核多体系物理学

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- 1. 有限温度・密度における場の理論入門
- (a) 経路積分・松原和・自由エネルギー
- (b)QCD におけるカイラル相転移、南部 - ヨナラシニョ模型、強結合格子 QCD
- 2. 状態方程式を記述する理論模型
- (a) 核力と位相差、有効相互作用
- (b) 核物質の状態方程式、平均場理論
- 3. 原子核反応理論
- (a) 核子 - 核子散乱、ハドロン - 原子核反応
- (b) 流体力学、輸送理論
- (c) ハイパー核・中間子核生成反応の概観と直接反応

シラバス

- [授業の概要・目的]
  - 核子・ハドロン・クォークからなる多体系の性質を量子色力学(QCD)、状態方程式、および核反応論の観点から議論する。強い相互作用の基本理論であるQCDの基本的性質、核物質の状態方程式を記述するために必要となる核多体理論(平均場理論、有効相互作用、有限温度・密度での場の理論、強結合格子QCD)、ハイパー核生成反応や重イオン反応を理解する上で必要とされる原子核反応理論(直接反応、輸送模型等)、等の理論の枠組について解説すると共に、これらについての最近の研究成果についても紹介する。
- [到達目標] 次の事項を習得する。
  - 有限温度・密度の場の理論、散乱理論、有効相互作用理論の基本を理解する。
  - 核力から有効相互作用、あるいは場の理論から状態方程式につながる理論体系を理解する。
  - 簡単な平均場理論・直接反応理論の範囲内で状態方程式・反応スペクトルを計算するための手法

Contents

- 核子･ハドロン･クォーク物質の相互作用と状態方程式について以下の内容で講義する。
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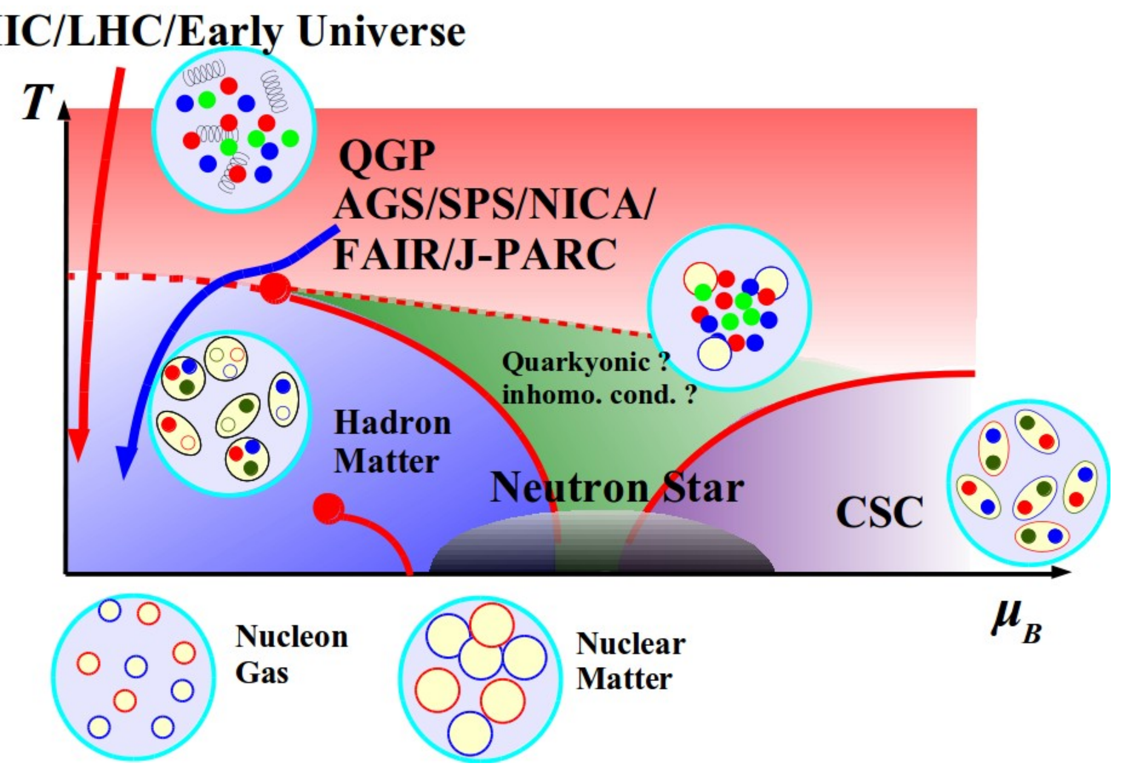
- 講義回数は 1(全体の概要講義 (初回) を含めて 6 回), 2(6 回), 3(4 回) を予定している。

概観

- 核物質の物理が関わる物理
  - 原子核質量･半径、励起状態 (巨大共鳴、 ...)
  - 原子核反応
    重イオン衝突、ストレンジンネス生成反応、中性子過剰核反応 ...
  - コンパクト天体現象
    中性子星、超新星爆発、連星中性子星合体、ブラックホール形成

- 核物質論の中で
  相図･状態方程式
  の理解を目指す。
  - クォーク物質から
  - 核子多体系から
  - 核反応から



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Field Theory at Finite T & ρ

– Short Course –

経路積分

- 量子力学での経路積分 (Path Integral)
  - 時刻 t_i に位置 q_i にいた粒子が時刻 t_f に位置 q_f に到着する振幅

$$S_{fi} = \langle q_f, t_f | \exp[-i\hat{H}(t_f - t_i)] | q_i, t_i \rangle = \int Dq \exp(iS[q])$$

$$S[q] = \int_{q(t_i)=q_i,\, q(t_f)=q_f} dt\, L(q, \dot{q})$$

  経路 q(t) についての和→経路積分

  - 特徴
    - 演算子の代わりに通常の数 (c-number) で表せる
    - 作用 S の構成時に正準交換関係を用いることにより「量子論」の性質を取り込む
- 場の理論 = 各点での場の振幅 φ(x,t) を座標とする量子力学

$$S_{fi} = \langle \Psi_f | \exp[-i\hat{H}(t_f - t_i)] | \Psi_i \rangle = \int D\phi \exp(iS[\phi])$$

$$S[\phi] = \int_{\Psi(t_i)=\Psi_i,\, \Psi(t_f)=\Psi_f} d^4x\, L(\phi, \partial_\mu \phi)$$

分配関数とユークリッド化

分配関数

$$Z=\sum_n \exp(-E_n/T)=\sum_n \langle n|\exp[-\hat{H}/T]|n\rangle$$

$$=\sum_n \langle n|\exp[-i\hat{H}(t_f-t_i)]|n\rangle_{t_f-t_i=-i/T}=\int D\varphi \exp(-S_E[\varphi])$$

$$S_E[\varphi]=\int_0^\beta d\tau\, d^3x\, L_E(\varphi,\partial_i\varphi,\partial_\tau\varphi)\,\Big|_{\varphi(x,\beta)=\varphi(x,0)}$$

$$L_E(\varphi,\partial_i\varphi,\partial_\tau\varphi)=-L(\varphi,\partial_i\varphi,i\partial_\tau\varphi)$$

$$t=-i\tau,\quad \partial_\tau=-i\partial_t,\ \beta=1/T$$

$$iS=i\int_0^{-i\beta} dt\int d^3x\, L=\int_0^\beta d\tau\, d^3x\, L=-\int_0^\beta d\tau\, d^3x\, L_E$$

- 統計力学の分配関数は虚時間発展の振幅の和である。
- 全ての状態について和 → **τ=0, β** で周期境界条件をつけて任意の ϕ **(x,t)** について足し合わせる。

Example: Scalar Field

- **Lagrangian density**

$$L=\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi-\frac{1}{2}m^2\phi^2-U(\phi)$$

Euler-Lagrange equation (principle of least action)

$$\partial_\mu\left[\frac{\partial L}{\partial(\partial_\mu\phi)}\right]-\frac{\partial L}{\partial\phi}=0\;\rightarrow\;\partial_\mu\partial^\mu\phi+m^2\phi+\frac{\partial U}{\partial\phi}=0\,(\text{Klein-Gordon eq.})$$

- **Euclidean Lagrangian**
  - **Euclid 化のルール** $t = -i\tau$, $x_4=\tau$, $g_{\mu\nu}=(1,1,1,1)$, $L_E=-L$

$$L_E=\frac{1}{2}\partial_\mu\phi\,\partial_\mu\phi+\frac{1}{2}m^2\phi^2+U(\phi)$$

→ 相互作用がない場合に実際に経路積分してみましょう。

Partition Func. of Free Scalar Field

- 自由スカラー場の分配関数
  - 有限のサイズの箱 (体積 V) の中で自由スカラー場 (U=0)
  - フーリエ変換

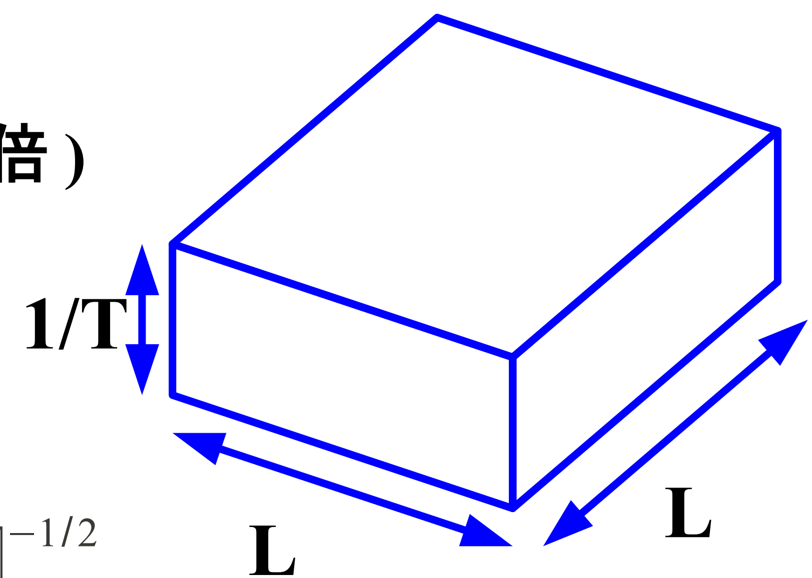
$$\phi(\tau,\boldsymbol{x})=\frac{1}{\sqrt{V/T}}\sum_{n,\boldsymbol{k}}\exp(-i\omega_n\tau+i\boldsymbol{k}\cdot\boldsymbol{x})\phi_n(\boldsymbol{k})$$

Periodic boudnary condition $\quad \omega_n=2\pi n T\,,\quad k_i=2\pi n_i/L$

Euclidean action $\quad S_E=\frac{1}{2}\sum_{n,\boldsymbol{k}}(\omega_n^2+\boldsymbol{k}^2+m^2)\phi_n^2(\boldsymbol{k})$

  - フーリエ変換はユニタリー変換だから、
    積分の測度は変わらない。(高々定数倍)

$$D\phi=N\prod_{n,\boldsymbol{k}}d\phi_n(\boldsymbol{k})$$



  - ガウス積分 → 分配関数

$$Z=\int D\phi\, e^{-S_E}=N\prod_{n,\boldsymbol{k}}\sqrt{2\pi}\left[\omega_n^2+\boldsymbol{k}^2+m^2\right]^{-1/2}$$

Partition Func. of Free Scalar Field (cont.)

- 自由エネルギー

$$\Omega = -T\log Z = \frac{1}{2}\sum_{\boldsymbol{k}}\left[T\sum_n \log(\omega_n^2 + \underbrace{\boldsymbol{k}^2 + m^2}_{E_k^2})\right] + \text{const.}$$

$$= \frac{1}{2}\sum_{\boldsymbol{k}} I(E_{\boldsymbol{k}}, T) + \text{const.}$$

- 松原和 (Matsubara Frequency summation)

$$\sum_n \frac{1}{a^2+\bar{n}^2} = \frac{\pi}{2a}\times\begin{Bmatrix}\coth(\pi a/2) & (\bar{n}=2\mathrm{n}) \\ \tanh(\pi a/2) & (\bar{n}=2\mathrm{n}+1)\end{Bmatrix}$$

$$\frac{\partial I(E_k, T)}{\partial E_k} = \sum_n \frac{2TE_k}{\omega_n^2 + E_k^2} = \cdots = \frac{1+\exp(-E_k/T)}{1-\exp(-E_k/T)}$$

$$I(E_k, T) = E_{\boldsymbol{k}} + 2T\log\left(1-\exp(-E_k/T)\right) + \text{const.}$$

Partition Func. of Free Scalar Field (cont.)

- 自由エネルギー (グランド･ポテンシャル)

$$\Omega = \sum_{\boldsymbol{k}} \left\{ \frac{E_{\boldsymbol{k}}}{2} + T \log\left(1 - e^{-E_{\boldsymbol{k}}/T}\right) \right\} + \text{const.}$$

$$= V \int \frac{d^3 k}{(2\pi)^3} \left[\underbrace{\frac{E_{\boldsymbol{k}}}{2}}_{\text{ゼロ点エネルギー} (\hbar\omega/2)} + \underbrace{T \log\left(1 - e^{-E_{\boldsymbol{k}}/T}\right)}_{\text{熱的励起}} \right]$$

ゼロ点エネルギー部分を無視して部分積分すると、
通常の圧力を得る。

$$P = -\Omega / V = \int \frac{d^3 k}{(2\pi)^3} \frac{\boldsymbol{k} \cdot \boldsymbol{v}}{3} \frac{e^{-E_{\boldsymbol{k}}/T}}{1 - e^{-E_{\boldsymbol{k}}/T}} \qquad \left(\boldsymbol{v} = \frac{\partial E_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \right)$$

場の理論 → Euclid 化 + Imag. Time → 統計力学

Matsubara Frequency Summation

- **Contour integral technique**

$$S = T\sum_n g(\omega_n = 2\pi n T, \pi(2n+1)T)$$

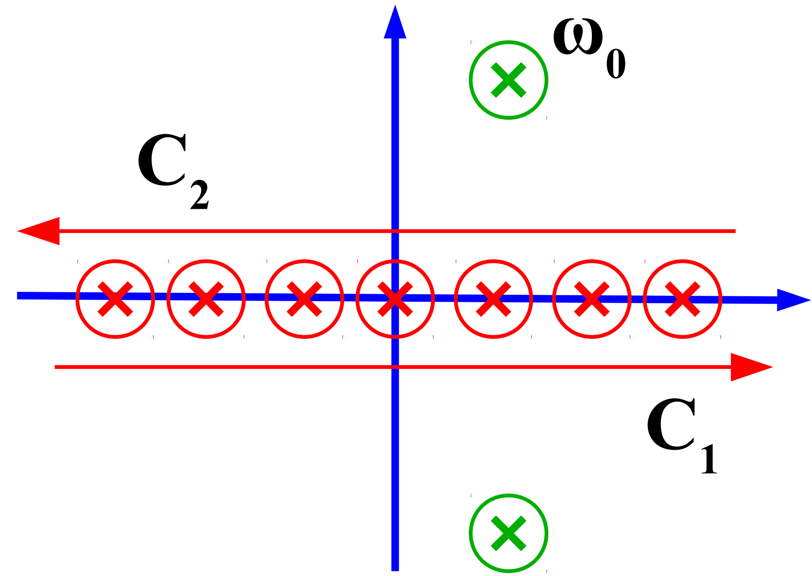
$$= \pm\int_{C_1+C_2} \frac{dz}{2\pi}\frac{g(z)}{e^{i\beta z}\mp 1} = \mp i\sum_{\omega_0}\frac{\mathrm{Res}\, g(\omega_0)}{e^{i\beta\omega_0}\mp 1}$$

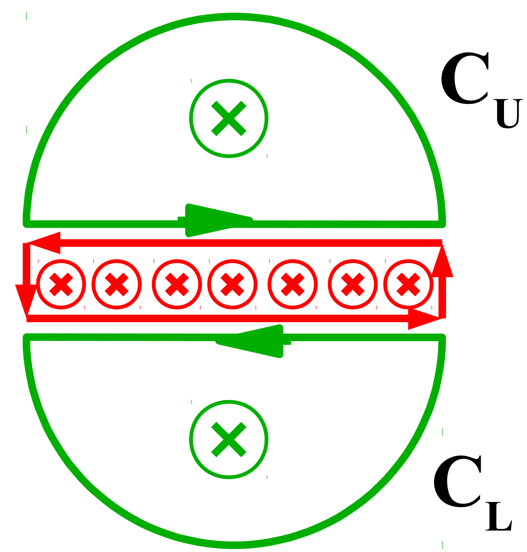
(g: meromorphic (有理型),
no pole on real axis,
decreases faster than 1/ω at ω → ∞)

