right]^n F(\{t_k\})$$

$$= F(\{\tilde{t}_k\})$$

$$\tilde{t}_k = t_k - g_{\mathrm{s}}(2k-1)!!\sum_i z_i^{-2k-1} \qquad \left(\xi_i = \frac{1}{2}z_i^2\right)$$

(The shift of this kind has been known since the 1980's and is generated by the infinitesimal Bäcklund transformation for the KdV equation.)

(Date-Jimbo-Kashiwara-Miwa '82)

Macroscopic loop operators, BA function and CD kernel

- Z_n's correspond to macroscopic loop operators

$$Z_1(\beta) = \int_{-\infty}^{x} dx' \langle x'|e^{\beta Q}|x'\rangle = \mathrm{Tr}\left[e^{\beta Q}\Pi\right]$$

$$(x := \hbar^{-1}t_0)$$

$$Q := \partial_x^2 + u, \qquad \Pi = \int_{-\infty}^{x} dx' |x'\rangle\langle x'|$$

(Okuyama-KS '19, '20)

$$Z_2(\beta_1,\beta_2) = \mathrm{Tr}\left[e^{(\beta_1+\beta_2)Q}\Pi - e^{\beta_1 Q}\Pi e^{\beta_2 Q}\Pi\right]$$

$$Z_3(\beta_1,\beta_2,\beta_3) = \mathrm{Tr}\left[e^{(\beta_1+\beta_2+\beta_3)Q}\Pi + e^{\beta_1 Q}\Pi e^{\beta_2 Q}\Pi e^{\beta_3 Q}\Pi + e^{\beta_1 Q}\Pi e^{\beta_3 Q}\Pi e^{\beta_2 Q}\Pi \right.$$
$$\left. - e^{\beta_1 Q}\Pi e^{(\beta_2+\beta_3)Q}\Pi - e^{\beta_2 Q}\Pi e^{(\beta_3+\beta_1)Q}\Pi - e^{\beta_3 Q}\Pi e^{(\beta_1+\beta_2)Q}\Pi\right]$$

general formula is known (Okuyama '18)

- This allows us to express Z_n in terms of Baker-Akhiezer function $\psi(E)$

$$\mathrm{Tr}(e^{\beta_1 Q}\Pi\cdots e^{\beta_n Q}\Pi) = \int_{-\infty}^{\infty} dE_1 \cdots \int_{-\infty}^{\infty} dE_n\, e^{-\sum_{i=1}^{n}\beta_i E_i} K_{12}K_{23}\cdots K_{n1}$$

$$K_{ij} \equiv K(E_i,E_j) = \langle E_i|\Pi|E_j\rangle = \int_{-\infty}^{x} dx' \psi(E_i)\psi(E_j) = \frac{\partial_x\psi(E_1)\psi(E_2) - \partial_x\psi(E_2)\psi(E_1)}{-E_1+E_2}$$

(Christoffel-Darboux kernel)

$$L\psi = -E\psi, \qquad \dot{\psi} = M\psi$$

$$\left(1 = \int_{-\infty}^{\infty} dE_i |E_i\rangle\langle E_i|\right) \qquad L = Q = \partial_x^2 + u, \qquad M = \frac{2}{3}\partial_x^3 + u\partial_x + \frac{1}{2}u'$$

General correlators of FZZT branes and macroscopic loops

- For even number of FZZT branes we find (the odd case is similar)

(Okuyama-KS '21)

$$\left\langle \prod_{i=1}^{n} Z(\beta_i) \prod_{j=1}^{k} \Psi(\xi_j)\Psi(\eta_j) \right\rangle = \det G \frac{\det\left(\tilde{K}(\xi_i,\eta_j)\right)}{\Delta(\xi)\Delta(\eta)}\Bigg|_{\mathcal{O}(w_1\cdots w_n)}$$

macroscopic loop

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta H}$$

FZZT brane

$$\Psi(\xi) = \det(\xi + H)$$

$$\tilde{K}(\xi,\eta) = \langle \eta | \Pi G^{-1} | \xi \rangle$$

$$Q|\xi\rangle = \xi|\xi\rangle$$

$$Q = \partial_x^2 + u \qquad (u = g_s^2 \partial_0^2 F)$$

$$\Pi = \int_{-\infty}^{x} dx' |x'\rangle\langle x'|$$

$$G = 1 + A\Pi$$

$$A = -1 + \prod_{i=1}^{n}\left(1 + w_i e^{\beta_i Q}\right)$$

$$\Delta(\xi) = \prod_{i<j}(\xi_i - \xi_j)$$

- Our expression here does not rely on the genus expansion and thus can be studied non-perturbatively.

6. Conclusions

- JT gravity is a special case of 2d topological gravity.
- Multi-boundary correlators of 2d topological gravity are computed by simply solving the KdV equation.
- The genus expansion of the SFF can be summed up in the 't Hooft and tau-scaling limits. The ramp and plateau behavior can be studied analytically.
- The effect of adding FZZT branes is clarified.

Outlook

- Non-perturbative effects
- "Swampland"
- Multi-matrix models