Surface Defect in $\mathcal{N}=4$ Super Yang-Mills Theory and Integrability

Hiroki Kawai
Based on the work with Adolfo Holguin from Trinity College Dublin
University of California, Santa Barbara
hkawai@ucsb.edu

Summary

In the $\mathcal{N}=4$ **super Yang-Mills theory**, it is well-known that the one-loop anomalous dimension operator for the single trace operators is equivalent to an **integrable spin chain**. Recent works have extended the application of this integrability to scenarios **involving a boundary or a defect**. The correlators of the single trace operators can be described as an overlap between the Bethe states and the corresponding defect state. The defect state is called an **integrable state** if it corresponds to the BCs not spoiling the bulk integrability. In that case, the overlap admits nice selection rules and can be exactly calculated for some cases. We show that **the state corresponding to the Gukov-Witten surface defect is integrable**. We also calculate the tree-level one-point function of the single trace operators in the $SU(2)$ sector under this defect background.

$\mathcal{N}=4$ Super Yang-Mills and Integrable Spin chain

$\mathcal{N}=4$ SYM is the 4D superconformal gauge theory with the global symmetry of $PSU(2,2|4)$ **superconformal group** (bosonic part: $\supset SO(1,5)\times SU(4)_R$).
Constructed with a $\mathcal{N}=4$ multiplet in the adjoint of $G=SU(N)$ consisting of

- Real scalars ϕ^I ($I=1,...,6$: R-symmetry $SO(6)_R (\simeq SU(4)_R)$ vector **6** irrep.)
- LH Weyl fermions λ^a ($a=1,...,4$: R-symmetry $SU(4)_R$ fund. **4** irrep.)
- Gauge fields A_μ

Single trace operators

The simplest local gauge invariant ($SU(N)$ singlet) objects:

$$\mathcal{O}(x)=\Psi_{I_1 I_2\cdots I_L}\operatorname{tr}\left[\phi^{I_1}(x)\phi^{I_2}(x)\cdots\phi^{I_L}(x)\right] \tag{1}$$

6 charges for the bosonic part : $(\underbrace{\Delta}_{\text{conf. dim.}};\ \underbrace{S_1,S_2}_{SO(1,3)\text{ spins}};\ \underbrace{J_1,J_2,J_3}_{\text{R-sym.}})$

Classically, $\Delta_0=L$ (length of the chain). Each complex field $Z\equiv\phi^1+i\phi^4$, $X\equiv\phi^2+i\phi^5$, $W\equiv\phi^3+i\phi^6$ has a unit J_i charge for each $i=1,2,3$.
$\mathcal{O}(x)$ with one of $J_i=\Delta_0$ (up to similarity transf.) $\implies$ 1/2-BPS are called the **chiral primary operators (CPOs)**. All CPOs have **traceless and symmetric** $\Psi_{I_1...I_L}$.
Dimension Δ is protected from the quantum corrections.
The non-CPOs' dimensions are corrected by the **anomalous dimension** $\Delta=\Delta_0+\gamma(\lambda)$.
Operator mixing $\implies$ need to diagonalize the **"anomalous dimension operator"** Γ with eigenvalues γ.

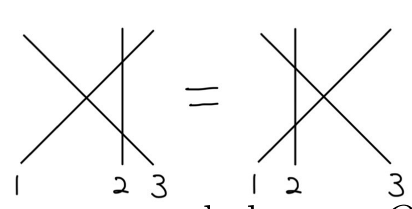
1-loop $\Gamma\sim$ integrable spin chains in large-N [Minahan-Zarembo (2002) [3]]

- The whole scalar ($SO(6)$) sector: $\Gamma_{SO(6)}\propto\sum_{\ell=1}^{L}(\underbrace{1}_{\text{s.t. zero eval. for CPOs}}-\underbrace{P_{\ell,\ell+1}}_{\text{perm. op.}}+\frac{1}{2}\underbrace{K_{\ell,\ell+1}}_{\text{trace op.}})$
- Smaller $SU(2)$ sector (only Z and X) ($SU(2)$ sector): **Heisenberg chain**.