  - **Applicable to more general cases !**
  - **Anti-periodic condition → Fermi-Dirac dist.**

- **Example: $g(\omega)=1/(\omega^2+E^2)$**
→ $\omega_0 = \pm iE$, Res $g = \pm 1/2iE$

$$S = \frac{1}{2E}\frac{e^{\beta E}\pm 1}{e^{\beta E}\mp 1}$$





$C_1+C_2+C_U+C_L=0$

Fermion

- **Lagrangian**

$$L=\bar{N}\left(i\gamma^{\mu}\partial_{\mu}-m\right)N$$

- **Euclidean**

$$(x_{\mu})_E=(\tau=it\,,\boldsymbol{x}),\quad (\gamma_{\mu})_E=(\gamma_4=i\gamma^0,\boldsymbol{\gamma})$$

$$L_E=\bar{N}\left(-i\gamma_{\mu}\partial_{\mu}+m\right)N$$

- **Grassman number**
  経路積分において、フェルミオンは反可換な Grassmann 数

$$\int d\chi\cdot 1=\text{anti-comm. constant}=0\quad,\quad \int d\chi\cdot\chi=\text{comm. constant}\equiv 1$$

$$\int d\chi\,d\bar{\chi}\exp\left[\bar{\chi}A\chi\right]=\int d\chi\,d\bar{\chi}\frac{1}{N!}(\bar{\chi}A\chi)^N=\cdots=\det A$$

$$=\exp\left[-(-\log\det A)\right]$$

Bi-linear Fermion action leads to -log(det A) effective action

- **Example: Relativistic Mean Field (RMF)**

$$L = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m - \Sigma)\psi + L_{\mathrm{meson}}(\Phi) \quad (\Phi = \sigma, \omega, \rho)$$

$$\Sigma = g_{\sigma}\sigma + \gamma^{0}(g_{\omega}\omega^{0} + g_{\rho}\rho^{0}\boldsymbol{\tau})$$

- **Euclid 化＋化学ポテンシャルの導入**

$$\begin{aligned} Z &= \int D\psi D\bar{\psi} D\Phi \ \exp\left[-\int d^4x (L - \mu\psi^{+}\psi)\right] \\ &= \int D\psi D\bar{\psi} D\Phi \ \exp\left[-\int d^4x \{\bar{\psi} D\psi + L_{\mathrm{meson}}(\Phi)\}\right] \\ &= \int D\Phi \exp\left[-S_{\mathrm{eff}}(\Phi; T, \mu)\right] \end{aligned}$$

$$D = -i\gamma\partial - \mu\gamma^{0} + m + \Sigma$$

- **有効作用**

$$S_{\mathrm{eff}} = S_{\mathrm{eff}}^{(F)} + S_{\mathrm{meson}} = -\sum_{n,\boldsymbol{k}} \log\det D_{n,\boldsymbol{k}} + \int d^4x\, L_{\mathrm{meson}}$$

RMF (cont.)

- 一様な場を仮定 → Fourier 変換により D をブロック対角化

$$D_{n,\boldsymbol{k}}=\gamma^0\left(-i\omega_n-\left(\mu-V^0\right)\right)+\boldsymbol{\gamma}\cdot\boldsymbol{k}+M+g_\sigma\sigma$$

$$\rightarrow \det D=\left[\left(\omega_n+i\mu^*\right)^2+E^{*2}\right]^2$$

$$\mu^*=\mu-g_\omega\omega^0-g_\rho\rho^0\boldsymbol{\tau},\quad E^*=\sqrt{\boldsymbol{k}^2+M^{*2}},\quad M^*=m+g_\sigma\sigma$$

- 松原振動数和を実行

$$F_{\text{eff}}^{(F)}=-\frac{d_f}{2}\int\frac{d^3k}{(2\pi)^3}\Big[\underbrace{E^*}_{\text{ゼロ点}}+\underbrace{T\log\left(1+e^{-(E^*-\mu^*)/T}\right)}_{\text{粒子(核子)}}+\underbrace{T\log\left(1+e^{-(E^*+\mu^*)/T}\right)}_{\text{反粒子(反核子)}}\Big]$$

- 温度 0 の場合

$$F_{\text{eff}}^{(F)}=-\frac{d_f}{2}\int^{\Lambda}\frac{d^3k}{(2\pi)^3}E^*+d_N\int^{k_F}\frac{d^3k}{(2\pi)^3}E^*-\mu^*\rho_B\quad\left(d_N=d_f/2\right)$$

ゼロ点エネルギーは核子のループから現れる
(RMF では通常無視)

Spontaneous Chiral Symmetry Breaking in NJL model

Chiral Symmetry in Quantum Chromodynamics

notation: Yagi, Hatsuda, Miake

- **QCD Lagrangian**

$$L=\bar{q}(i\gamma^\mu D_\mu - m)q - \frac{1}{2}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}$$

- **Chiral symmetry: SU(N_f)$_L$ x SU(N_f)$_R$**
  - **Left- and Right-handed quarks can rotate independently**

$$q_L=(1-\gamma_5)q/2,\quad q_R=(1+\gamma_5)q/2 \rightarrow V_L q_L,\quad V_R q_R$$

$$L_q=\underbrace{\bar{q}_L(i\gamma^\mu D_\mu)q_L+\bar{q}_R(i\gamma^\mu D_\mu)q_R}_{\text{invariant}}-\underbrace{m}_{\text{small (for u, d)}}(\bar{q}_L q_R+\bar{q}_R q_L)$$

- **Chiral transf. of hadrons**

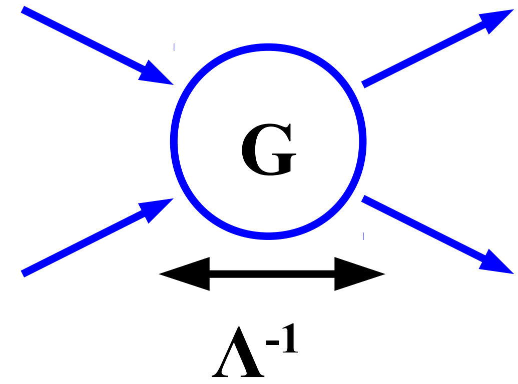
$$\sigma=\bar{q}q \quad,\quad \pi^a=\bar{q}i\gamma_5\tau^a q \;\rightarrow\; \begin{pmatrix}\sigma'\\ \pi'\end{pmatrix}=\begin{pmatrix}1 & -\theta\\ \theta & 1\end{pmatrix}\begin{pmatrix}\sigma\\ \pi\end{pmatrix}$$

  - **σ ($J^\pi=0^+$) and π ($J^\pi=0^-$) mix via chiral transf. but have diff. masses.**
    → Spontaneous breaking of chiral symmetry.
    (As in Bogoliubov shown in superconductor of electrons.)

Nambu-Jona-Lasinio (NJL) model

- **NJL Lagrangian**

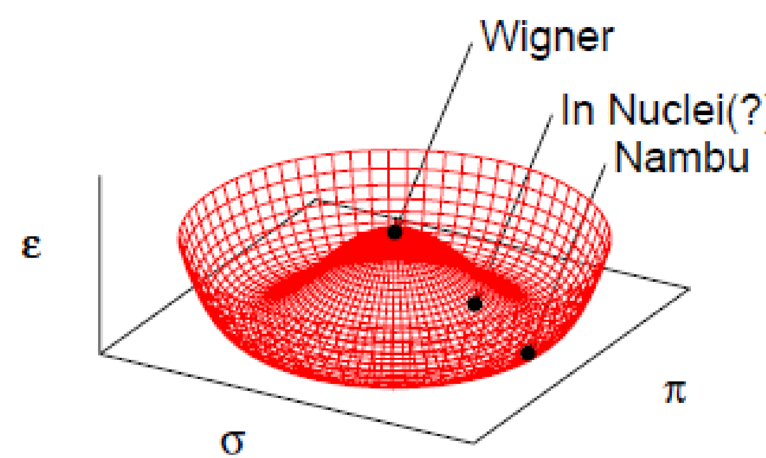
$$L = \bar{q}(i\gamma^{\mu}\partial_{\mu} - m)q + \frac{G^2}{2\Lambda^2}\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5\boldsymbol{\tau}q)^2\right]$$



- **Integrating out gluons and hard quarks in QCD**
  → Effective theory of quarks
  with the same symmetry as QCD

$$S = \bar{q}q\,,\ \boldsymbol{P} = \bar{q}i\gamma_5\boldsymbol{\tau}q$$

$$\rightarrow S^2 + \boldsymbol{P}^2 = \text{inv. under chiral transf.}$$



- **Euclidean action**

$$(x_{\mu})_E = (\tau = it\,, \boldsymbol{x}),\quad (\gamma_{\mu})_E = (\gamma_4 = i\gamma^0\,, \boldsymbol{\gamma})$$

$$L_E = \bar{q}(-i\gamma_{\mu}\partial_{\mu} + m)q - \frac{G^2}{2\Lambda^2}\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5\boldsymbol{\tau}q)^2\right]$$

Nambu, Jona-Lasinio ('61), Hatsuda, Kunihiro ('94)

Partition Function in NJL

- **Bosonization (Hubbard-Stratonovich transf.)**

$$-\frac{G^2}{2\Lambda^2}\left[(\bar{q}q)^2+(\bar{q}i\gamma_5\boldsymbol{\tau}q)^2\right] \to \frac{\Lambda^2}{2}(\sigma^2+\boldsymbol{\pi}^2)+G\bar{q}(\underbrace{\sigma+i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}}_{\Sigma})q$$

- **Partition Function**

$$G = G_0 - G_0\Sigma G$$

$$Z_{\mathrm{NJL}}=\int Dq\,D\bar{q}\exp\left[-\int d^4x\,L_{\mathrm{NJL}}\right]$$

$$=\int Dq\,D\bar{q}\,D\Sigma\exp[-\int d^4x\{\bar{q}(\underbrace{-i\gamma\partial+\overbrace{m+G\Sigma}^{M}}_{D})q+\frac{\Lambda^2}{2}(\sigma^2+\boldsymbol{\pi}^2)\}]$$

$$=\int D\Sigma\exp\left[-S_{\mathrm{eff}}(\sigma,\boldsymbol{\pi};T)\right]$$

- **Effective Action**

$$S_{\mathrm{eff}}(\Sigma;T)=-\log\det D+\int d^4x\frac{\Lambda^2}{2}\left[\sigma^2(x)+\boldsymbol{\pi}^2(x)\right]$$

Bosonization & Grassman Integral

- **Bosonization (Hubbard-Stratonovich transf.)**

$$\exp\left[\frac{G^2 S^2}{2\Lambda^2}\right] = \int d\sigma \exp\left[-\frac{\Lambda^2}{2}\left(\sigma - \frac{GS}{\Lambda^2}\right)^2 + \frac{G^2 S^2}{2\Lambda^2}\right]$$

$$\exp\left[\frac{G^2 (P^a)^2}{2\Lambda^2}\right] = \int d\pi^a \exp\left[-\frac{\Lambda^2}{2}\left(\pi^a - \frac{GP^a}{\Lambda^2}\right)^2 + \frac{G^2 (P^a)^2}{2\Lambda^2}\right]$$

- **Grassman number**

$$\int d\chi \cdot 1 = \text{anti-comm. constant} = 0 \quad , \quad \int d\chi \cdot \chi = \text{comm. constant} \equiv 1$$

$$\int d\chi\, d\bar{\chi} \exp\left[\bar{\chi} A \chi\right] = \int d\chi\, d\bar{\chi} \frac{1}{N!} (\bar{\chi} A \chi)^N = \cdots = \det A$$

$$= \exp\left[-(-\log \det A)\right]$$

Bi-linear Fermion action leads to -log(det A) effective action

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Fermion Determinant in Mean Field Approximation

- **Mean Field approx.+Fourier transf.→ Diagonal Fermion matrix**

$$D = -i\boldsymbol{\gamma}\cdot\nabla - i\gamma_4\partial_\tau + M \rightarrow \begin{pmatrix} -i\omega + M & \boldsymbol{k}\cdot\boldsymbol{\sigma} \\ -\boldsymbol{k}\cdot\boldsymbol{\sigma} & i\omega + M \end{pmatrix} \quad (M = G\sigma = \text{const.})$$

$$\det D = \prod_{n,\boldsymbol{k}} (\omega_n^2 + \boldsymbol{k}^2 + M^2)^{d_f/2} \quad (d_f = 4N_c N_f = \text{Fermion d.o.f.})$$

- **Effective Potential**

$$F_{\text{eff}} = \Omega/V = -\frac{T}{V}\log Z = \frac{\Lambda^2}{2}\sigma^2 - \frac{T}{V}\sum_{n,\boldsymbol{k}} \log(\omega_n^2 + \boldsymbol{k}^2 + M^2)^{d_f/2}$$

$$= \frac{\Lambda^2}{2}\sigma^2 - d_f \int \frac{d^3k}{(2\pi)^3}\left[\underline{\frac{E_k}{2}} + \underline{\frac{k^2}{3E_k}\frac{1}{e^{E_k/T}+1}}\right]$$

Matsubara sum

Fermion det. → Zero point energy (ħ ω/2)+ Thermal pressure

- **Matsubara Frequency Summation**

$$I(E,T)=T\sum_n \log\left(\omega_n^2+E^2\right)$$

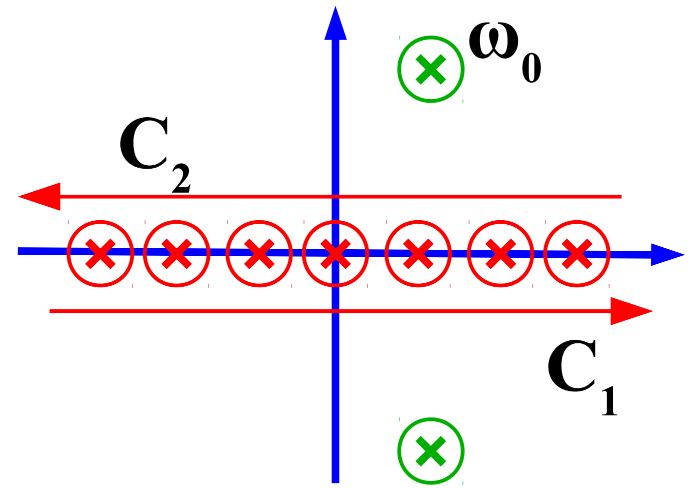
$$\omega_n=2n\pi T\,,\ \pi T(2n-1)$$

$$(\text{for bosons, fermions})$$

This is it !

$$\frac{\partial I(E,T)}{\partial E}=\sum_n \frac{2TE}{\omega_n^2+E^2}=\frac{e^{E/2T}\pm e^{-E/2T}}{e^{E/2T}\mp e^{-E/2T}}$$

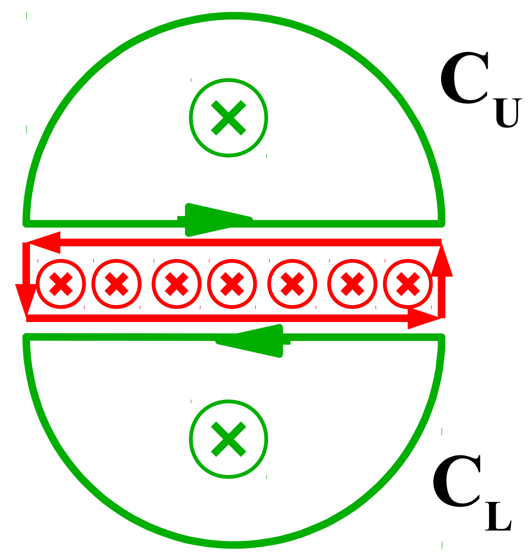
$$I(E,T)=2T\log\left[e^{E/2T}\mp e^{-E/2T}\right]$$
$$=E+2T\log\left(1\mp\exp(-E/T)\right)+\text{const.}$$



