WKB analysis of the linear problem for modified affine Toda field equations

Mingshuo Zhu

Tokyo Institute of Technology

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in collaboration with Katsushi Ito

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Contents

Introduction

An integrable model (IM) is a Hamiltonian system with

number of degrees of freedom = number of conserved charges

- The integrable model has (infinite) numbers of conserved charges.
- The equations of motion for an integrable field theory can be rewritten into Lax pairs, which lead to two **linear problems**.
- [Drinfeld, Sokolov] It is possible to diagonalize the linear problems with affine Lie algebra structures, where the diagonal elements turn to be classical conserved currents.
- S-matrix in integrable field theory (satisfied by TBA Equations) is exactly solvable. The Y-function in the TBA equation is a generating function for conserved charges (The ODE/IM correspondence).

Introduction

The ODE/IM correspondence [Dorey-Tateo 9812211]

- It is a relation between the spectral analysis of the ordinary differential equation and the “functional relations” in quantum IM.
- The generating function of quantum conserved charges (Y-function) corresponds to the WKB period of the ODE.
- The simplest one is between $[\epsilon^2\partial_z^2 + V(z) - E]\psi(z,\epsilon) = 0$ and the Sine-Gordon model (V(z) is a polynomial in z).
- The correspondence can be assumed to be $I_{\text{classical}}(z) \sim I_{\text{quantum}}(z)$.

Motivation

There must be some relations between the WKB solutions to the ODEs and the classical conserved currents.

Introduction

The classical conserved currents for the $A_1^{(1)}$ Toda field theory

$$I_2(z) = \frac{T(z)}{2},$$
$$I_4(z) = \frac{\partial_z^2 T(z) - T^2(z)}{8},$$
$$I_6(z) = \frac{1}{32}\left(-5T'(z)^2 - 6T(z)u''(z) + T^{(4)}(z) + 2T(z)^3\right),$$

The WKB solutions for the $A_1^{(1)}$-type ordinary differential equation $(\epsilon^2\partial_z^2 + \epsilon^2 u_2(z) - p(z))\psi(z,\epsilon) = 0$ with $\psi(z,\epsilon) = \exp(\frac{1}{\epsilon}\int^z dz\, P(z,\epsilon))$

$$P_0(z) = \sqrt{p(z)},$$
$$P_1(z) = -\frac{1}{2}\partial_z \ln P_0,$$
$$P_2(z) = \frac{P_0''}{16P_0^2} + \frac{u_2(z)}{2P_0} + \partial_z(\frac{3P_0'}{16P_0^2}),$$
$$P_3(z) = -\partial_z(-\frac{u_2(z)}{4P_0^2} + \frac{3P_0'^2}{16P_0^4} - \frac{P_0''}{8P_0^3}),$$

Contents

Affine Toda field equations

The action of $\hat{\mathfrak{g}}$ affine Toda field theory in $2d$ complex plane:

$$S = \int d^2z \Big\{ \frac{1}{2}\partial_z\phi \cdot \bar{\partial}_{\bar{z}}\phi + \Big(\frac{m^2}{\beta}\Big)\Big[\sum_{i=1}^{r} \exp\left(\beta\alpha_i \cdot \phi\right) + \exp\left(\beta\alpha_0 \cdot \phi\right)\Big]\Big\}.$$

Its equation of motion: the $\hat{\mathfrak{g}}$ affine Toda field equation is

$$\bar{\partial}_{\bar{z}}\partial_z\phi(z,\bar{z}) - \Big(\frac{m^2}{\beta}\Big)\Big[\sum_{i=1}^{r} \alpha_i \exp\left(\beta\alpha_i \cdot \phi\right) + \alpha_0 \exp\left(\beta\alpha_0 \cdot \phi\right)\Big] = 0.$$

$$\phi(z,\bar{z}) = \sum_{i=1}^{r} \alpha_i^{\vee}\phi_i(z,\bar{z}),$$

$\alpha_i(\alpha_i^{\vee})$: roots(coroots) of $\hat{\mathfrak{g}}$,

β : a coupling constant,

m : a mass parameter.