Quantum Integrability and Integrable States

Both Γ as a spin-chain Hamiltonian H can be constructed from the R-matrix ($\sim$ S-matrix) $V\otimes V\to V\otimes V$ satisfying the **Yang-Baxter equation**

$$R_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu)=R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda-\mu)$$



The integrability can be said by the fact that there exist many conserved charges Q_n.

$$t(\mu)\equiv\operatorname{Tr}_a(T_a(\mu))=\operatorname{Tr}_a(R_{aL}(\mu)\cdots R_{a1}(\mu))=\sum_n Q_{n+1}\mu^n \qquad (Q_2=H) \tag{2}$$

$$\text{YB eqn.}\implies[t(\lambda),t(\mu)]=0\implies[Q_n,Q_m]=0\ \forall m,n \tag{3}$$

BCs keeping the integrability satisfy the **boundary Yang-Baxter equation**.
Rotate (open $\to$ closed) then this is an initial condition of the quantum state.

Integrable boundary state [Piroli-Pozsgay-Vernier (2017) [4]]

$$t(u)\left|\psi\right\rangle=\Pi t(u)\Pi\left|\psi\right\rangle\implies Q_{2n+1}\left|\psi\right\rangle=0\quad(n=1,2,...) \tag{4}$$

This gives nice selection rules: the Bethe states with nonzero overlap with $|\psi\rangle$ must have **zero total rapidity**.

References

[1] I would like to thank Shota Komatsu for providing a part of the Mathematica code.
[2] S. Gukov and E. Witten. Gauge Theory, Ramification, And The Geometric Langlands Program. 12 2006.
[3] J. A. Minahan and K. Zarembo. The Bethe ansatz for N=4 superYang-Mills. *JHEP*, 03:013, 2003.
[4] L. Piroli, B. Pozsgay, and E. Vernier. What is an integrable quench? *Nucl. Phys. B*, 925:362–402, 2017.

Integrable states and integrable defects

Expectation: **Integrable defect in $\mathcal{N}=4$ SYM $\overset{?}{\longleftrightarrow}$ Integrable boundary state of** Γ

To see this, let us think of a 1pt function at the tree level:

$$\begin{aligned}\langle\mathcal{O}(x)\rangle_{\mathcal{D}}&=\Psi_{I_1...I_L}\operatorname{tr}\left[\langle\phi^{I_1}\rangle_{\mathcal{D}}\cdots\langle\phi^{I_L}\rangle_{\mathcal{D}}\right]\\&=\langle I_1...I_L|\Psi_{I_1...I_L}\operatorname{tr}\left[\langle\phi^{J_1}\rangle_{\mathcal{D}}\cdots\langle\phi^{J_L}\rangle_{\mathcal{D}}\right]|J_1...J_L\rangle=\langle\Psi|\mathcal{D}\rangle\end{aligned} \tag{5}$$

The state $|\mathcal{D}\rangle=\operatorname{tr}\left[\langle\phi^{J_1}\rangle_{\mathcal{D}}\cdots\langle\phi^{J_L}\rangle_{\mathcal{D}}\right]|J_1...J_L\rangle$ describes the defect.

$|\mathcal{D}\rangle$ is characterized by the nonzero classical configurations of the fields $\langle\phi^I\rangle_{\mathcal{D}}$.

Gukov-Witten defect [Gukov-Witten 2006 [2]]

Gukov-Witten defect is the codim-2 defect keeping maximal (1/2) SUSY supported on Σ.
$\Sigma=\mathbb{R}^2$ or S^2 for maximal symmetry.
Integrability ✓on the string side by [Dekel-Oz (2011)] as the probe D3-brane config.

$$\textbf{Bosonic symmetry breaking: } SO(1,5)\times SU(4)\to\underbrace{SO(1,3)}_{s.t.}\times\underbrace{SO(2)}_{s.t.+R}\times\underbrace{SO(4)}_{R} \tag{6}$$

$$\textbf{Gauge symmetry breaking: } SU(N)\to\mathbb{L}=S\left[\prod_{l=1}^{M}U(N_l)\right]\quad\left(\sum_{l=1}^{M}N_l=N\right) \tag{7}$$

Classical configs. satisfy the 4D SD equation $\longrightarrow$ 2D Hitchin's equations

$$\text{Solutions: } A=\alpha d\psi,\quad \Phi=\phi^1+i\phi^2=\frac{1}{z}(\beta+i\gamma) \tag{8}$$

where α,β,γ are **diagonal**, and there is a 2D theta-parameter η, which is also diagonal:

$$(\alpha,\beta,\gamma,\eta)\in(\mathbb{T},\mathfrak{t},\mathfrak{t},\mathbb{T}^\vee)/W_{\mathbb{L}} \tag{9}$$

$z=re^{i\psi}$ is the complex coordinate of the transverse direction to Σ.