- **Contour integral technique**

$$S=T\sum_n g\left(\omega_n=2\pi nT\,,\ \pi(2n+1)T\right)$$

$$=\pm\int_{C_1+C_2}\frac{dz}{2\pi}\,\frac{g(z)}{e^{i\beta z}\mp 1}=\mp i\sum_{\omega_0}\frac{\operatorname{Res} g(\omega_0)}{e^{i\beta\omega_0}\mp 1}$$



$$C_1+C_2+C_U+C_L=0$$

Effective potential of NJL model

- **Effective potential (Grand pot. density)**

$$F_{\mathrm{eff}} = \Omega/V = -d_f \int \frac{d^3k}{(2\pi)^3} \left[\frac{E_k}{2} + T \log\left(1 + e^{-E_k/T}\right) \right] + \frac{\Lambda^2}{2}\sigma^2$$

Zero point energy + Thermal (particle) excitation + Aux. Fields

- **Effective potential in vacuum (T=0, μ=0) in the chiral limit (m=0)**

$$F_{\mathrm{eff}} = -\frac{d_f}{2} \underbrace{\int^{\Lambda} \frac{d^3k}{(2\pi)^3} E_k}_{\Lambda^4 I(M/\Lambda)} + \frac{\Lambda^2}{2}\sigma^2 = \Lambda^4 \left[-\frac{d_f}{2} I(x) + \frac{x^2}{2G^2} \right] \quad (x = M/\Lambda)$$

$$\frac{F_{\mathrm{eff}}}{\Lambda^4} = -\frac{d_f}{16\pi^2} + \frac{x^2}{2} \boxed{\left[\frac{1}{G^2} - \frac{1}{G_c^2} \right]} + O(x^4 \log x) \quad (G_c^2 = 8\pi^2/d_f)$$

G>G$_c$ → 2nd coef. < 0 → Spontaneous Chiral Sym. Breaking

$$I(x) = \frac{1}{16\pi^2} \left[\sqrt{1+x^2}(2+x^2) - x^4 \log \frac{1+\sqrt{1+x^2}}{x} \right] \simeq \frac{1}{8\pi^2} \left[1 + x^2 + \frac{1}{8} x^4 \left(1 + 4 \log \frac{x}{2} \right) + O(x^6) \right]$$

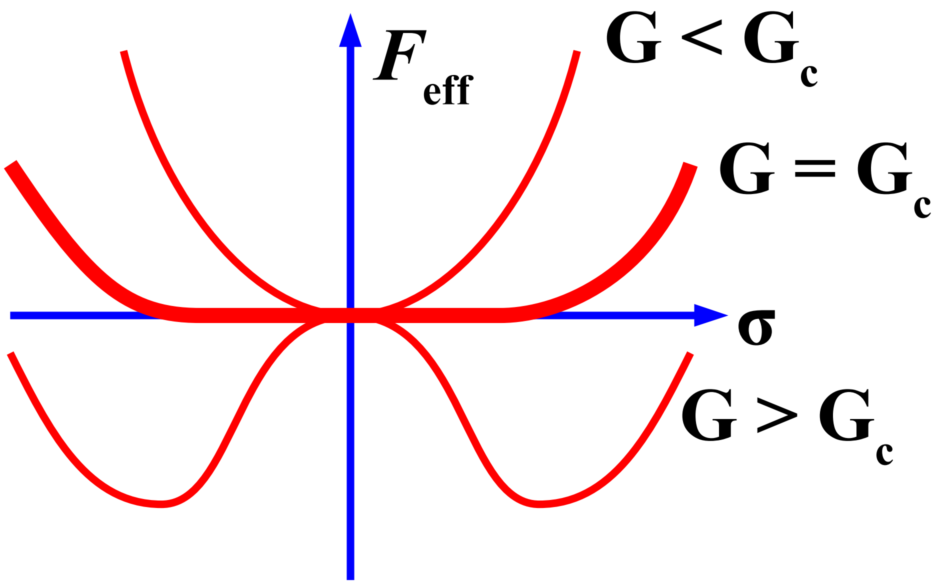
Spontaneous breaking of chiral symmetry

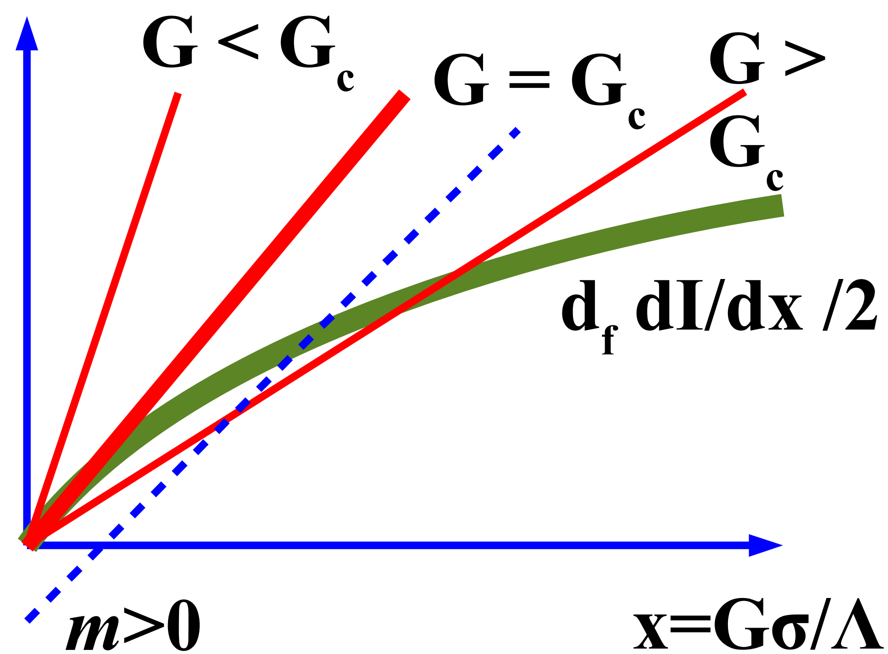
- **σ is chosen to minimize F_{eff} (Gap equation)**

$$\frac{1}{\Lambda^4}\frac{\partial F_{\mathrm{eff}}}{\partial x} = -\frac{d_f}{2}\frac{dI(x)}{dx} + \frac{x}{G^2} = 0$$

For $G>G_c$ → finite σ(~ $q^{bar}q$) solution gives min. energy state.

> ***If the interaction is strong enough, σ(~$q^{bar}q$) condensates and quark mass is generate.(Nambu, Jona-Lasinio ('61))***





Chiral phase transition at finite T and μ (Chiral Limit)

NJL model with μ

NJL Lagrangian

$$L = \bar{q}(i\gamma^{\mu}\partial_{\mu} - m + \gamma_0 \mu) q + \frac{G^2}{2\Lambda^2}\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5 \boldsymbol{\tau} q)^2\right]$$

$$L_E = \bar{q}(-i\gamma_{\mu}\partial_{\mu} + m - \gamma_0 \mu) q - \frac{G^2}{2\Lambda^2}\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5 \boldsymbol{\tau} q)^2\right]$$

Effective Action

$$S_{\text{eff}}(\Sigma; T) = -\log\det D + \int d^4x \frac{\Lambda^2}{2}\left[\sigma^2(x) + \boldsymbol{\pi}^2(x)\right]$$

$$D = -i\gamma\partial - \gamma_0\mu + M \quad , \quad m + G\Sigma$$

$$\text{MF+Fourier} \rightarrow D = -\gamma_0(i\omega + \mu) + \boldsymbol{\gamma}\cdot\boldsymbol{k} + M \quad , \quad M = m + G\sigma$$

Free energy density

$$F_{\text{eff}} = \frac{\Lambda^2}{2}\sigma^2 - \frac{T}{V}\sum_{n,\boldsymbol{k}} \log\left((\omega_n - i\mu)^2 + \boldsymbol{k}^2 + M^2\right)^{d_f/2}$$

$$= \frac{\Lambda^2}{2}\sigma^2 - d_f \int \frac{d^3k}{(2\pi)^3}\left[\frac{E_k}{2} + \frac{k^2}{3E_k}\frac{1}{2}\left(\frac{1}{e^{(E_k-\mu)/T}+1} + \frac{1}{e^{(E_k+\mu)/T}+1}\right)\right]$$

T, μ and m dependence of thermal pressure

- **Thermal pressure as a function of T, μ, and m (Fermions)**
  Kapusta ('89), Kapusta, Gale (2006)

$$P^F/d_F = \frac{7}{8}\frac{\pi^2}{90}T^4 + \frac{1}{24}\mu^2 T^2 + \frac{\mu^4}{48\pi^2}$$

Stefan-Boltzmann (m=0)

$$- \frac{m^2}{16\pi^2}\left[\frac{\pi^2}{3}T^2 + \mu^2\right]$$

m^2 term → phase transition

$$- \frac{m^4}{32\pi^2}\left[\log\left(\frac{m}{\pi T}\right) - \frac{3}{4} + \gamma_E \boxed{- H^\nu\left(\frac{\mu}{T}\right)}\right] + \mathcal{O}\left(m^6\right)$$

New

m^4 term → critical point

$$H^\nu(\nu) = \frac{7}{4}\zeta(3)\left(\frac{\nu}{\pi}\right)^2 - \frac{31}{16}\zeta(5)\left(\frac{\nu}{\pi}\right)^4 + \frac{127}{64}\zeta(7)\left(\frac{\nu}{\pi}\right)^6 + \cdots$$

Mass reduces pressure (enh. Feff) → phase transition ?

Chiral Transition at Finite T

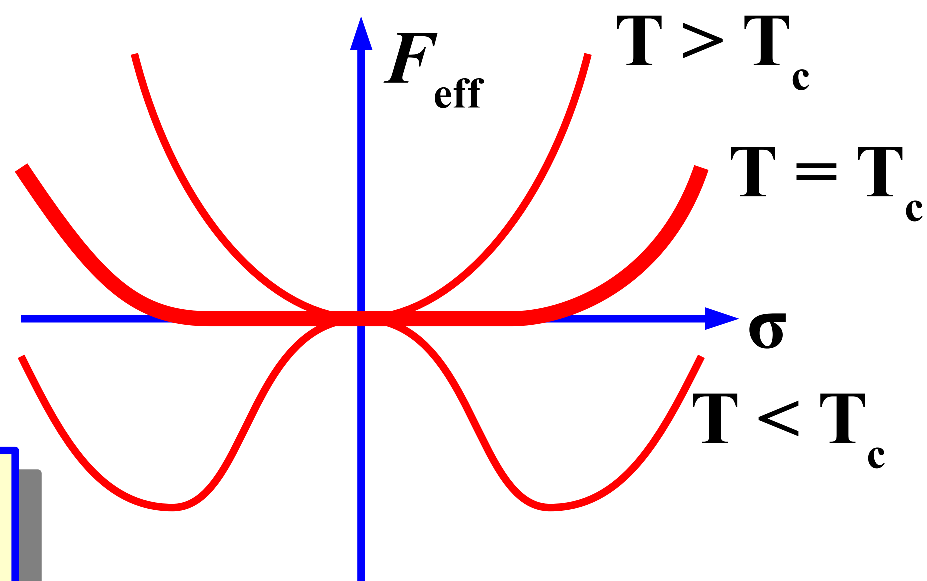
- **Effective potential at finite *T* in NJL**

$$\frac{F_{\rm eff}}{\Lambda^4} = -\frac{d_f}{2} I(x) + \frac{x^2}{2G^2} - \frac{P^F}{\Lambda^4}$$

$$= -\frac{d_f}{16\pi^2} - \underline{\frac{d_f \pi^2}{90}\frac{7}{8}\left(\frac{T}{\Lambda}\right)^4} + \frac{x^2}{2}\left[\frac{1}{G^2} - \frac{1}{G_c^2}\left(1 - \underline{\frac{\pi^2}{3}\left(\frac{T}{\Lambda}\right)^2}\right)\right] + O(x^4)$$

Stefan-Boltzmann **Correction from T**

- **Chiral transition should occur at T < $3^{1/2}$ Λ/π.**



Chiral Transition at finite T is suggested by NJL !

High-Temperature Expansion (1)

- **Thermal pressure (Fermions)**

$$P^F = \frac{d_F}{2}\int \frac{d^3p}{(2\pi)^3}\,\frac{p^2}{3\omega}\left[\frac{1}{e^{(\omega-\mu)/T}+1}+\frac{1}{e^{(\omega+\mu)/T}+1}\right]$$

$$\omega = \sqrt{p^2+m^2}$$

- **High-Temperature Expansion = Expansion in *m/T***
  - **Important to discuss chiral transition (*m* = *Gσ*)**
  - **Naive expansion does not work (non-analytic term in *m*)**
- **Kapusta method**
  - **Recursion formula: simpler integral → pressure**

$$P^F = \frac{4T^4 d_F}{\pi^2}\,h_5^F\left(y=\frac{m}{T},\nu=\frac{\mu}{T}\right)\ ,\quad \frac{dh_{n+1}}{dy} = -\frac{y}{n}h_{n-1}$$

  - **Replace integrand**

$$\frac{1}{2\omega}\left[\frac{1}{e^{\omega-\nu}+1}+\frac{1}{e^{\omega+\nu}+1}\right] = \frac{1}{2\omega} - \sum_{l=-\infty}^{\infty}\frac{1}{\omega^2+[\pi(2l-1)-i\nu]^2}$$

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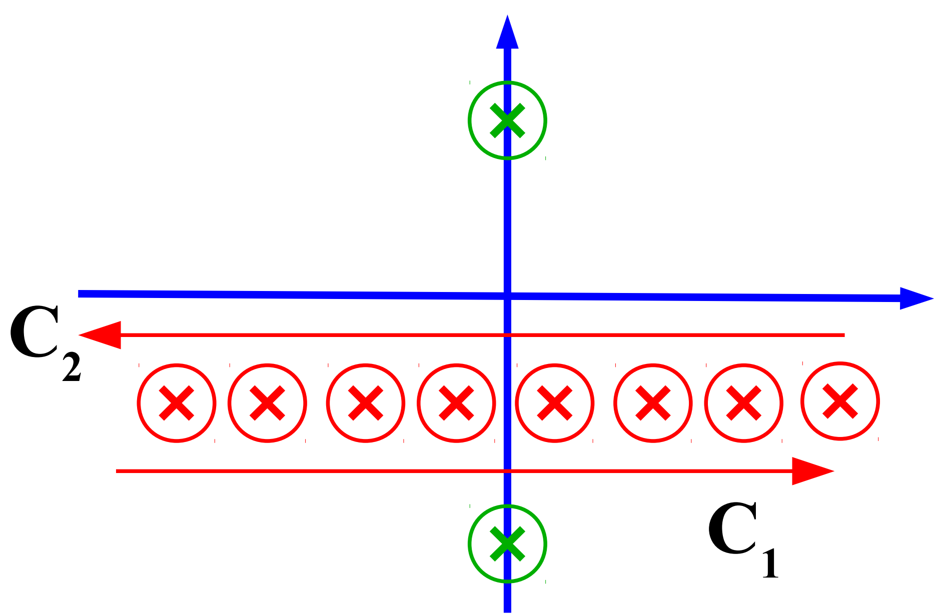
High-Temperature Expansion (2)

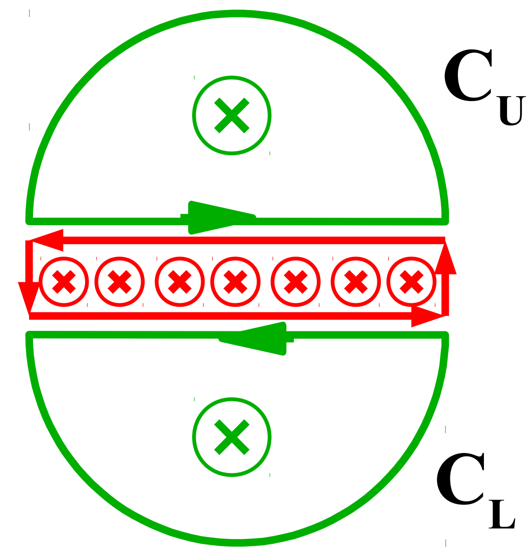
- **Following identity is obtained from contour integral.**

$$\frac{1}{2\omega}\left[\frac{1}{e^{\omega-\nu}+1}+\frac{1}{e^{\omega+\nu}+1}\right]=\frac{1}{2\omega}-\sum_{l=-\infty}^{\infty}\frac{1}{\omega^2+[\pi(2l-1)-i\nu]^2}$$

$$\oint_{\mathcal{C}_\mathrm{U}+\mathcal{C}_\mathrm{L}}\frac{dz}{2\pi}\,\frac{1}{e^{iz-\nu}+1}\,\frac{1}{z^2+\omega^2}=-\oint_{\mathcal{C}}\frac{dz}{2\pi}\,\frac{1}{e^{iz-\nu}+1}\,\frac{1}{z^2+\omega^2}$$

pole at $z = \pm i\,\omega$ **pole at $z = \pi(2\,l - 1) - i\,\nu$**





$C_1+C_2+C_U+C_L=0$

High-Temperature Expansion (3)