Affine Toda field equations

The affine Toda field equations can be separated into Lax pairs:

$$\mathcal{L} = \partial_z + \beta \sum_{i=1}^{r} \partial_z \phi_i(z, \bar{z}) H_i + m\lambda\Lambda,$$
$$\bar{\mathcal{L}} = \partial_{\bar{z}} + e^{-\beta \sum_{i=1}^{r} \phi_i H_i} (m\lambda^{-1}\bar{\Lambda})\, e^{\beta \sum_{i=1}^{r} \phi_i H_i}.$$

λ: a spectral parameter,
E_{α_i}, $E_{-\alpha_i}$, $H_i = \alpha_i^\vee \cdot H$ $(i = 0, \ldots, r)$: the Chevalley generators
$\Lambda = \sum_{i=0}^{r} E_{\alpha_i}$ and $\bar{\Lambda} = \sum_{i=0}^{r} E_{-\alpha_i}$

The flatness condition

$$[\mathcal{L}, \bar{\mathcal{L}}] = 0$$

is the integrability condition of the linear problem

$$\mathcal{L}\Psi = \bar{\mathcal{L}}\Psi = 0$$

Affine Toda field equations

Take the conformal transformation ($\rho^\vee$ is the co-Weyl vector)

$$z \to w(z), \quad \bar{z} \to \bar{w}(\bar{z}), \quad \phi \to \hat{\phi} = \phi - \rho^\vee \log(\partial_z w \partial_{\bar{z}} \bar{w}),$$

then the affine Toda field equations will be modified into

$$\partial_{\bar{z}}\partial_z \phi(z,\bar{z}) - [\sum_{i=1}^{r} \alpha_i \exp\left(\alpha_i \cdot \phi\right) + p(z)\bar{p}(\bar{z})\alpha_0 \exp\left(\alpha_0 \cdot \phi\right)] = 0$$

with $p(z) = (\partial_z w)^h, \quad \bar{p}(\bar{z}) = (\partial_{\bar{z}} \bar{w})^h$. The modified Lax operators are

$$\mathcal{L}_m = \partial_z + \sum_{i=1}^{r} \partial_z \phi_i(z,\bar{z}) H_i + \lambda(\sum_{i=1}^{r} E_{\alpha_i} + p(z) E_{\alpha_0}),$$

$$\bar{\mathcal{L}}_m = \partial_{\bar{z}} + \lambda^{-1} e^{-\phi_i H_i}(\bar{p}(\bar{z}) E_{\alpha_0} + \sum_{i=1}^{r} E_{-\alpha_i}) e^{\phi_i H_i}.$$

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The diagonalization approach

It is possible to diagonalize the linear problem and the diagonal elements are classical conserved currents [Drinfeld, Sokolov (1984)].

Let us focus on the holomorphic part $\mathcal{L}_m$ ($[\mathcal{L}_m, \bar{\mathcal{L}}_m] = 0$).
We replace the spectral parameter λ with Planck constant $\epsilon = \lambda^{-1}$.

$$\epsilon\mathcal{L}_m = \epsilon\partial_z + \epsilon\sum_{i=1}^{r}\partial_z\phi_i(z)H_i + \sum_{i=1}^{r}E_{\alpha_i} + p(z)E_{\alpha_0}.$$

One can view it as a covariant derivative with connection:

$$A(z) = \epsilon\sum_{i=1}^{r}\partial_z\phi_i(z)H_i + \sum_{i=1}^{r}E_{\alpha_i} + p(z)E_{\alpha_0}$$

Then the gauge transformation is given by

$$\mathbf{Gau}_T[A(z)] = T^{-1}(z)A(z)T(z) + \epsilon T^{-1}(z)\partial_z T(z).$$

The diagonalization approach

The transformation matrix T can be decomposed into

$$T(z) = T_d T_{d-1} \dots T_3 T_2 T_1.$$

d is the representation dimension and $T_i(z)$ are $d \times d$ matrices satisfying

$$T_i(z)_{ab} = \begin{cases} 1, & \text{if} \quad a = b, \\ g_{i,b}(z, \epsilon), & \text{if} \quad a = i, \quad b \neq i, \quad 1 \leq b \leq d, \\ 0, & \text{otherwise.} \end{cases}$$

The decomposition means we diagonalize the connection row by row from the bottom to the top. For instance

$$T_d = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & \\ g_{d,1} & g_{d,2} & \cdots & g_{d,d-1} & 1 \end{pmatrix},$$