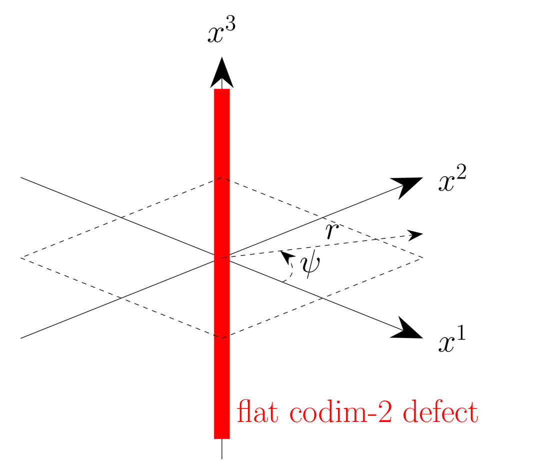


Fig. 1: Slice of the codim-2 defect (red) with $\Sigma=\mathbb{R}^2$ extended to the $x^{3,4}$ directions.

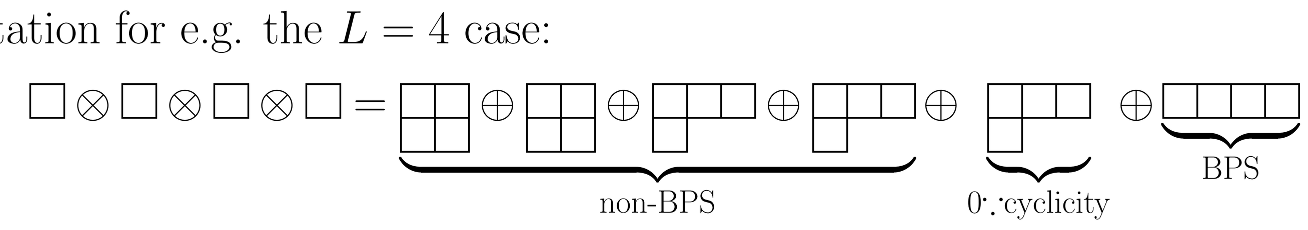
Defect State Integrability and Overlaps

$\langle\phi^{I=1,2}\rangle_{\mathcal{D}}$ are diagonal matrices $\supset\beta,\gamma$.

- $SU(2)$ sector: We can prove that the defect state $|\mathcal{D}\rangle$ satisfies the integrability $t(u)\left|\mathcal{D}\right\rangle=\Pi t(u)\Pi\left|\mathcal{D}\right\rangle$
- $SO(6)$ sector: We do not know how to show the integrability exactly, at least checked that it is annihilated by the first odd charge $Q_3\left|\mathcal{D}\right\rangle=0$ (enough for MPS?)

No contradiction against the expectation of its integrability!

Let us consider the tree-level 1pt func. $\langle\mathcal{O}(x)\rangle_\Sigma=\langle\Psi|\text{MPS}\rangle$ for the $SU(2)$ sector.
Representation for e.g. the $L=4$ case:



- **CPOs:** corresponding to the states in the symmetric traceless rep. They can have a **nonzero 1pt function**. E.g.

$$\left\langle\operatorname{tr}\left[Z^L\right]\right\rangle_\Sigma=\frac{1}{r^L}(\beta\cos\psi+\gamma\sin\psi)^L \tag{10}$$

- **Non-CPOs:** Non-CPOs constructed via the Bethe ansatz have antisymmetric indices, and the matrices are all diagonal (i.e. commutative), so the overlaps with $|\text{MPS}\rangle$ are **all zero**.

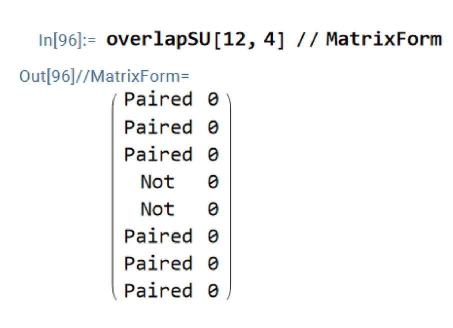


Fig. 2: E.g. $L=12$ with $n_{\text{Bethe roots}}=4$, all the Bethe states have zero overlaps with $|\text{MPS}\rangle$ [1].