- **Recursion relation of h-functions**

$$h_n^F(y,\nu) = \frac{1}{2(n-1)!}\int_0^\infty \frac{x^{n-1}dx}{\omega}\left\{\frac{1}{e^{\omega-\nu}+1}+\frac{1}{e^{\omega+\nu}+1}\right\}$$

$$\frac{dh_{n+1}}{dy} = -\frac{y}{n}h_{n-1}$$

  - **From $h_1(y, \nu)$, $h_3(0, \nu)$, $h_5(0, \nu)$, we obtain $h_5(y, \nu)$ and pressure.**

- **Key function= $h_1(y, \nu)$**

$$h_1^F(y,\nu) = \lim_{L\to\infty}\int_0^{2\pi L} dx\left[\frac{1}{2\omega} - \sum_{l=-\infty}^{\infty}\frac{1}{\omega^2+[\pi(2l-1)-i\nu]^2}\right]$$

$$= -\frac{1}{2}\log\frac{y}{\pi} - \frac{1}{2}\gamma_{\mathrm{E}} - \frac{1}{2}\sum_{l=1}^{\infty}\left[\frac{\pi}{\omega_l}+\frac{\pi}{\omega_l^*}-\frac{2}{2l-1}\right]$$

$$(\omega_l = \sqrt{y^2+[\pi(2l-1)-i\nu]^2})$$

Chiral Transition at Finite μ

- **Effective potential at finite μ in NJL**

$$F_{\text{eff}}(m;T,\mu) = F_{\text{eff}}(0;T,\mu) + \frac{c_2(T,\mu)}{2} m^2 + \frac{c_4(T,\mu)}{24} m^4 + \mathcal{O}(m^6)$$

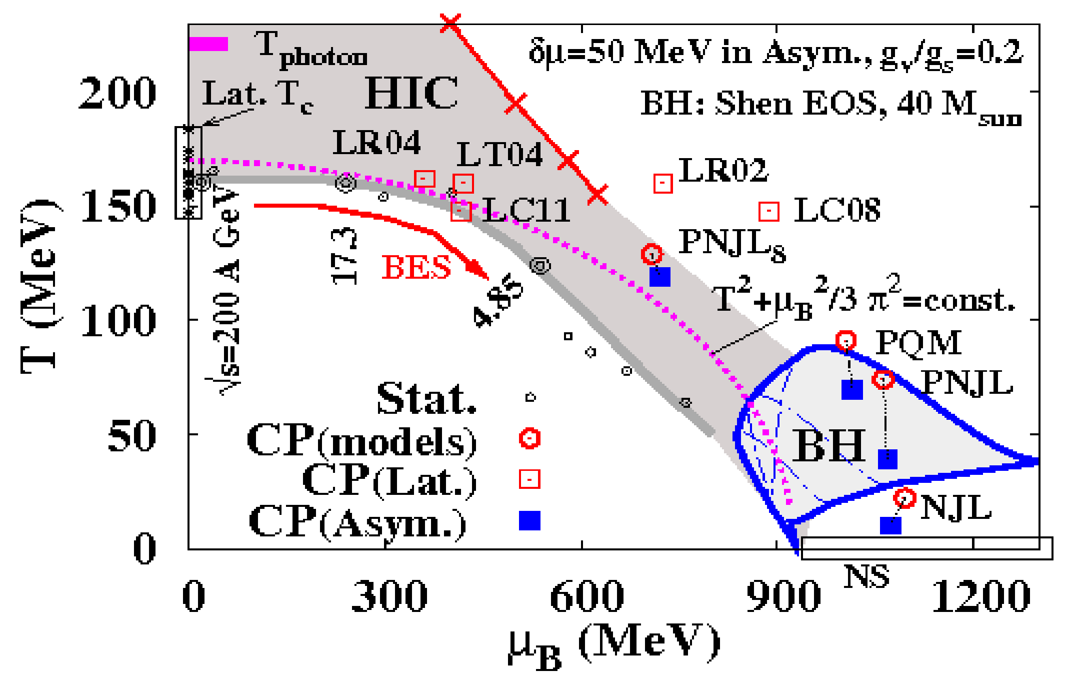
$$c_2(T,\mu) = -\frac{d_F}{24}\left[\boxed{\frac{3}{\pi^2}\Lambda^2\left(1-\frac{8\pi^2}{d_F G^2}\right)} - \left(T^2 + \frac{3}{\pi^2}\mu^2\right)\right]$$

$T_c^2(\mu=0)$

- **2nd order phase boundary**

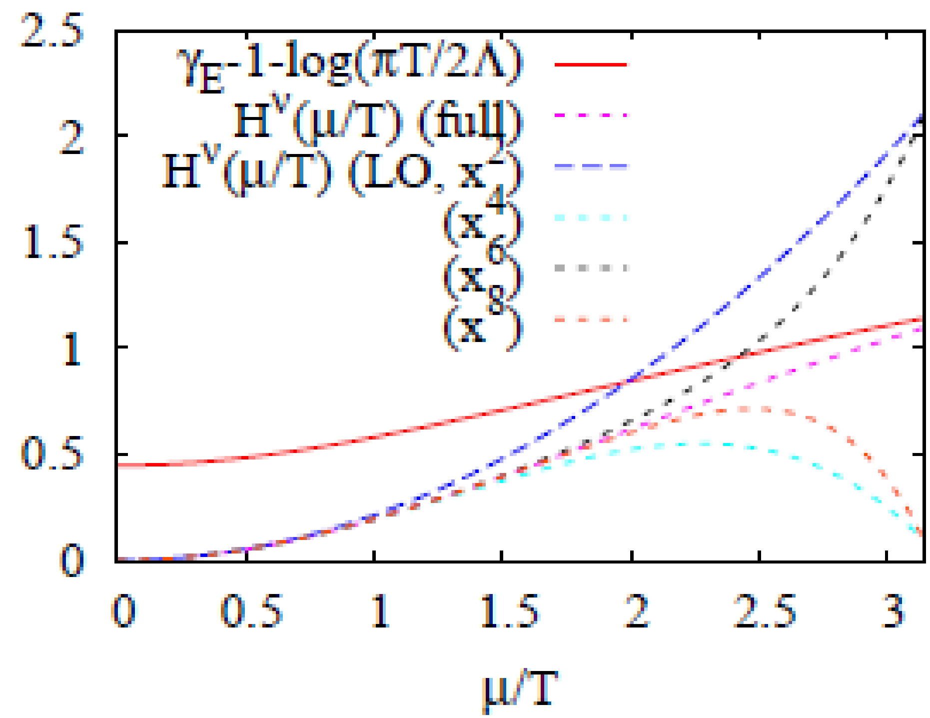
$$T^2 + \frac{3}{\pi^2}\mu^2 = T_c^2(\mu=0)$$

Roughly matches chem. freeze-out line.



(Tri)Critical Point

- **Do we expect the existence of (Tri)Critical Point in NJL ?**
  - **Yes, as first shown by Asakawa, Yazaki ('89)**
  - **TCP in the chiral limit → CP at finite bare quark mass**
- **Estimate from high-temperature expansion**
  - **TCP: $c_2 = 0$ and $c_4 = 0$ simultaneously.**
  - **c_4 decreases as μ/T increases.**
  - **Existence is probable,
Position is sensitive
to parameters and treatment.**



Chiral Transition at Finite μ

- **Effective potential at finite μ in NJL**

$$F_{\text{eff}}(m;T,\mu) = F_{\text{eff}}(0;T,\mu) + \frac{c_2(T,\mu)}{2}m^2 + \frac{c_4(T,\mu)}{24}m^4 + \mathcal{O}(m^6)$$

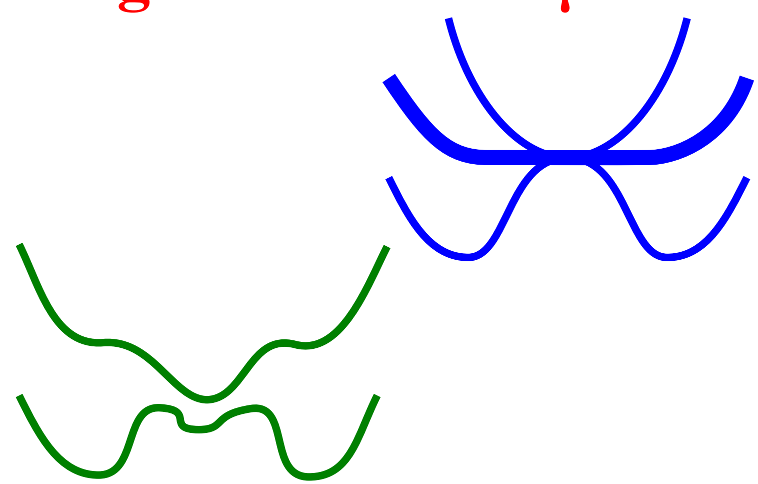
$$c_2(T,\mu) = -\frac{d_F}{24}\left[\frac{3}{\pi^2}\Lambda^2\left(1-\frac{8\pi^2}{d_F G^2}\right) - \left(T^2 + \frac{3}{\pi^2}\mu^2\right)\right]$$

$$c_4(T,\mu) = \frac{3d_F}{4\pi^2}\left[\gamma_E - 1 - \log\left(\frac{\pi T}{2\Lambda}\right) - H^\nu(\mu/T)\right]$$

μ=0

negative at finite μ

- **$c_2 = 0$ and $c_4 > 0 \rightarrow$ 2nd order**
- **$c_2 \geq 0$ and $c_4 < 0 \rightarrow$ 1st order**
- **$c_2 = 0$ and $c_4 = 0 \rightarrow$ tricritical point**



Short Summary

- **We expect the existence of QCD phase transition and the critical point from chiral effective model studies. This point is discussed based on the Nambu-Jona-Lasinio model**
  - **When qq interaction is strong enough, chiral symmetry is spontaneously broken in vacuum.**
  - **Chiral symmetry should be restored at high temperature.**
  - **Density effect reduces the 4-th coeff. in m (or σ), and we can expect the first order transition at high density.**
  - **Technical part**
    Matsubara sum, Hubbard-Stratonovich transformation, High-temperature expansion, ...
- **Since the first principle calculation of QCD has difficulties at finite densities, we need studies using effective models, approximate treatment of QCD, and of course, experiments.**

QCD Symmetries

- **QCD Lagangian**

$$\mathcal{L} = \bar{q}(i\gamma^\mu D_\mu - m)q - \frac{1}{2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}$$

$$D_\mu = \partial_\mu \pm igA_\mu\ , \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \mp g f^{abc} A^b_\mu A^c_\nu$$

- **Classical Symmetry = Symmetry of Action (Lagrangian)**

$$\mathrm{SU}(N_c) \otimes \mathrm{U}(N_f)_L \otimes \mathrm{U}(N_f)_R$$

- **Quantum theory → Action + Measure (path integral)**
  Chiral anomaly breaks $\mathrm{U(1)}_\mathrm{A}$

$$\mathrm{SU}(N_c) \otimes \mathrm{U}(1)_B \otimes \mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R$$

- **Spontaneous + Explicit breaking of chiral sym.**

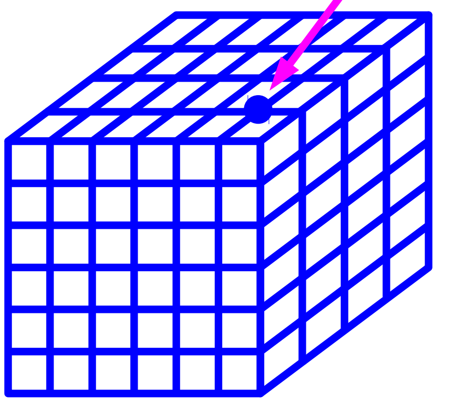
$$\mathrm{SU}(N_c) \otimes \mathrm{U}(1)_B \otimes \mathrm{SU}(N_f)_V$$

→(N_f^2-1) Goldstone bosons

Introduction to Lattice Field Theory

格子上の場の理論

- 場の理論＝無限自由度
  - 解析的・厳密にとくことは一般には困難 → 数値的に解く
  - 求めたいものは非常に複雑な積分
    →「区分求積」＝ 有限の格子上で解き、連続極限をとる。



- スカラー場
  - 連続理論 (Euclidean) の作用 (ϕ^4理論)

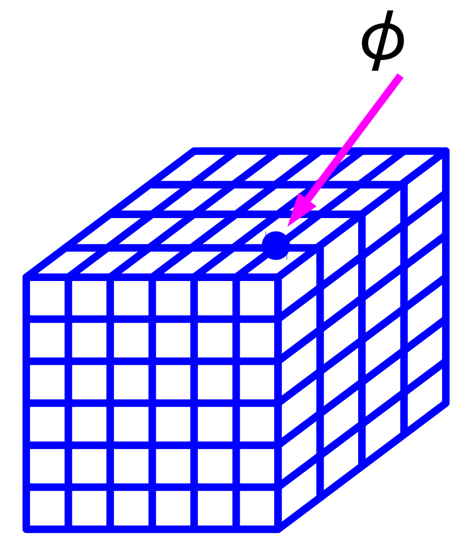
$$S_{\mathrm{cont}}=\int d^4x\left[\frac{1}{2}\partial_\mu\phi\,\partial_\mu\phi+\frac{1}{2}m^2\phi^2+\frac{1}{4!}\lambda\phi^4\right]$$

  - 格子上の作用
    - 連続極限で S_{cont} に一致
    - S_{cont} とできるだけ同じ対称性を持つ

$$S_{\mathrm{lat}}=-\frac{a^4}{2}\sum_{n,\mu}\phi(n)\frac{\phi(n+\hat{\mu})+\phi(n-\hat{\mu})-2\phi(n)}{a^2}+a^4\sum_n\left[\frac{1}{2}m^2\phi^2(n)+\frac{\lambda}{4!}\phi^4(n)\right]$$

格子上の場の理論 (cont.)

- 格子上の作用：スカラー場理論



$$S_{\mathrm{lat}} = -\frac{a^4}{2}\sum_{n,\mu}\phi(n)\frac{\phi(n+\hat{\mu})+\phi(n-\hat{\mu})-2\phi(n)}{a^2} + a^4\sum_n\left[\frac{1}{2}m^2\phi^2(n)+\frac{\lambda}{4!}\phi^4(n)\right]$$

$n=(n_x, n_y, n_z)$: spacetime point on the lattice

$\hat{\mu}$: unit vector in the positive μ direction.