The diagonalization approach

The connection after the first gauge transformation:

$$A'(z) = \begin{pmatrix} & & \ddots & & \\ & & & & \\ \color{red}{\mathbf{Gau}_{T_d}[A]_{d,1}} & \color{red}{\mathbf{Gau}_{T_d}[A]_{d,2}} & \cdots & \color{red}{\mathbf{Gau}_{T_d}[A]_{d,d-1}} & \mathbf{Gau}_{T_d}[A]_{d,d} \end{pmatrix},$$

For each step of the gauge transformation $\mathbf{Gau}_{T_i}$, we fix $g_{i,b}(z)$ such that the connection $A'(z)$ satisfies

$$A'_{ij} = 0, \quad 1 \leq j \leq d, \quad j \neq i.$$

There are finally $d-1$ constraints to diagonalize the i-th row in $A(z)$ and fix the diagonal elements perturbatively.

The final diagonalized connection $A_{\text{diag}}(z)$ is given by

$$A_{\text{diag}}(z) = \mathbf{Gau}_{T_1} \circ \mathbf{Gau}_{T_2} \ldots \mathbf{Gau}_{T_{d-2}} \circ \mathbf{Gau}_{T_{d-1}} \circ \mathbf{Gau}_{T_d}[A(z)].$$

The diagonalization of $A_1^{(1)}$

The holomorphic part of the modified Lax operator in $A_1^{(1)}$ is of the form

$$\mathcal{L}_m = \epsilon\partial_z + \epsilon\partial_z\phi(z)H + E_\alpha + p(z)E_{-\alpha}$$

with

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We decompose the transformation matrix: $T(z) = T_1 T_2$ by

$$T_2(z) = H + g_{2,1}E_{-\alpha} = \begin{pmatrix} 1 & 0 \\ g_{2,1}(z,\epsilon) & 1 \end{pmatrix}, \quad T_1(z) = \begin{pmatrix} 1 & g_{1,2}(z,\epsilon) \\ 0 & 1 \end{pmatrix}.$$

$T_2(z)$ is determined to diagonalize the second row

$$\mathbf{Gau}_{T_2}[A(z)] = \begin{pmatrix} g_{2,1} + \epsilon\phi' & 1 \\ -2\epsilon g_{2,1}\phi' + \epsilon g_{2,1}' - g_{2,1}^2 + p & -g_{2,1} - \epsilon\phi' \end{pmatrix}.$$

It gives the condition for $g_{2,1}(z)$:

$$g_{2,1}^2(z,\epsilon) + 2\epsilon g_{2,1}(z,\epsilon)\phi'(z,\bar{z}) - \epsilon g_{2,1}'(z,\epsilon) - p(z) = 0.$$

The diagonalization of $A_1^{(1)}$

Diagonal element $f(z,\epsilon) := -g_{2,1}(z,\epsilon) - \epsilon\phi'(z)$ satisfies **Riccati** equation

$$f^2(z,\epsilon) + \epsilon f'(z,\epsilon) - \epsilon^2 u_2(z) - p(z) = 0.$$

$u_2(z) = \phi'(z)^2 - \phi''(z)$ is the classical energy-momentum tensor in sine-Gordon theory, where the subscript denotes the spin.

The second gauge transformation $T_1(z)$ gives

$$\mathbf{Gau}_{T_1} \circ \mathbf{Gau}_{T_2}[A(z)] = \begin{pmatrix} -f(z,\epsilon) & 1 - 2g_{1,2}(z,\epsilon)f(z,\epsilon) \\ 0 & f(z,\epsilon) \end{pmatrix}.$$

We do not need to extract the diagonalization condition from the first row since $g_{1,2}$ is independent of the diagonal elements (traceless condition).

