Topological Vertex Formalism for BCD-type Gauge Theories and qq-characters

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Introduction/Motivation

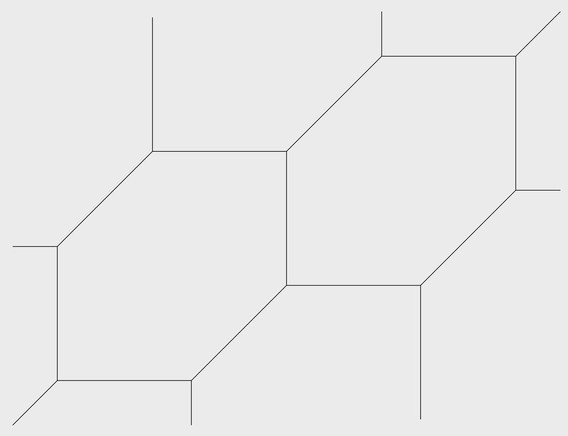
• The *Quantum Field Theory* is so far the best framework to describe our universe. However, due to the technical difficulties, what we usually do is to compute physical quantities perturbatively, which requires the existence of small expansion parameter, and this method does not apply to the strong coupling region of theories such as QCD at low energy.

• *String Theory* provides us powerful tools to study *Quantum Field Theories*, especially those with supersymmetries.

• One of the most interesting and abundant objects in string theory is the D-brane.

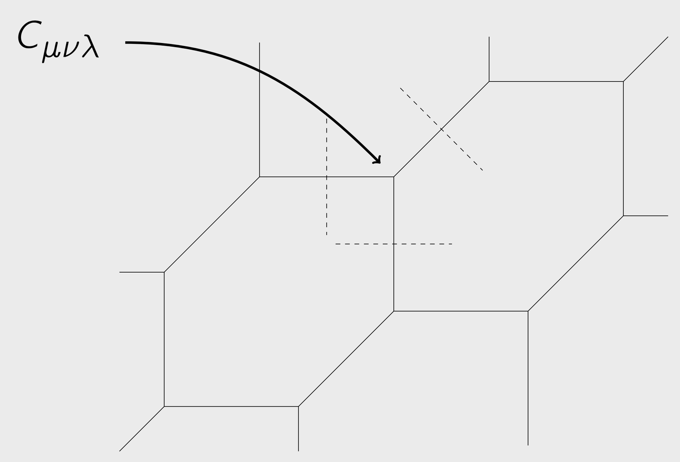
• In particular, engineering the web of D-branes gives the string-theory construction for a large class of gauge theories, with most of the familiar examples and even some theories without known Lagrangian description included.

• For example, 5d $\mathcal{N} = 1$ gauge theories are characterized by a diagram describing the location of different types of 5-branes as follows.



• The simplest observable that can be computed for these 5d gauge theories is the (Nekrasov) partition function on $\mathbb{C}^2_{\epsilon_1,\epsilon_2} \times S^1$.

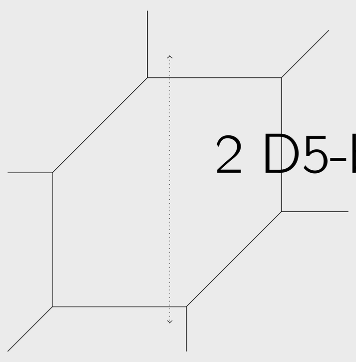
• This partition function can be computed by using localization technique in the pure QFT approach, but it can also be calculated directly in the string-theory setup via the topological vertex formalism.



$$Z = \sum_{\mu,\nu,\lambda,\cdots:Young\ diagrams} Q_1^{|\mu|} Q_2^{|\nu|} Q_3^{|\lambda|} \ldots C_{\mu\nu\lambda} \ldots \tag{1}$$

The topological vertex is known to be written in terms of symmetric functions, and Q_i's are Kähler parameters.