- $\mathbf{a} \to \mathbf{0}$ の極限で、連続理論の作用と一致

$$S_{\mathrm{lat}} \to a^4\sum_n\left[-\frac{1}{2}\phi(n)\sum_\mu\frac{\partial^2\phi}{\partial x_\mu^2}+\frac{m^2}{2}\phi(n)^2+\frac{\lambda}{4!}\phi(n)^4\right]+O(a^6)$$

$$=\int d^4x\left[-\frac{1}{2}\phi(x)\partial^\mu\partial_\mu\phi(x)+\frac{m^2}{2}\phi(x)^2+\frac{\lambda}{4!}\phi(x)^4\right]$$

- $\phi \to -\phi$ 等の対称性を保つ。

Gauge field

- **Gauge action (Euclidean)**

$$S_G = \frac{1}{2g^2}\int d^4x\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu},\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu],$$

$$A_\mu = A_\mu^a t^a \quad \left([t^a, t^b] = i f_{abc} t^c,\ \mathrm{tr}(t^a t^b) = \frac{1}{2}\delta_{ab}\right)$$

(経路積分では変数が c 数なので、*gA* → *A* とスケール)

- **Gauge transformation**

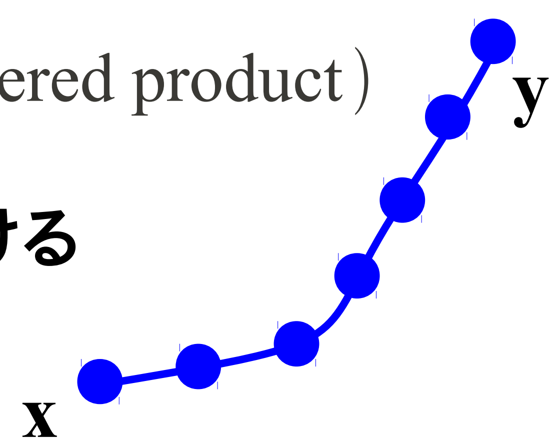
$$A_\mu(x) \to V(x)(A_\mu(x) - i\partial_\mu)V^+(x),\quad F_{\mu\nu}(x) \to V(x)F_{\mu\nu}V^+(x)$$

- **ゲージ不変性をもつ格子上の作用をどのように作るか？**
  → リンク変数

$$U(x,y) \equiv P\exp\left[i\int_x^y dz_\mu A_\mu(z)\right]\quad (P:\text{path ordered product})$$

リンク変数は両端の点でのゲージ変換を受ける

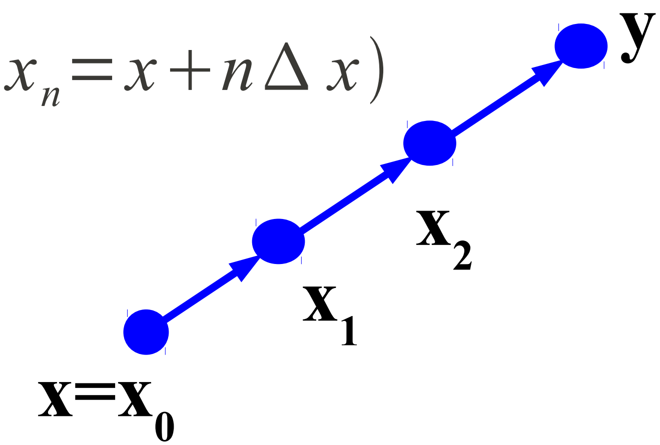
$$U(x,y) \to U'(x,y) = V(x)U(x,y)V^+(y)$$



Appendix: Gauge transformation of U

- **Proof of U(x,y) → V(x)U(x,y)V⁺(y)**

$$U(x,y) = \lim_{N\to\infty} \prod_{n=0}^{N-1} [1 + i A_\mu(x_n) \Delta x_\mu] \quad (x_n = x + n\Delta x)$$



(multiply (1+i A Δx) to the right !)
By using the gauge transformation of *A*,

$$A_\mu(x) \to V(x)(A_\mu(x) - i\partial_\mu) V^+(x)$$

and the unitarity of *V*, *V*(*x*) *V*⁺(*x*)=1, we get

$$\begin{aligned}
&1 + iA'_\mu(x_n)\Delta x_\mu \\
&= 1 + iV(x_n) A_\mu(x_n) V^+(x_n)\Delta x_\mu + V(x_n)\partial_\mu V^+(x_n)\Delta x_\mu \\
&= V(x_n)V^+(x_{n+1}) + iV(x_n)A_\mu(x_n)V^+(x_{n+1})\Delta x_\mu + O((\Delta x)^2) \\
&= V(x_n)[1 + iA_\mu(x_n)\Delta x_\mu] V^+(x_{n+1}) + O((\Delta x)^2) \\
&\to U'(x,y) = V(x)U(x,y)V^+(y)
\end{aligned}$$

Gauge action

- **リンク変数**

$$U_{n,\mu} \equiv U(n, n+\hat{\mu}) = \exp[ia A_\mu(n)] \in \mathrm{SU(N)}$$

$$U_{n,\mu} = U(n, n+\hat{\mu})$$

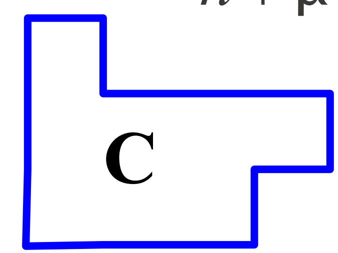
$n \longrightarrow n+\hat{\mu}$

- **リンク変数は両端の点でのゲージ変換を受けるので、「閉じた経路」にそって積をとると、その trace はゲージ不変。**

$$U^{+}_{n,\mu} = U(n+\hat{\mu}, n)$$

$n \longleftarrow n+\hat{\mu}$

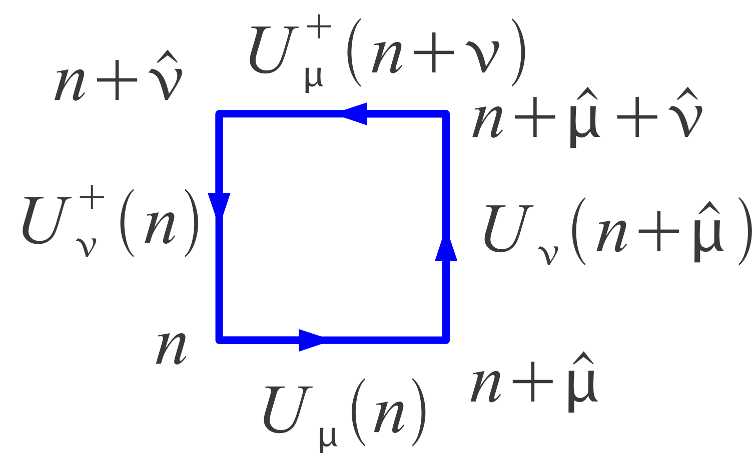
$$\prod_{n\in C} U \rightarrow V(n) \Big(\prod_{n\in C} U \Big) V^{+}(n)$$



- **Plaquette**
  Lattice 上で最も小さな loop は単位正方形

$$n \rightarrow n+\hat{\mu} \rightarrow n+\hat{\mu}+\hat{\nu} \rightarrow n$$

$$U_{\mu\nu}(n) \equiv U_{n,\mu} U_{n+\hat{\mu},\nu} U^{+}_{n+\nu,\mu} U^{+}_{n,\nu}$$



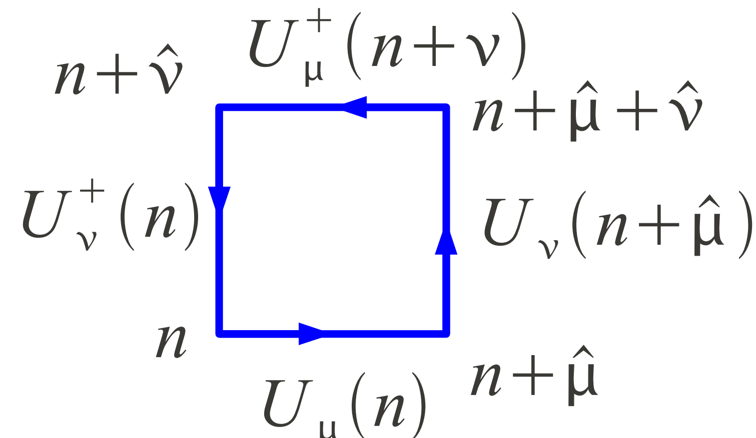
- **Gauge action (plaquette action)**

$$S_G = \beta_g \sum_{plaq.} \left[1 - \frac{1}{N_c} \mathrm{Re\,tr}\, U_{\mu\nu}(n) \right] \quad (\beta_g = 2N_c/g^2)$$

Appendix: Plaquette and continuum action

- **ゲージ場の格子作用**

$$S_G=\beta_g \sum_{plaq.}\left[1-\frac{1}{N_c}\mathrm{Re\,tr}\,U_{\mu\nu}(n)\right]$$



- **U(1) (電磁場) の場合： 周積分 =rotation の面積分 → $F_{\mu\nu}F_{\mu\nu}$**
- **非可換ゲージ場の場合 : Hausdorff 公式の利用** $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$

$$\mathrm{tr}\,U_{\mu\nu}(x)=\mathrm{tr}\,e^{ia\{A_\mu(x)+A_\nu(x+\hat{\mu})+ia[A_\mu,A_\nu]/2+\cdots\}}\times e^{-ia\{A_\mu(x+\hat{\nu})+A_\nu(x)-ia[A_\mu,A_\nu]/2+\cdots\}}$$

$$=\mathrm{tr}\,e^{ia[(A_\nu(x+\hat{\mu})-A_\nu(x))-(A_\mu(x+\hat{\nu})-A_\mu(x))+i\,a[A_\mu,A_\nu]+O(a^3)]}$$

$$=\mathrm{tr}[1+\underbrace{ia^2F_{\mu\nu}+a^4X_4}_{0}-a^4F^2_{\mu\nu}/2+O(a^6)]$$

$$\lim_{a\to 0} S_G=\sum_{n,\mu\neq\nu}\beta_g\,\mathrm{tr}\left[1-\frac{a^4}{2}F^2_{\mu\nu}\right]$$

S. Aoki, Text

Link Integral

- ゲージ場の経路積分

$$Z_G = \int \prod_{n,\mu} dU_{n,\mu} \exp(-S_G) = \int \prod_{n,\mu} dU_{n,\mu} \exp\left[\beta_g \sum_{P \in \text{plaq.}} \mathrm{tr}(U_P + U_P^{+})\right]$$

  - **dU は群上の不変測度（Haar measure）**
    → ゲージ変換

$$U_{n,\mu} \to V(n) U_{n,\mu} V^{+}(n+\hat{\mu})$$

    に対して不変な積分の測度が必要

- リンク積分 **SU(N)**
  - ゲージ不変性のみで、リンク変数の多項式の積分はほぼ決まる。

$$\int dU\, 1 = 1 \quad (\text{normalization}), \quad \int dU\, U_{ab} = 0$$

$$\int dU\, U_{ab} U^{+}_{ij} = \frac{1}{N} \delta_{aj} \delta_{bi}$$

$$\int dU\, U_{ai} U_{bj} U_{ck} = \frac{1}{N!} \varepsilon_{abc} \varepsilon_{ijk} \quad (N=3)$$

Proof of the one-link integral formulae

- $\int dU\, U_{ab} = 0$
  - LHS = R_{ab} とおく。両辺に左から SU(N) 行列 V をかけると

$$\mathrm{LHS} = \int dU\,(VU)_{ab} = \int d(VU)\,(VU)_{ab} = R_{ab}, \quad \mathrm{RHS} = (VR)_{ab}$$

V は任意の SU(N) の元だから、$R_{ab}=0$

- $\int dU\, U_{ab} U^{+}_{ij} = \frac{1}{N}\delta_{aj}\delta_{bi}$
  - LHS=T^{aj}_{bi} とおく。U, U^{+} が LU, $U^{+}L^{+}$と変換するよう L, L^{+}をかける。

$$\mathrm{LHS} = \int dU\,(LU)_{ab}(LU)^{+}_{ij} = \int d(LU)\,(LU)_{ab}(LU)^{+}_{ij} = T^{aj}_{bi}$$

$$\mathrm{RHS} = L_{ac} T^{ck}_{bi} L^{+}_{kj} \;\rightarrow\; L\,T_{bi} = T_{bi}\,L \;\rightarrow\; T^{aj}_{bi} = S_{bi}\,\delta^{aj}$$

任意の SU(N) の元と交換するので上添字について T は単位行列。
同様に右変換して S も単位行列に比例。
a=j とおいて和をとると、比例係数が 1/N と分かる。

Proof of the one-link integral formulae

- $\int dU\, U_{ai} U_{bj} U_{ck} = \frac{1}{N!} \varepsilon_{abc} \varepsilon_{ijk} \quad (N=3)$
  - **左辺を T^{abc}_{ijk} とおく。U を左変換。**

    $$T^{abc}_{ijk} = L_{ad} L_{be} L_{cf} T^{def}_{ijk}$$

    任意の L に対して不変な 3 階のテンソルは完全反対称テンソルのみ。右変換も同様。

    $$T^{abc}_{ijk} = c\, \varepsilon^{abc} \varepsilon_{ijk}$$

  - **両辺に abc をかけて和をとり、det U=1 を使うと c=1/N!**

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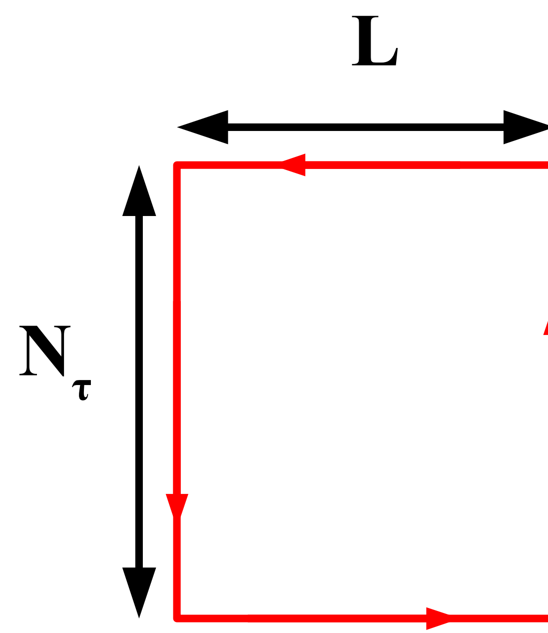
Wilson Loop

- **One link integral formulae の応用として、強結合領域での Wilson loop の期待値を求めてみます。**
- **Wilson loop**



$$W(C=L\times N_\tau)=\mathrm{tr}\left[\prod_{i\in C} U_i\right]$$

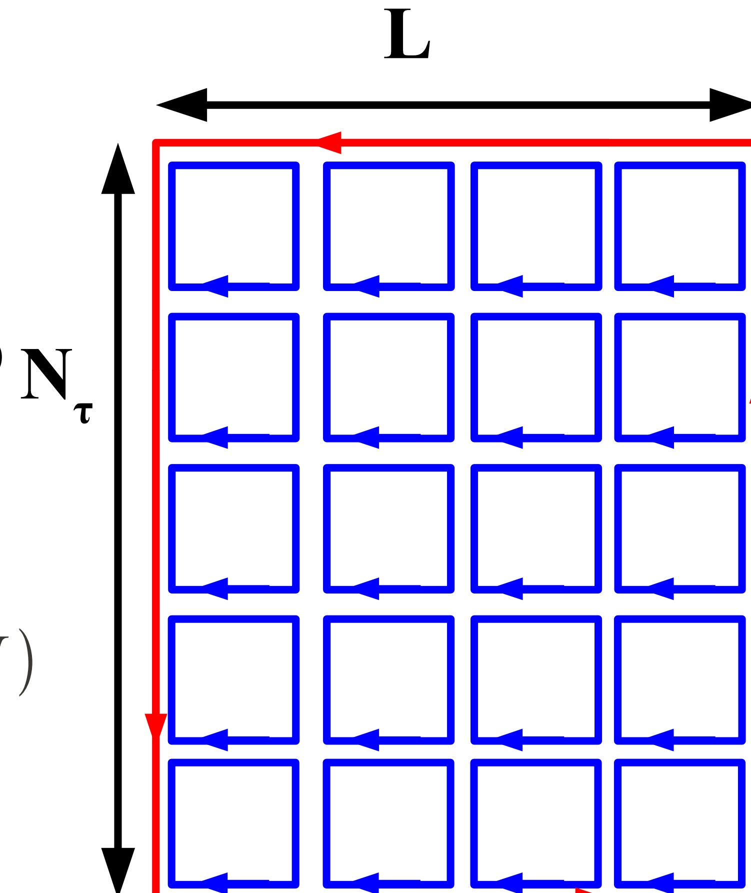
  - **空間方向 L、時間方向 N_τ のループにそって、リンク変数を掛け合わせたもの。**
  - **意味づけ**
    ある時刻に両端が重いクォークからなり、L だけ伸びたストリングを作る。虚時間 Nτ の後に同じ位置で観測する確率。

$$\langle O_L(N_\tau)O_L^+(0)\rangle \propto \exp(-V(L)N_\tau) \quad (\text{for large } N_\tau)$$
$$V(L)=\text{Interquark potential}$$