The diagonalization of $A_1^{(1)}$

Let us substitute $f(z,\epsilon) = \sum_{i=0}^{\infty} \epsilon^i f_i(z)$ into the Riccati equation. The first four orders of diagonal elements $\pm f(z,\epsilon)$ are listed below

$$f_0(z) = \sqrt{p(z)}, \qquad -f_0(z) = -\sqrt{p(z)},$$

$$f_1(z) = -\frac{1}{2}\partial_z \ln f_0, \qquad -f_1(z) = f_1(z) + \partial_z \ln f_0,$$

$$f_2(z) = \frac{f_0''}{16 f_0^2} + \frac{u_2(z)}{2 f_0} + \partial_z\left(\frac{3 f_0'}{16 f_0^2}\right), \qquad -f_2(z) = -f_2(z),$$

$$f_3(z) = \partial_z\left(\frac{u_2(z)}{4 f_0^2} - \frac{3 f_0^{'2}}{16 f_0^4} + \frac{f_0^{''}}{8 f_0^3}\right), \qquad -f_3(z) = f_3(z) - \partial_z\left(\frac{u_2(z)}{2 f_0^2} - \frac{3 f_0^{'2}}{8 f_0^4} + \frac{f_0^{''}}{4 f_0^3}\right).$$

Therefore, the diagonal elements can be summarized as

$$\epsilon\partial_z + A_{\text{diag}}(z) = \epsilon\partial_z + \begin{pmatrix} -f(z,-\epsilon) + d(*) & 0 \\ 0 & f(z,\epsilon) \end{pmatrix},$$

where $d(*)$ denotes total derivatives.

The traceless condition implies $f_{2i+1}(z)$ are all total derivatives.

The diagonalization of $A_1^{(1)}$

Let us pay attention back to $f(z,\epsilon)$ and the Riccati equation

$$f^2(z,\epsilon) + \epsilon f'(z,\epsilon) - \epsilon^2 u_2(z) - p(z) = 0.$$

The diagonal element $f(z,\epsilon)$ can also be obtained from the $A_1^{(1)}$ ordinary differential equation

$$[\epsilon^2 \partial_z^2 - \epsilon^2 u_2(z) - p(z)]\psi(z,\epsilon) = 0$$

with the WKB ansatz $\psi_1(z,\epsilon) = \exp(\frac{1}{\epsilon}\int^z dz\, P(z,\epsilon))$

$$P^2(z,\epsilon) + \epsilon P'(z,\epsilon) - \epsilon^2 u_2(z) - p(z) = 0.$$

There exists an equivalence between the diagonal elements and the WKB solutions

$$f(z,\epsilon) = P(z,\epsilon)$$

The diagonalization of $A_2^{(1)}$

The equality $f(z,\epsilon) = P(z,\epsilon)$ can be generalized into $A_2^{(1)}$ types.
There are two scalar fields: $\phi_1(z)$ and $\phi_2(z)$. The modified Lax operator is

$$\mathcal{L}_m = \epsilon\partial_z + \sum_{i=1}^{2} \epsilon\partial_z\phi_i(z,\bar{z})H_i + \sum_{i=1}^{2} E_{\alpha_i} + p(z)E_{\alpha_0}$$

Perform the diagonalization by $T = T_3T_2T_1$.
The gauge transformation by T_3 leads to

$$\mathbf{Gau}_{T_3}[A(z)] = \begin{pmatrix} \epsilon\phi_1' & 1 & 0 \\ g_{3,1} & g_{3,2} + \epsilon(\phi_2' - \phi_1') & 1 \\ \mathbf{Gau}_{T_3}[A(z)]_{3,1} & \mathbf{Gau}_{T_3}[A(z)]_{3,2} & -g_{3,2} - \epsilon\phi_2' \end{pmatrix},$$

Set $f(z,\epsilon) \equiv -g_{3,2} - \epsilon\phi_2'$, $\mathbf{Gau}_{T_3}[A(z)]_{3,1} = \mathbf{Gau}_{T_3}[A(z)]_{3,2} = 0$ gives

$$f^3 + 3\epsilon ff' - \epsilon^2 u_2 f + \epsilon^2 f'' - \epsilon^3 u_3 - p = 0.$$

The diagonalization of $A_2^{(1)}$

After the second gauge transformation T_2,

$$\mathbf{Gau}_{T_2 T_3}[A(z)] = \begin{pmatrix} \epsilon\phi_1' + g_{2,1} & 1 & g_{2,3} \\ \mathbf{Gau}_{T_2 T_3}[A(z)]_{2,1} & g_{3,2} + \epsilon(\phi_2' - \phi_1') - g_{2,1} & \mathbf{Gau}_{T_2 T_3}[A(z)]_{2,3} \\ 0 & 0 & f \end{pmatrix}$$