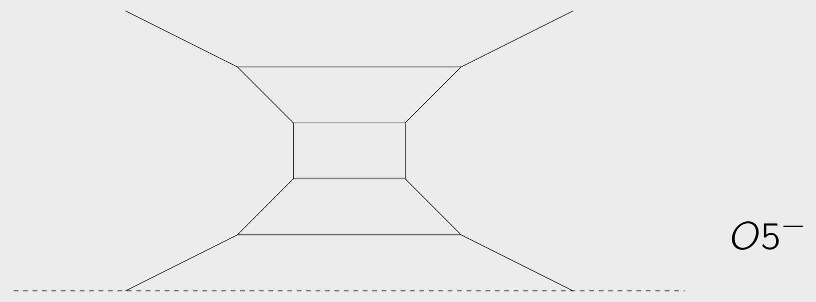
• The gauge group is one of the most important (characterizing) data of gauge theories. The number of D5 branes (horizontal lines) N gives rise to the SU(N) gauge group in the corresponding gauge theory. This class of theories are called A-type gauge theories.



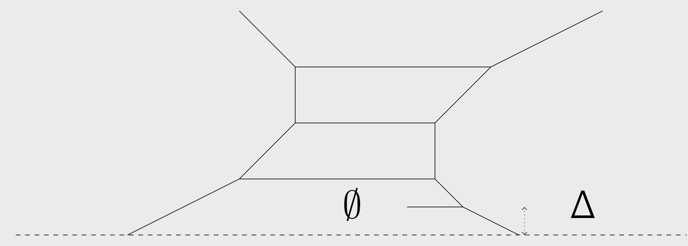
• Motivation: We want to generalize this convenient tool, the topological vertex formalism, to brane web with orientifolds.

Orientifold and BCD-type Gauge Groups

Gauge theories with BD-type (SO-type) gauge groups can be constructed with the so-called $O5^-$ orientifold plane. An example in the D-series of pure SO(8) gauge theory is given by



and theories in the B-series such as SO(7) gauge theory requires to put a (fractional) D5-brane on the top of $O5^-$ plane:

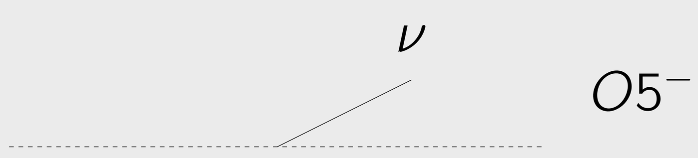


where the distance Δ should be taken to be zero at the end.

The construction for the C-series involves $O5^+$ orientifold, and it is also essentially a "combination" of $O5^-$ and a D5 brane.

Topological Vertex for Orentifold and Results

It now seems that the only unknown building block we need to complete the $ABCD$-series is the vertex for



We found its expression through string dualities to take the form

$$V_\nu = \frac{P_\nu(q)}{(q;q)_{|\nu|/2}}, \quad V_\nu(-Q)^{|\nu|} = \langle 0| \mathbb{O}(Q,q) |\nu\rangle, \tag{2}$$

where $P_\nu(q)$ is a polynomial of q of degree at most $m(|\nu|) = \frac{n(n+1)}{2}$ for $n = \frac{|\nu|}{2}$ and can only be non-zero when $|\nu|$ is even. We proposed a vertex operator $\mathbb{O}(Q,q)$ to determine this polynomial uniquely [1],

$$\mathbb{O}(Q,q) = \exp\Big(\sum_{n=1}^{\infty} -\frac{Q^{2n}(1+q^n)}{2n(1-q^n)} J_{2n} + \frac{Q^{2n}}{2n} J_n J_n\Big). \tag{3}$$

Our results:

1. We checked the algebraic relations for both the partition function and the qq-character: $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, $\mathfrak{so}(6) \simeq \mathfrak{su}(4)$, $\mathfrak{su}(2) \simeq \mathfrak{sp}(1)$, $\mathfrak{so}(5) \simeq \mathfrak{sp}(2)$. In particular, highly non-trivial examinations have been down for Sp(1) and Sp(2) cases respectively up to 27 (28 in the 4d limit) and 16 instantons. [2] [3]

3. Perturbative part of partition function of G_2 and SU(3)$_{\kappa_{cs}=9}$.

4. One- and two-instanton partition functions for G_2 checked against the literature (analytic closed-form can be obtained). The equivalence between two expectedly equivalent brane webs for G_2 is confirmed.