強結合極限での評価

$$\langle W(C=L\times N_\tau)\rangle=\int DU\,W(C)\exp\left[\frac{1}{g^2}\sum_P \mathrm{tr}(U_P+U_P^+)\right]$$

- リンク変数が残っていると積分して 0 。
  → Wilson loop に含まれるすべての
  リンクを plaquette からのリンクと
  組み合わせて消す必要がある。
- 結合が強いとき、できるだけ少ない数の
  plaq. で消すには、Wilson loop を
  平面的に plaq. で埋めればよい。



$$\langle W(C)\rangle=N\left(\frac{1}{g^2N}\right)^{LN_\tau}\to V(L)=L\log(g^2N)$$

強結合極限では面積則
→ クォークの閉じ込め

K.G.Wilson, PRD10('74),2445

Strong Coupling Lattice QCD: Pure Gauge

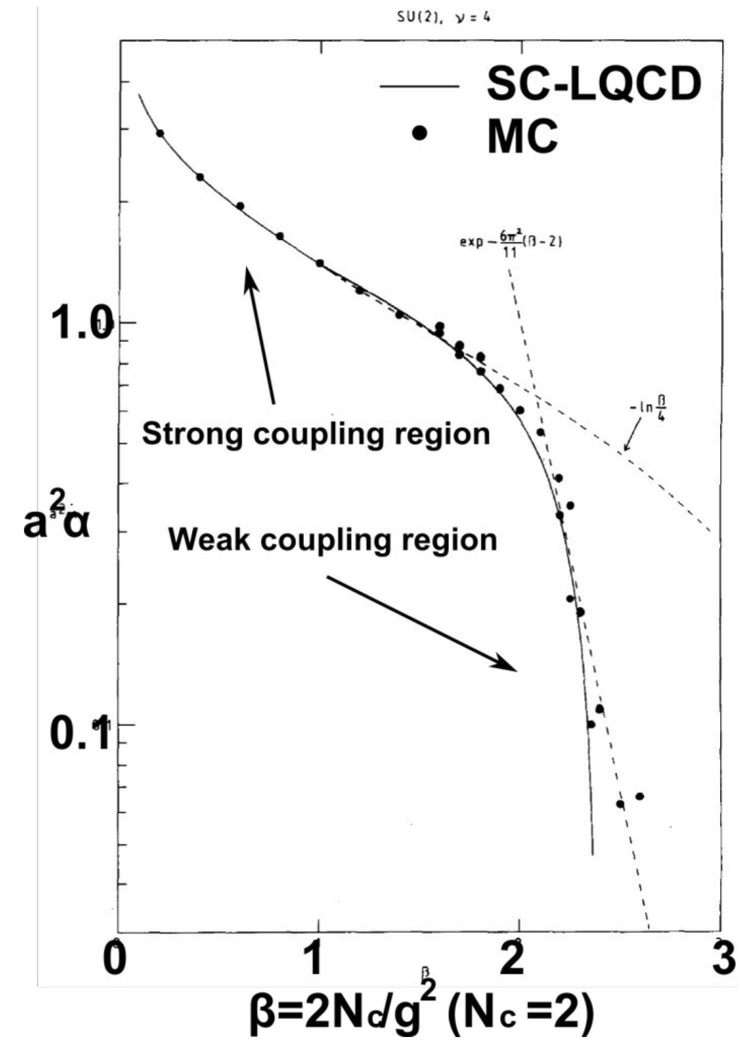
- Quarks are confined in Strong Coupling QCD

K. G. Wilson, PRD10(1974),2445
M. Creutz, PRD21(1980), 2308.
G. Munster, (1980, 1981)

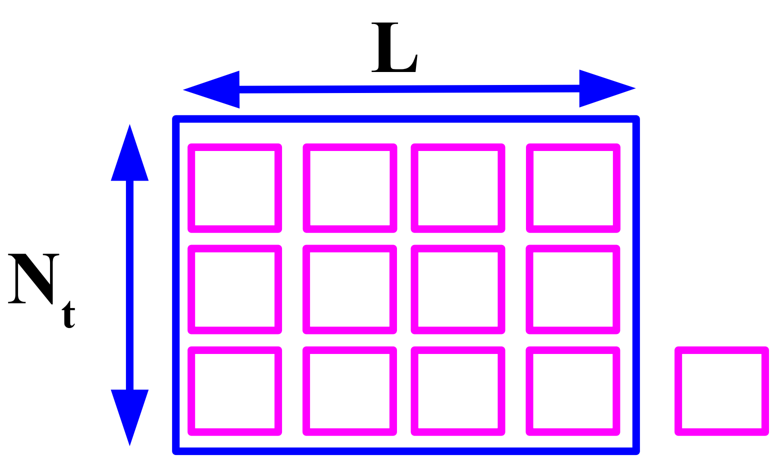
  - Strong Coupling Limit (SCL)
    → Fill Wilson Loop with Min. # of Plaquettes
    → Area Law (Wilson, 1974)

$$S_{\mathrm{LQCD}} = -\frac{1}{g^2}\sum_{\Box} \mathrm{tr}\left[U_{\Box} + U_{\Box}^{\dagger}\right]$$

  - Smooth Transition from SCL to pQCD in MC (Creutz, 1980; Munster 1980)



Munster, '80



$\Box = 1/N_c\, g^2$

Fermions on the Lattice

- **Fermion action (Euclidean)**

$$S_{q,cont}=\int d^4x\,\bar{q}(-i\gamma_\mu D_\mu+m)q\,,\quad D_\mu=\partial_\mu+iA_\mu$$

- **格子上の action → Link 変数の利用**

$$S_{F,\mathrm{lat}}=a^4\sum_x\left[\sum_\mu\frac{\left(\bar{q}(x)\Gamma_\mu U_{x,\mu}q(x+\hat{\mu})-\bar{q}(x+\hat{\mu})\Gamma_\mu U^+_{x,\mu}q(x)\right)}{2a}+m\bar{q}(x)q(x)\right]$$

  - **q, U の変換性からゲージ不変**

$$q(x)\to V(x)q(x),\quad U_{x,\mu}\to V(x)U_{x,\mu}V^+(x+\hat{\mu})$$

  - **連続極限で $S_{q,\,cont}$ へ**

$$U=1+iA_\mu a\,,\quad S_{F,\mathrm{lat}}\to a^4\sum_x\bar{q}(x)\left[\Gamma_\mu\left(\frac{\partial}{\partial x_\mu}+iA_\mu q\right)+m\right]q(x)$$

$$\Gamma_\mu=-i\gamma_\mu$$

これ以降、Γ をγ , $x_4\to x_0$ とします。

Fermions on the Lattice (cont.)

- 一見よさそうだが、問題点が ... → ダブラー
  - 自由場の場合、Fermion の hopping matrix を Fourier 変換すると

$$D = i\Gamma_\mu \frac{\sin(p_\mu a)}{a} \quad (p_\mu = 2\pi n_\mu / La, n_\mu = 0, 1, \ldots, L-1)$$

  - (3+1) 次元格子上で、D は 16 回 0 となる。($p_\mu = 0, \pi/a$)
    → 低エネルギーで現れる Fermion の種類が 16 倍増える。
- Nielsen-Ninomiya の定理

「適当な仮定 (平行移動不変性、カイラル対称性、局所性、エルミート性、双線形性) を満たす格子 Fermion にはダブラーが存在」

- 解決方法
  - Wilson Fermion : a → 0 でダブラーが無限に重くなるように 2 階微分に対応する項を加える。(カイラル対称性がない)
  - Domain wall Fermion, Overlap Fermion, ...
  - Staggered (Kogut-Susskind) Fermion

Staggered Fermion

- **Staggered Fermion: Spinor 構造・ γ 行列を数因子 η で表せる。**

$$q=\gamma_0^{x_0}\gamma_1^{x_1}\gamma_2^{x_2}\gamma_3^{x_3}\chi$$

$$\to\bar{q}(x)\gamma_\mu q(x+\hat{\mu})=\bar{\chi}(x)\gamma_3^{x_3}\gamma_2^{x_2}\gamma_1^{x_1}\gamma_0^{x_0}\gamma_\mu\gamma_0^{x_0}\cdots\gamma_\mu^{x_\mu+1}\cdots\gamma_3^{x_3}\chi(x+\hat{\mu})$$

$$=\eta_\mu(x)\bar{\chi}(x)\gamma_3^{x_3}\cdots\gamma_{\mu+1}^{x_{\mu+1}}\gamma_\mu^{2x_\mu+2}\gamma_{\mu+1}^{x_{\mu+1}}\cdots\gamma_3^{x_3}\chi(x+\hat{\mu})=\eta_\mu(x)\bar{\chi}(x)\chi(x+\hat{\mu})$$

$$\eta_\mu(x)=(-1)^{x_0+x_1+\cdots+x_{\mu-1}}$$

  - **Lattice action with staggered Fermion**

$$S_F=\frac{1}{2}\sum_{x,\mu}\eta_\mu(x)\left[\bar{\chi}_x U_{x,\mu}\chi_{x+\hat{\mu}}-\bar{\chi}_{x+\hat{\mu}}U^+_{x,\mu}\chi_x\right]+\sum_x m\bar{\chi}_x\chi_x$$

Fermion の 4 成分が全て等価。1 成分のみを考えてよい。
→ 16 個のダブラーが、(Dirac Fermion で)4 つのダブラーとなる。

  - **カイラル変換 : χ, χ^{bar} について同じ、隣り合った χ で逆の位相**

$$\chi_x\to\exp(i\theta\varepsilon(x))\chi_x,\quad \bar{\chi}_x\to\exp(i\theta\varepsilon(x))\bar{\chi}_x,\quad \varepsilon(x)=(-1)^{x_0+x_1+x_2+x_3}$$

→ 厳密なカイラル対称性をもつ

Lattice QCD with staggered Fermion

- Lattice QCD action with (unrooted) staggered Fermion

$$S_{LQCD} = S_F + S_G$$

$$S_F = \frac{1}{2} \sum_{x,\mu} \eta_\mu(x) \left[\bar{\chi}_x U_{x,\mu} \chi_{x+\hat{\mu}} - \bar{\chi}_{x+\hat{\mu}} U^+_{x,\mu} \chi_x \right] + \sum_x m \bar{\chi}_x \chi_x$$

$$S_G = -\frac{1}{g^2} \sum_{plaq.} \mathrm{tr} \left[U_P + U_P^+ \right]$$

- Spinor 構造が simple(無い) → 解析的･数値的な計算が簡単
- *m*=0 (chiral limit) で厳密な chiral 対称性をもつ
  → カイラル相転移の議論が可能
- 連続領域 (g → 0, a → 0) では N_f=4 だが、有限の *a* ではフレーバー対称性は破れている。
- Chiral anomaly ($(U(1)_A)$) については controversial

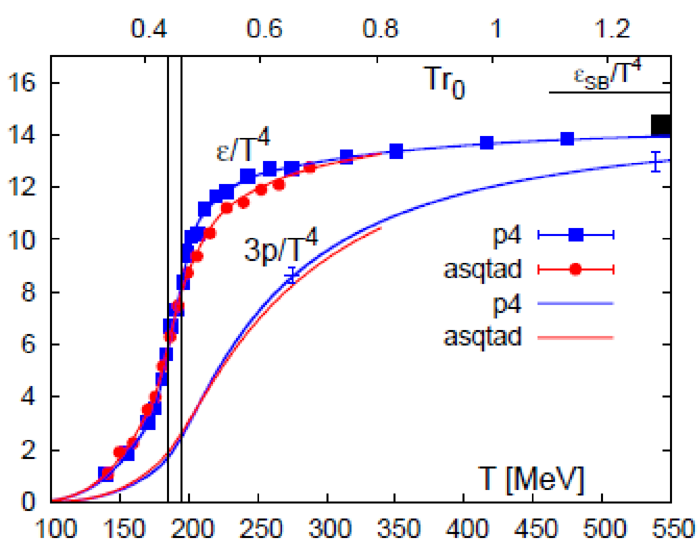
Monte-Carlo simulation in Lattice QCD

- 分配関数 (or 生成汎関数)

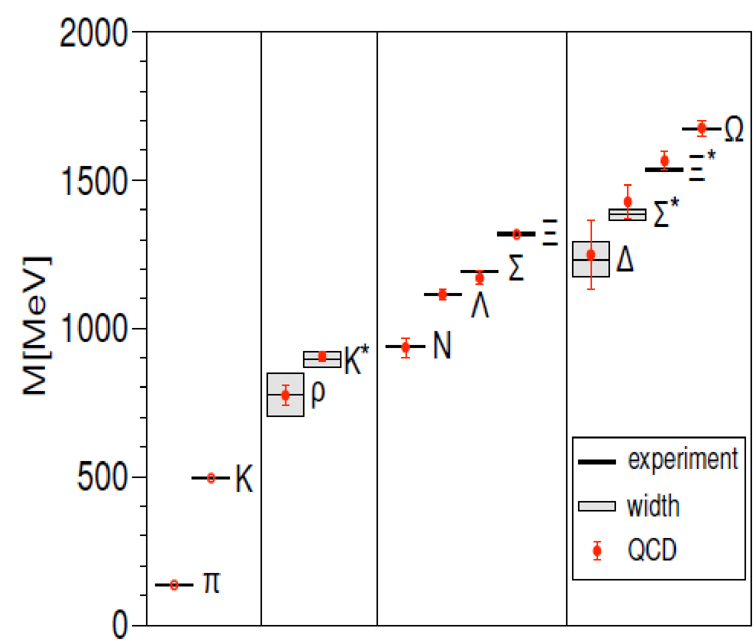
$$S=S_G(U)+\bar{q}\,D\,q\,,\quad Z[J]=\int DU \det D(U)\exp[-S_G(U)+J\hat{O}]$$

$$\to\langle O\rangle=\frac{\int DU \det D(U)\,O(U)\exp[-S_G(U)]}{\int DU \det D(U)\exp[-S_G(U)]}=\frac{\delta Z[J]}{\delta J}$$

- **Monte-Carlo 法では、通常先に Fermion determinant を評価し、リンク変数の配位を MC 法で求める。クォークを含む演算子の場合には、propagator をあらわに評価。**



Hot QCD (2009)



BMW collaboration, Sceience 322(2008)1224

格子上の場の理論 *Short Summary*

- 格子 QCD
  - リンク変数の導入により、完全なゲージ対称性を保持。
  - グルーオン作用 : Plaquette (プラケット) 作用 (or its improved ver.)
    → 連続極限 ($a \to 0$) で連続理論のゲージ作用
  - クォーク作用 : リンク変数を用いてゲージ対称性を保てる。
- Monte-Carlo simulation
  - 非摂動論的 QCD を厳密に解く第一原理計算。
  - 大きな成功 : カラーの閉じ込め、ハドロン質量、QCD 相転移 (μ=0)
  - カイラル対称性には多少の問題あり
    - Staggered fermion: Fast, but ugly (N_f=4 → quarter root, anomaly, …)
    - Wilson fermion: Explicit chiral symmetry breaking at finite *a*.
    - DW/Overlap fermion: large numerical cost.
- 有限密度での格子 QCD MC simulation は残された大きな問題。
  - 「大学院生や postdoc に与えてはいけないテーマ」(青木さん)