Set $h(z,\epsilon) \equiv g_{2,1} + \epsilon\phi_1'$, $\mathbf{Gau}_{T_2 T_3}[A(z)]_{2,1} = 0$ leads to the equation

$$h^2 + fh + f^2 - \epsilon h' + \epsilon f' - \epsilon^2 u_2 = 0.$$

$u_2(z)$ and $u_3(z)$ are classical energy-momentum tensor and $\mathcal{W}_3$ field in $A_2^{(1)}$ affine Toda field theory with

$$u_2(z) = \phi_1'(z)^2 - \phi_2'(z)\phi_1'(z) + \phi_2'(z)^2 - \phi_1''(z) - \phi_2''(z),$$
$$u_3(z) = 2\phi_2'(z)\phi_2''(z) - \phi_1'(z)\phi_2''(z) - \phi_1'(z)\phi_2'(z)^2 + \phi_1'(z)^2\phi_2'(z) - \phi_2^{(3)}(z).$$

The diagonalization of $A_2^{(1)}$

The Riccati equation satisfied by $f(z,\epsilon)$ can also be obtained from

$$(-\epsilon)^3(\partial_z-\partial_z\phi_1)(\partial_z-\partial_z\phi_2+\partial_z\phi_1)(\partial_z+\partial_z\phi_2)\psi+p(z)\psi=0.$$

with $\psi(z,\epsilon)=\exp(\frac{1}{\epsilon}\int dz\, f(z,\epsilon))$.

Expand $f=\sum_{n=0}^{\infty}f_n\epsilon^n$ and $h=\sum_{n=0}^{\infty}h_n\epsilon^n$. The first four terms are

$$f_0(z)=p^{\frac{1}{3}}, \qquad h_0(z)=e^{-\frac{2\pi i}{3}}f_0,$$

$$f_1(z)=-\frac{f_0'}{f_0}, \qquad h_1(z)=f_1(z)+2\partial_z(\ln f_0),$$

$$f_2(z)=\frac{f_0''}{6f_0^2}+\frac{u_2(z)}{3f_0}+\partial_z(\frac{f_0'}{2f_0^2}), \qquad h_2(z)=e^{\frac{2\pi i}{3}}f_2(z),$$

$$f_3(z)=-\frac{u_3(z)}{3f_0^2}+\frac{f_0'u_2(z)}{3f_0^3}-\partial_z(-\frac{f_0'^2}{2f_0^4}+\frac{f_0''}{3f_0^3}-\frac{u_2(z)}{3S_0^2}), \qquad h_3(z)=e^{\frac{4\pi i}{3}}(f_3(z)-\partial_z(\frac{u_2}{3f_0^2})).$$

The diagonalization of $A_2^{(1)}$

The diagonal connection is summarized as

$$A_{\mathrm{diag}}(z) = \begin{pmatrix} e^{-\frac{i2\pi}{3}} f(z, e^{\frac{i2\pi}{3}}\epsilon) + d(*) & 0 & 0 \\ 0 & e^{-\frac{i4\pi}{3}} f(z, e^{\frac{i4\pi}{3}}\epsilon) + d(*) & 0 \\ 0 & 0 & f(z,\epsilon) \end{pmatrix}.$$

- The traceless condition implies $f_{1+3i}(z)$ are total derivatives.
- The diagonalization helps us solve some complicated pseudo-differential equations ($D_r^{(1)}$ and $D_{r+1}^{(2)}$).

Generalized to other affine Lie algebras

The ODEs satisfied by $\psi(z,\epsilon)=\exp(\frac{1}{\epsilon}\int dz\, f(z,\epsilon))$

$$A_r^{(1)}:\ (-\epsilon)^h(\partial_z-\partial_z\phi_1)(\partial_z-\partial_z\phi_2+\partial_z\phi_1)\cdots(\partial_z+\partial_z\phi_r)\psi(z,\epsilon)=p(z)\psi(z,\epsilon)$$

$$A_{2r-1}^{(2)}:\ \epsilon^{(2r-1)}(\partial_z-\partial_z\phi_1)\cdots(\partial_z-\partial_z\phi_r+\partial_z\phi_{r-1})(\partial_z+\partial_z\phi_r-\partial_z\phi_{r-1})\cdots(\partial_z+\partial_z\phi_1)\psi-2\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi=0$$