References

[1] H. Hayashi and Rui-Dong Zhu, arXiv:2012.14197
[2] S. Nawata and Rui-Dong Zhu, arXiv:2107.03656
[3] S. Nawata, K. Zhang and Rui-Dong Zhu, arXiv:2302.00525

Quasinormal Modes of C-metric from the Quantized Seiberg-Witten Theory

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Introduction/Motivation

• An interesting connection was discovered by [Aminov, Grassi, Hatsuda (2020)] between the perturbation theory (especially the quasinormal modes, a.k.a. QNMs) around black holes and Nekrasov's partition function in the Nekrasov-Shatashvili (NS) limit. The QNMs are important physical quantity to study, which can be detected during the ring-down phase of the gravitational wave observation. Traditionally one often uses numerical methods to compute the QNMs, e.g. discretization, WKB method etc., but the NS partition function approach can give non-perturbative and semi-classical analysis to QNMs.

In [Aminov, Grassi, Hatsuda (2020)], the B-cycle quantization condition in the quantized Seiberg-Witten (SW) theory is proposed to compute the QNMs,

$$\Pi_B\left(E(\omega), \boldsymbol{m}(\omega), \mathfrak{q}(\omega), 1\right) = \pi\left(2n+1\right), \quad n \in \mathbb{N}, \tag{1}$$

where the quantum B-period Π_B can be determined purely from the gauge theory inputs (in the NS limit),

$$\Pi_B = \partial_a\left(\mathcal{F}_{NS}^{pert} + \mathcal{F}_{NS}^{inst}\right). \tag{2}$$

This can be understood from the resonant state nature of the QNMs.

• Dictionary: the Klein-Gordon equation (for the scalar perturbations) can be converted to the Schrödinger form,

$$\left(\frac{\partial^2}{\partial r^2} + Q(r)\right)\psi(r) = 0, \tag{3}$$

with singularities contained in the potential $Q(r)$. Reading off the singularity structure, it can be mapped to the Class-S construction of 4d $\mathcal{N}=2$ theories.

• C-metric: realization of accelerating black holes in relativity,

$$ds^2 = \frac{1}{(1-\alpha r\cos\theta)^2}\left(-f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2 d\theta^2}{P(\theta)} + P(\theta)r^2\sin^2\theta d\varphi^2\right), \tag{4}$$

where

$$f(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)(1-\alpha^2 r^2), \tag{5}$$

$$P(\theta) = 1 - 2\alpha M\cos\theta + \alpha^2 Q^2\cos^2\theta. \tag{6}$$

Bringing the Klein-Gordon equation to the Schrödinger form, one finds four singularities at $r = r_\pm$ and $r = \pm\frac{1}{\alpha}$, where $r_\pm = M \pm \sqrt{M^2 - Q^2}$. To map it to the quantized SW curve, one can choose dictionary 1,

$$z = -\frac{2\alpha(r-r_-)}{(\alpha r_- - 1)(\alpha r + 1)}, \quad r = \frac{1}{\alpha} \leftrightarrow z = 1, \quad r = r_+ \leftrightarrow z = t, \tag{7}$$

or dictionary 2,

$$z = \frac{1 + r_-\alpha}{1 + r_+\alpha}\frac{r - r_+}{r - r_-}, \quad r = r_+ \leftrightarrow z = 0, \quad r = \frac{1}{\alpha} \leftrightarrow z = t. \tag{8}$$

• QNMs in C-metric: three families, 1) Photon-sphere modes: (ℓ, m_0) similar to angular momentum, ω_{PS}, which can be computed from the photon sphere of the black hole, 2) accelerating modes,

$$\omega_\alpha \sim -\alpha\left(m_0(P(\pi)-1) + \ell + n + 1\right), \quad n = 0, 1, 2, \ldots \tag{9}$$

and 3) near extremal modes,

$$\omega_{NE} \sim -\frac{|f'(r_+)|}{2}\left(m_0(P(\pi)-1) + \ell + n + 1\right), \quad n = 0, 1, 2, \ldots \tag{10}$$

• **An apparent puzzle**: How one quantum number can realize three families of QNMs?