Monte-Carlo Integral: Importance Sampling

- **Metropolis sampling**
  = One of the typical (popular) method of importance sampling

$$P_{B\to A} = 1$$

Config. A		Config. B
$S_{eff}(A)$	$\longleftarrow$ $\longrightarrow$	$S_{eff}(B)$

$$S_{eff}(A) < S_{eff}(B)$$

$$P_{A\to B} = \exp[S_{eff}(A) - S_{eff}(B)]$$

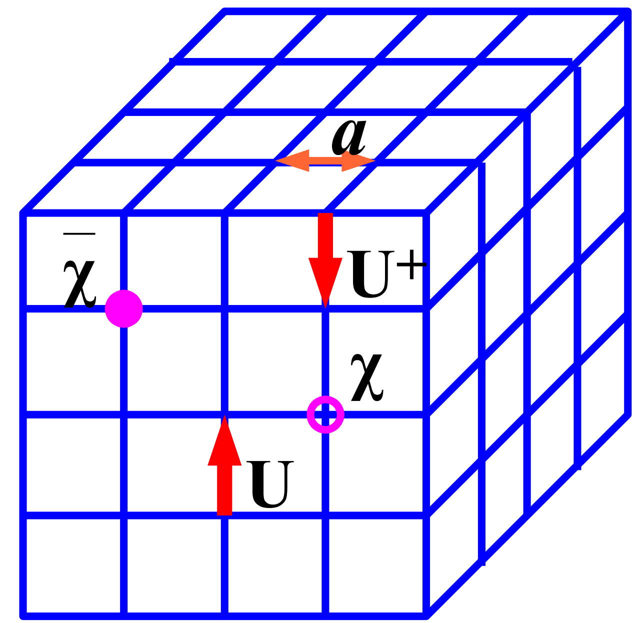
- **Trial prob.:** $P^{try}_{A\to B} = P^{try}_{B\to A}$ **(detailed balance)**
- **Pickup prob.: According to** S_{eff}.
- **In equilibrium,** $P(A)\, P_{A\to B} = P(B)\, P_{B\to A} \to P(A) \propto \exp[-S_{eff}(A)]$

QCD phase diagram in strong-coupling lattice QCD

Lattice QCD

- **Space-time discretization of fields**
- **Quarks = Grassmann number on sites**

$$\chi_i \chi_j = -\chi_j \chi_i, \quad \int d\chi\, 1 = 0, \quad \int d\chi\, \chi = 1$$
$$\rightarrow \int d\chi_1 d\chi_2 \cdots d\bar{\chi}_1 d\bar{\chi}_2 \cdots \exp(\bar{\chi} D \chi) = det(D)$$



- **Gluons → Link variable**

$$U_\mu(x) = \exp\left[ig \int_x^{x+\hat{\mu}} dx\, A(x) \right] \sim \exp(i g A_\mu)$$

$$\int dU\, U_{ab} = 0, \quad \int dU\, U_{ab} U^+_{cd} = \delta_{ad}\delta_{bc}/N_c, \quad \int dU\, U_{ab} U_{cd} U_{ef} = \varepsilon_{ace}\varepsilon_{bdf}/N_c!$$

- **Gauge transf.**

$$\chi(x) \rightarrow V(x)\chi(x), \quad \bar{\chi}(x) \rightarrow \bar{\chi}(x) V^+(x),$$
$$U_\mu(x) \rightarrow V(x) U_\mu(x) V(x+\hat{\mu})$$
$$\bar{\chi}(x) U_\mu(x) \chi(x+\hat{\mu}) = \text{invariant}$$

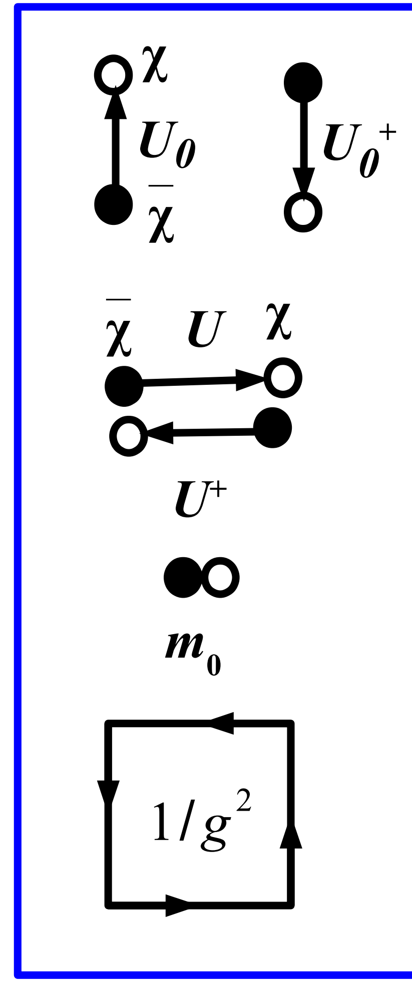
Lattice spacing = a
→ Lattice unit: a=1

Lattice QCD action

- **Lattice QCD action (unrooted staggered fermion)**

$$L=\frac{1}{2}\sum_x\left[\bar{\chi}_x U_0(x)e^{\mu}\chi_{x+\hat{0}}-\bar{\chi}_{x+\hat{0}}U_0^{+}(x)e^{-\mu}\chi_x\right]$$
$$+\frac{1}{2}\sum_{x,j}\eta_j(x)\underline{\left[\bar{\chi}_x U_j(x)\chi_{x+\hat{j}}-\bar{\chi}_{x+\hat{j}}U_j^{+}(x)\chi_x\right]}$$
$$\to \chi\,(\partial + i\,gA)\,\chi$$
$$+m_0\sum_x\bar{\chi}_x\chi_x$$
$$+\frac{2N_c}{g^2}\sum_{plaq.}\left[1-\frac{1}{N_c}\mathrm{Re\,tr}\,\underline{U_{\mu\nu}(x)}\right]$$

Stokes theorem
$\to$ rotation



- **Staggered sign factor (~ γ matrix)**
  $\eta_j(x)=(-1)^{**}(x_0+\ldots+x_{j-1})$
- **Chiral transf.**
  $\chi_x \to \exp[i\,\theta\varepsilon(x)]\,\chi_x$, $\varepsilon(x)=(-1)^{**}(x_0+x_1+x_2+x_3)$

χ quark (Grassmann #)
U link ~ exp(igA)

Sign problem in lattice QCD

- **Fermion determinant (= stat. weight of MC integral) becomes complex at finite μ in LQCD.**

$$Z=\int D[U,q,\bar{q}]\ \exp\left(-\bar{q}D(\mu,U)q-S_G(U)\right)$$
$$=\int D[U]\ \mathrm{Det}\left(D(\mu,U)\right)\ \exp\left(-S_G(U)\right)$$

$$\boxed{\begin{array}{r}\left[\gamma_5 D(\mu)\gamma_5\right]^{+}=D(-\mu^*)\rightarrow\left[\mathrm{Det}\left(D(\mu)\right)\right]^*=\mathrm{Det}\left(D(-\mu^*)\right)\\ \left(\gamma_5\ \ \text{hermiticity}\right)\end{array}}$$

  - **Note: Euclidean $D=\gamma_\mu D_\mu+m-\mu\gamma_0$ (γ = Hermite, D_μ= anti-Hermite)**
  - **Fermion det. (Det D) is real for zero μ (and pure imag. μ)**
  - **Fermion det. is complex for finite real μ.**

- **Approximate methods:**
  - **Taylor expansion, Imag. μ , Canonical, Re-weighting, Fugacity expansion, Histogram method, Complex Langevin, Strong-coupling lattice QCD**

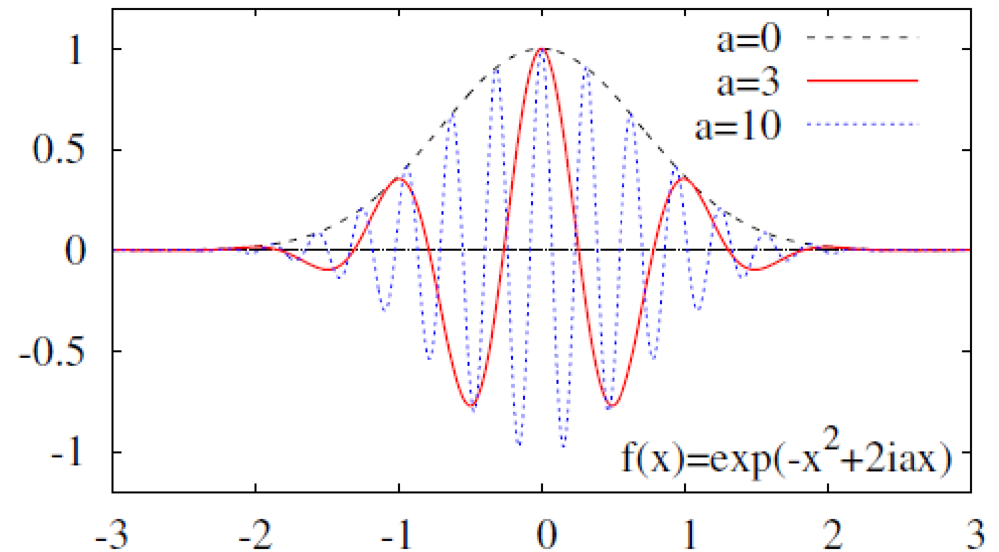
Sign Problem

- **Monte-Carlo integral of oscillating function**

$$Z=\int dx \exp(-x^2+2iax)=\sqrt{\pi}\exp(-a^2)$$

$$\langle O\rangle=\frac{1}{Z}\int dx\, O(x)\, e^{-x^2+2iax}$$

Easy problem for human is not necessarily easy for computers.



- **Complex phase appears from fluctuations of H and N.**
  de Forcrand

$$Z=\sum\langle\psi|\exp[-(H-\mu N)/T]|\psi\rangle=\sum\prod\langle\psi_\tau|\exp[-(H-\mu N)/(N_\tau T)]|\psi_{\tau+1}\rangle$$

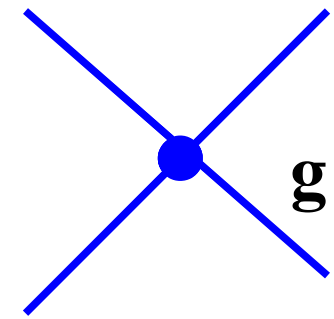
→ **Description based on "Hadronic" (color singlet) action would be helpful to reduce fluctuations.**

→ **Strong coupling lattice QCD**

Sign Problem (cont.)

- **Generic problem in quantum many-body problems**
  - **Example: Euclid action of interacting Fermions**

$$S=\sum_{x,y} \bar{\psi}_x D_{x,y} \psi_y + g \sum_x \left(\bar{\psi}\psi\right)_x \left(\bar{\psi}\psi\right)_x$$



  - **Bosonization and MC integral (g>0 → repulsive)**

$$\exp\left(-g M_x M_x\right)=\int d\sigma_x \exp\left(-g\sigma_x^2-2igσ_x M_x\right) \quad \left(M_x=\left(\bar{\psi}\psi\right)_x\right)$$

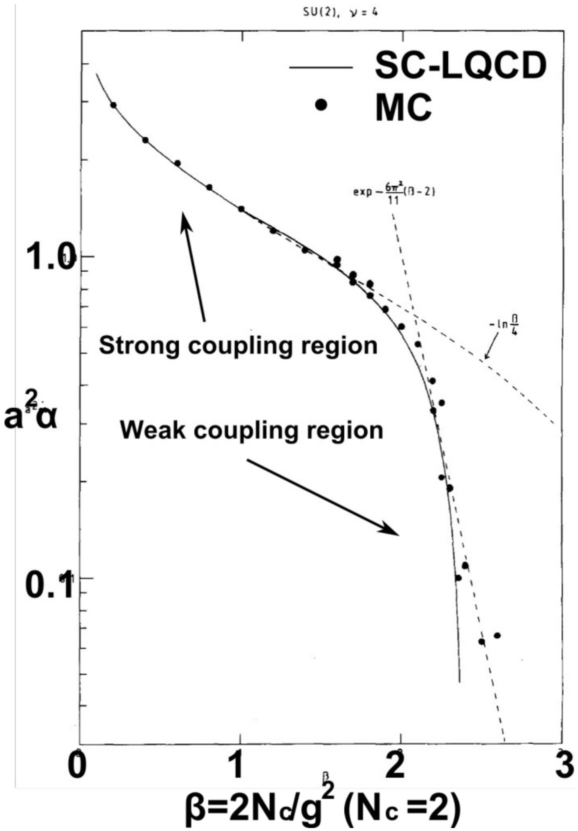
$$Z=\int D[\psi,\bar{\psi},\sigma]\exp\left[-\bar{\psi}\left(D+2ig\sigma\right)\psi-g\sum_x \sigma_x^2\right]$$

$$=\int D[\sigma]\ \underline{\mathrm{Det}\left(D+2ig\sigma\right)}\exp\left[-g\sum_x \sigma_x^2\right]$$

complex Fermion det.
→ complex stat. weight
→ sign problem

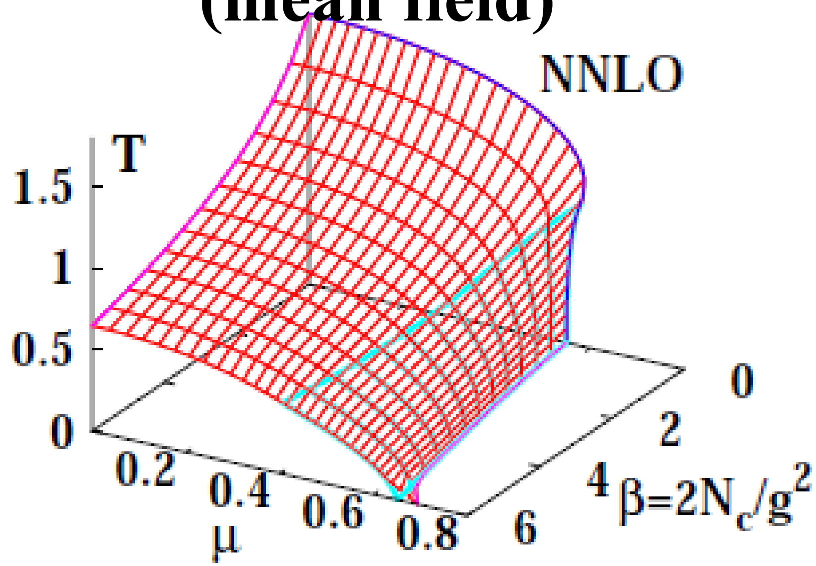
Strong Coupling Lattice QCD

Pure YM



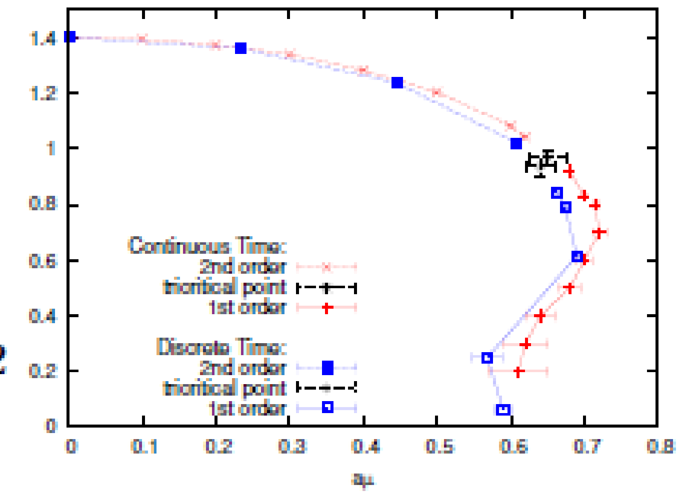
Wilson ('74), Creutz ('80), Munster ('80, '81), Lottini, Philipsen, Langelage's ('11)

Phase diagram (mean field)



Kawamoto ('80), Kawamoto, Smit ('81), Damagaard, Hochberg, Kawamoto ('85), Ilgenfritz, Kripfganz ('85), Bilic, Karsch, Redlich ('92), Fukushima ('03); Nishida ('03), Kawamoto, Miura, AO, Ohnuma ('07). Miura, Nakano, AO, Kawamoto ('09), Nakano, Miura, AO ('10)

Fluctuations



Mutter, Karsch ('89), de Forcrand, Fromm ('10), de Forcrand, Unger ('11), AO, Ichihara, Nakano ('12), Ichihara, Nakano, AO ('14), de Forcrand, Langelage, Philipsen, Unger ('14)

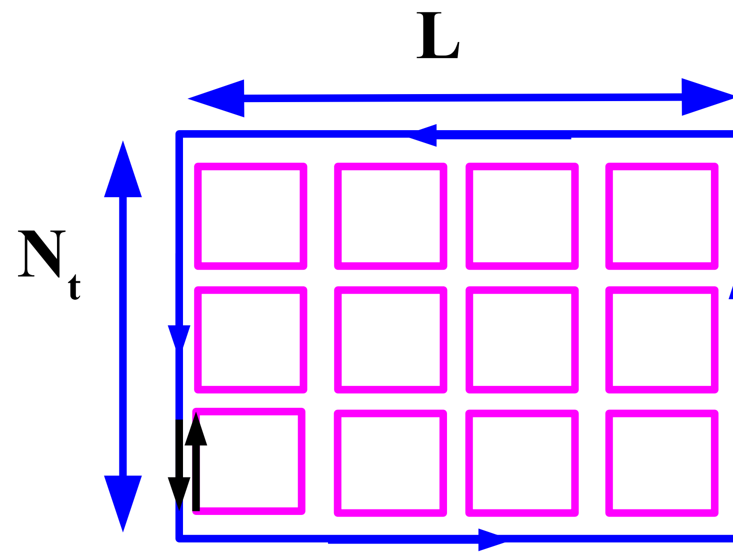
Area Law

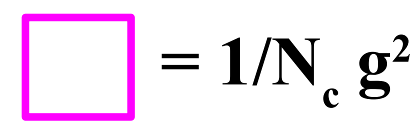
Wilson ('74), Creutz ('80), Munster ('80, '81)

- **Wilson loop in pure Yang-Mills theory**

$$\langle W(C=L\times N_\tau)\rangle$$
$$=\frac{1}{Z}\int DU\, W(C)\exp\left[\frac{1}{g^2}\sum_P \mathrm{tr}\left(U_P+U_P^+\right)\right]$$
$$=\exp\left(-V(L)N_\tau\right)$$

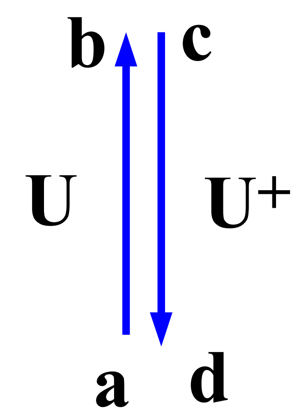
V(L)= heavy-qq pot.