$$B_r^{(1)}:\ \epsilon^{2r}(\partial_z-\partial_z\phi_1)\cdots(\partial_z-2\partial_z\phi_r+\partial_z\phi_{r-1})\partial_z(\partial_z+2\partial_z\phi_r-\partial_z\phi_{r-1})\cdots(\partial_z+\partial_z\phi_1)\psi-4\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi=0$$

$$D_{r+1}^{(2)}:\ \epsilon^{(2r+2)}(\partial_z-\partial_z\phi_1)\cdots(\partial_z-2\partial_z\phi_r+\partial_z\phi_{r-1})\partial_z(\partial_z+2\partial_z\phi_r-\partial_z\phi_{r-1})\cdots(\partial_z+\partial_z\phi_1)\psi-4p(z)\partial_z^{-1}p(z)\psi=0$$

$$D_r^{(1)}:\ \epsilon^{(2r-2)}(\partial_z-\partial_z\phi_1)\cdots(\partial_z-\partial_z\phi_r-\partial_z\phi_{r-1}+\partial_z\phi_{r-2})\partial_z^{-1}(\partial_z+\partial_z\phi_r+\partial_z\phi_{r-1}-\partial_z\phi_{r-2})\cdots(\partial_z+\partial_z\phi_1)\psi-4\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi=0$$

These (pseudo)-ODEs have been found in the ODE/IM correspondence [Dorey, Dunning, Masoero, Suzuki, Tateo (2007); Ito, Locke (2015)].

Contents

Conserved density vs. WKB solution

The classical conserved densities for the sine-Gordon equations

$$
\begin{aligned}
I_2(z) &= \frac{T(z)}{2}, \\
I_4(z) &= \frac{\partial_z^2 T(z) - T^2(z)}{8}, \\
I_6(z) &= \frac{1}{32}\left(-5T'(z)^2 - 6T(z)u''(z) + T^{(4)}(z) + 2T(z)^3\right),
\end{aligned}
$$

The WKB solutions for the $A_1^{(1)}$-type ordinary differential equation

$$
\begin{aligned}
f_0(z) &= \sqrt{p(z)}, \\
f_1(z) &= -\frac{1}{2}\partial_z \ln f_0, \\
f_2(z) &= \frac{f_0''}{16f_0^2} + \frac{u_2(z)}{2f_0} + \partial_z(\frac{3f_0'}{16f_0^2}), \\
f_3(z) &= -\partial_z(-\frac{u_2(z)}{4f_0^2} + \frac{3{f_0'}^2}{16f_0^4} - \frac{f_0''}{8f_0^3}),
\end{aligned}
$$

Conserved density vs. WKB solution

Recall the appearance of $p(z)$: the conformal transformation $z \to w(z)$

$$dw = \sqrt{p(z)}dz, \quad \hat{u}_2(w(z)) = \frac{1}{p(z)}\left[u_2(z) + \frac{4pp'' - 5p'^2}{16p^2}\right]$$

After the conformal transformation,

$$\begin{aligned}
\hat{f}_0(w) &= 1, \\
\hat{f}_2(w) &= \frac{\hat{u}_2(w)}{2}, \\
\hat{f}_4(w) &= \frac{\partial_w^2 \hat{u}_2(w) - \hat{u}_2^2(w)}{8},
\end{aligned}$$

They are nothing but the commonly conserved currents. In conclusions, the quantum period Π_i and conserved charges $\mathcal{Q}_i$ are related as follows:

$$\Pi_i \equiv \oint dz\ f_i(z) = \oint dz\ \sqrt{p(z)}\hat{f}_i(z) = \oint dw\ \hat{f}_i(w) \equiv \mathcal{Q}_i.$$

Contents

Summary and future directions

Summary

- A WKB method is found to diagonalize the linear problem. The diagonal elements are the WKB solutions to the (pseudo) ordinary differential equations appearing in the ODE/IM correspondence.
- There is a relation via the conformal transformation between the conserved currents and the WKB solutions.

Futhre directions

- It is possible to take exact WKB analysis on the $f(z, \epsilon)$.
- Combine the diagonalization approach with $T\bar{T}$-deformation.

Thank you for watching.