Connection formula approach

The boundary condition for QNMs:

$$\phi(r) \sim \begin{cases} e^{-i\omega r_*} & r_* \to -\infty, \quad r \to r_+ \\ e^{i\omega r_*} & r_* \to +\infty, \quad r \to \frac{1}{\alpha} \end{cases}, \tag{11}$$

where $dr_* = \frac{dr}{f(r)}$. The solutions to the QNM equation are basically given by series expansions, so solutions at two boundaries can be connected to each other as [Bonelli et al. (2022)]

$$\psi^{(t)\pm}(z) = \sum_{\theta'=\pm} A_{\pm\theta'}\psi^{(1)\theta'}(z), \tag{12}$$

and the **boundary condition** is translated to $A_{-+} = 0$, where

$$A_{-\theta'} = \left(\sum_{\sigma=\pm}\mathcal{M}_{-\sigma}(a_t, a; a_0)\mathcal{M}_{(-\sigma)\theta'}(a, a_1; a_\infty)t^{\sigma a}e^{-\frac{\sigma}{2}\partial_a F}\right) t^{\frac{1}{2}-a_0-a_t}(1-t)^{a_t-a_1}e^{-\frac{1}{2}(\partial_t + \theta'\partial_{a_1})F}, \tag{13}$$

with

$$\mathcal{M}_{\theta\theta'}(\alpha_0, \alpha_1; \alpha_2) = \frac{\Gamma(-2\theta'\alpha_1)}{\Gamma(\frac{1}{2} + \theta\alpha_0 - \theta'\alpha_1 + \alpha_2)}\frac{\Gamma(1 + 2\theta\alpha_0)}{\Gamma(\frac{1}{2} + \theta\alpha_0 - \theta'\alpha_1 - \alpha_2)}. \tag{14}$$

If all terms are non-vanishing, we can rewrite the boundary condition into

$$\frac{\mathcal{M}_{--}(a_t, a; a_0)\mathcal{M}_{++}(a, a_1; a_\infty)}{\mathcal{M}_{-+}(a_t, a; a_0)\mathcal{M}_{-+}(a, a_1; a_\infty)}t^{-2a}e^{\partial_a F} = -1. \tag{15}$$

Taking log of the above equation gives nothing but the B-cycle quantization condition.

Numerical results [LSZZ(2023)]

Using the equation obtained from the connection formula, we can calculate the QNMs and compare it with the numerical results obtained with *QNMspectral* package. Note that our calculation is based on instanton expansion (i.e. t-expansion). Dictionary 1:

• $\alpha M = 0.05$, $Q/M = 0.3$, $t = 0.174209$, our results (3-inst): $\omega_{PS_1} = 0.111416 - 0.10266i$, $\omega_{\alpha_1} = -0.0505002i$, $\omega_{\alpha_2} = -0.10341i$ vs num. results: $\omega_{PS_1} = 0.111 - 0.104i$, $\omega_{\alpha_1} = -0.0506i$, $\omega_{\alpha_2} = -0.103i$.

• $\alpha M = 0.3$, $Q/M = 0.999$, $t = 0.0572592$, our results (3-inst): $\omega_{PS_1} = 0.111516 - 0.0839528i$, $\omega_{\alpha_1} = -0.0412224i$, $\omega_{\alpha_2} = -0.0836822i$ vs num. results: $\omega_{PS_1} = 0.111 - 0.104i$, $\omega_{\alpha_1} = -0.0412i$, $\omega_{\alpha_2} = -0.084i$.

When t is large, one need to go to very large number of instantons (or even not convergent), e.g. $\alpha M = 0.5$, $Q/M = 0.8$, $t = 0.83333$ with our result (5-inst) $\omega_{PS_1} = 0.0347785 - 0.0364286i$ vs num. result $\omega_{PS_1} = 0.03945 - 0.04122i$. However if we use Dictionary 2, t will be small enough: $t = 0.166667$. Using the connection formula in this dictionary, we obtain (3-inst): $\omega_{PS_1} = 0.0394654 - 0.0412316i$, $\omega_{PS_2} = 0.0387606 - 0.123697i$ vs num. results: $\omega_{PS_1} = 0.03945 - 0.04122i$, $\omega_{PS_2} = 0.03876 - 0.1237i$.

References

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[LSZZ(2023)]: Yang Lei, Hongfei Shu, Kilar Zhang, RZ, to appear.