- **One-link integral**

$$\int dU\, U_{ab} U^+_{cd} = \frac{1}{N_c}\delta_{ad}\delta_{bc}$$



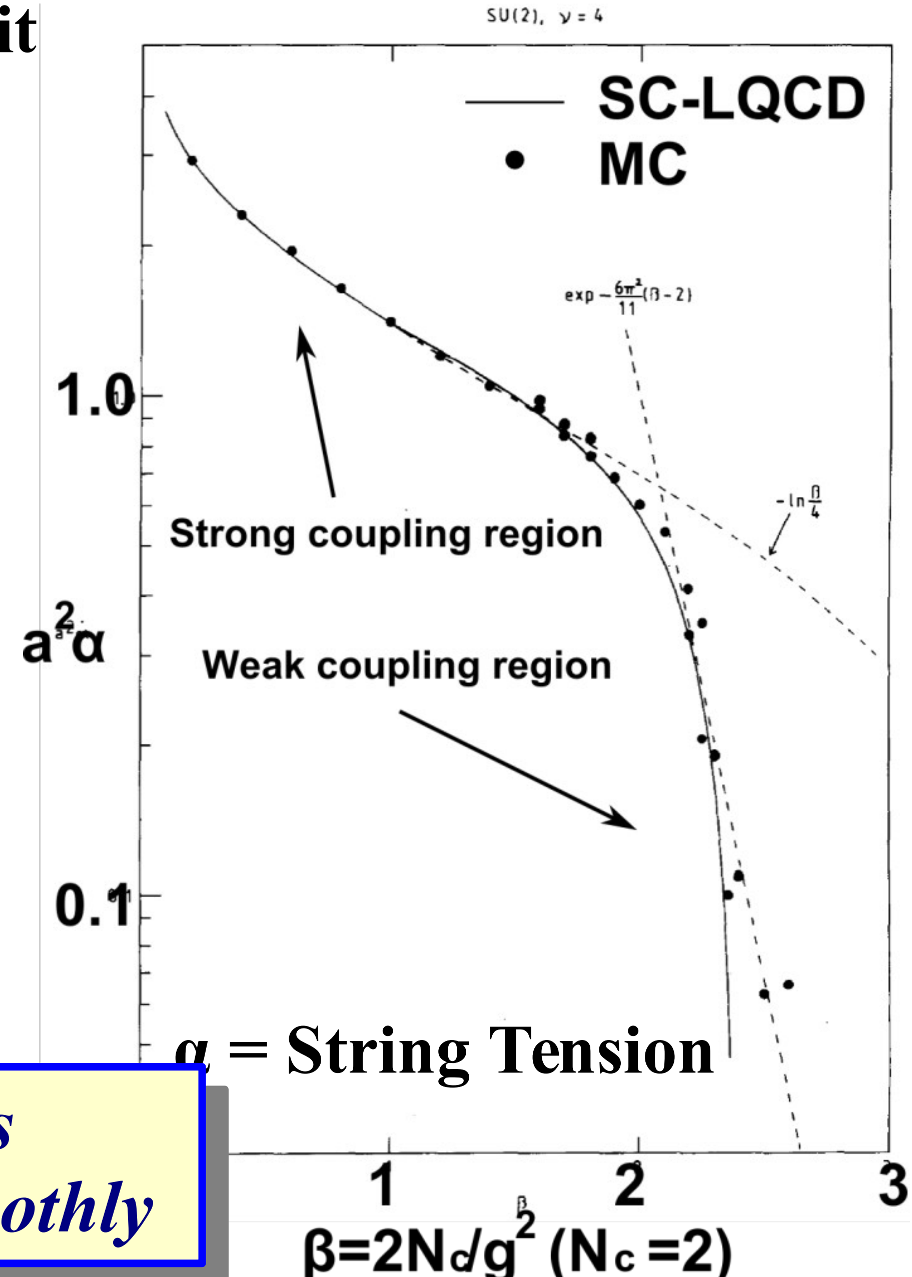
- **In the strong coupling limit**

$$\langle W(C)\rangle = N\left(\frac{1}{g^2 N}\right)^{L N_\tau} \rightarrow V(L)=L\log\left(g^2 N\right)$$

Linear potential between heavy-quarks → Confinement (Wilson, 1974)

Area Law

- **Area law in the strong coupling limit**
  Wilson ('74)
- **Verification of the area law in Lattice MC simulation**
  Creutz ('80)
- **Strong coupling expansion to higher orders**
  Munster ('80, '81),
  Lottini, Philipsen, Langelage ('11)
- **Weak coupling region**
  $\rightarrow g^2/4\pi = 1 / \beta_0 \log (q^2/\Lambda^2)$
  $\rightarrow a \sim 1/q \sim \exp(2\pi/g^2\beta_0)/\Lambda$



Strong coupling expansion connects SCL and Weak coupling region smoothly

Strong Coupling Lattice QCD

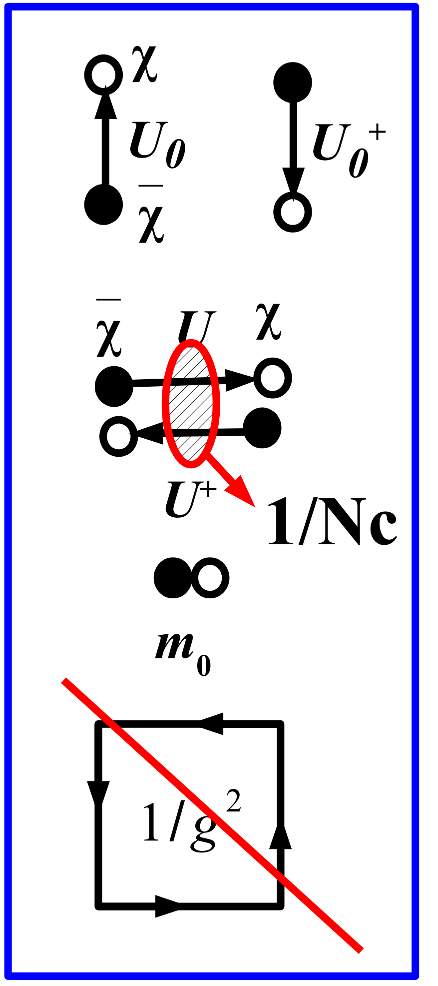
- **Strong coupling limit**
  Damgaard, Kawamoto, Shigemoto ('84)

$$S_{\mathrm{SCL}} = S_F^{(t)} - \frac{1}{4N_c} \sum_{x,j} M_x M_{x+\hat{j}} + m_0 \sum_x M_x$$

$$\left(M_x = \bar{\chi}_x \chi_x \right)$$

  - **Integrate out spatial links using one-link formula, and pick up diagrams with min. quark numbers.**

$$\int dU\, U_{ab} U^{+}_{cd} = \delta_{ad} \delta_{bc} / N_c$$



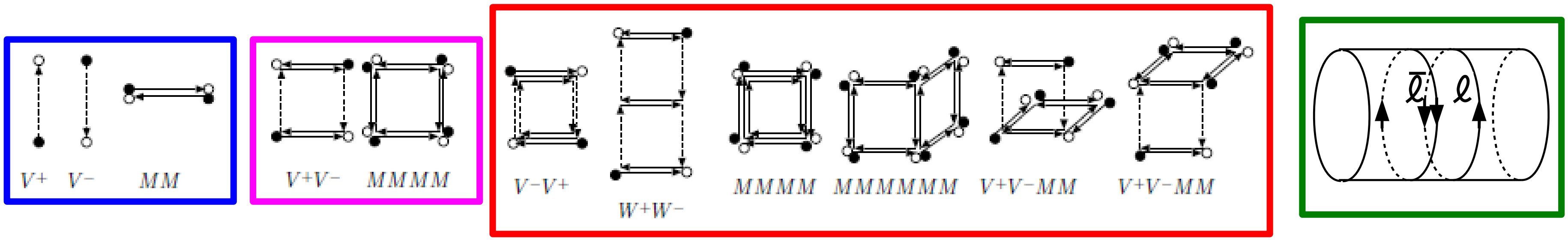
Lattice QCD in SCL
→ Fermion action with nearest neighbor four Fermi interaction

Finite Coupling Effects

- **Effective Action with finite coupling corrections**
  Integral of exp(-S_G) over spatial links with exp(-S_F) weight → $S_{\rm eff}$

$$S_{\rm eff} = S_{\rm SCL} - \log\langle \exp(-S_G)\rangle = S_{\rm SCL} - \sum_{n=1} \frac{(-1)^n}{n!}\langle S_G^n\rangle_c$$

$\langle S_G{}^n\rangle_c$=Cumulant (connected diagram contr.) *c.f. R.Kubo('62)*



$$S_{\rm eff} = \frac{1}{2}\sum_x (V_x^+ - V_x^-) - \frac{b_\sigma}{2d}\sum_{x,j>0}[MM]_{j,x}$$

SCL (Kawamoto-Smit, '81)

$$+\frac{1}{2}\frac{\beta_\tau}{2d}\sum_{x,j>0}[V^+V^- + V^-V^+]_{j,x} - \frac{1}{2}\frac{\beta_s}{d(d-1)}\sum_{x,j>0,k>0,k\neq j}[MMMM]_{jk,x}$$

NLO (Faldt-Petersson, '86)

$$-\frac{\beta_{\tau\tau}}{2d}\sum_{x,j>0}[W^+W^- + W^-W^+]_{j,x} - \frac{\beta_{ss}}{4d(d-1)(d-2)}\sum_{\substack{x,j>0,|k|>0,|l|>0\\|k|\neq j,|l|\neq j,|l|\neq|k|}}[MMMM]_{jk,x}[MM]_{j,x+\hat{l}}$$

$$+\frac{\beta_{\tau s}}{8d(d-1)}\sum_{x,j>0,|k|\neq j}[V^+V^- + V^-V^+]_{j,x}\left([MM]_{j,x+\hat{k}} + [MM]_{j,x+\hat{k}+\hat{0}}\right)$$

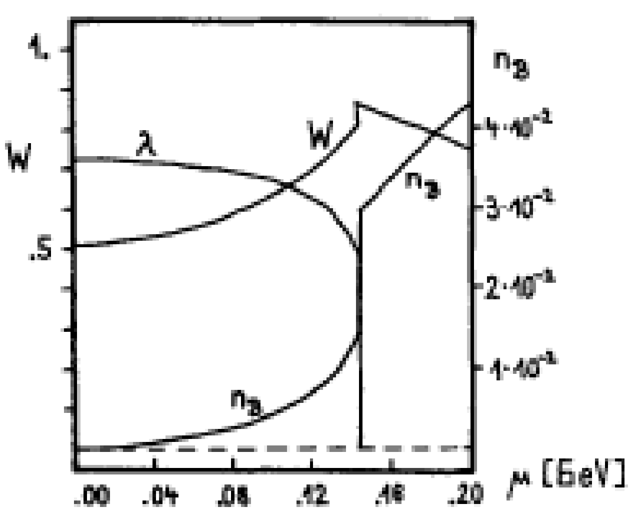
NNLO (Nakano, Miura, AO, '09)

$$-\left(\frac{1}{g^2N_c}\right)^{N_\tau} N_c^2 \sum_{\mathbf{x},j>0}\left(\bar{P}_{\mathbf{x}}P_{\mathbf{x}+\hat{j}} + h.c.\right)$$

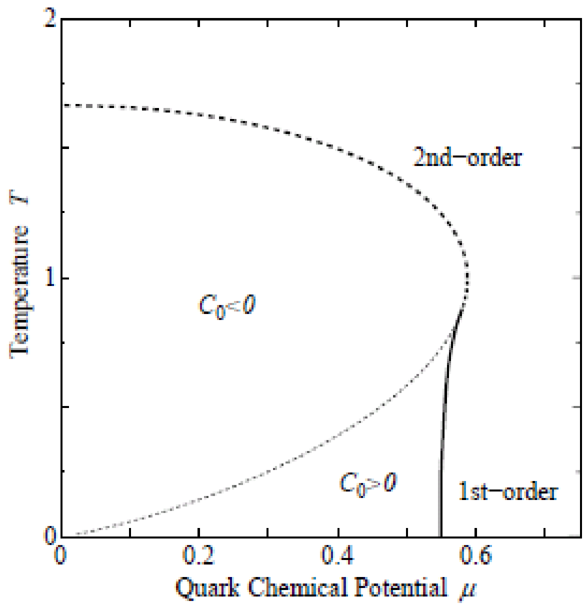
Polyakov loop (Gocksch, Ogilvie ('85), Fukushima ('04)
Nakano, Miua, AO ('11))

Phase diagram in SC-LQCD (mean field)

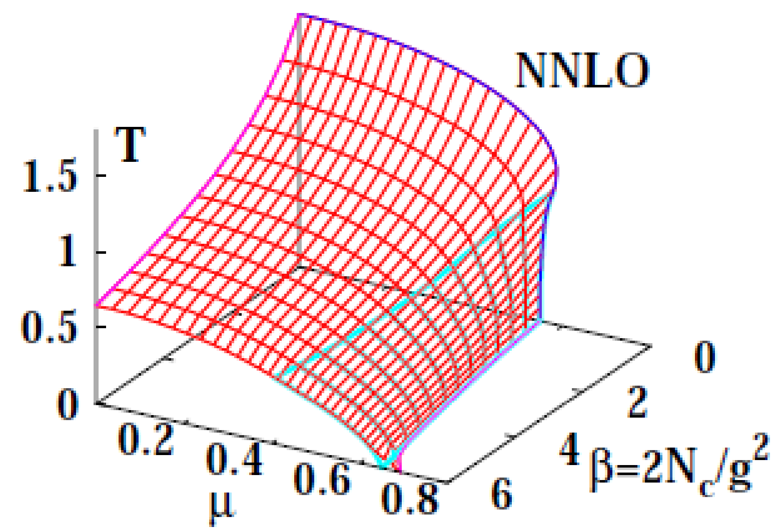
- **"Standard" simple procedure in Fermion many-body problem**
  - **Bosonize interaction term (Hubbard-Stratonovich transformation)**
  - **Mean field approximation (constant auxiliary field)**
  - **Fermion & temporal link integral**
    Damgaard, Kawamoto, Shigemoto ('84); Ilgenfritz, Kripfganz ('85); Faldt, Petersson ('86); Bilic, Karsch, Redlich ('92); Fukushima ('04); Nishida ('04); Miura, Nakano, AO, Kawamoto ('09); Nakano, Miura, AO ('10, '11)



Ilgenfritz, Kripfganz ('85)



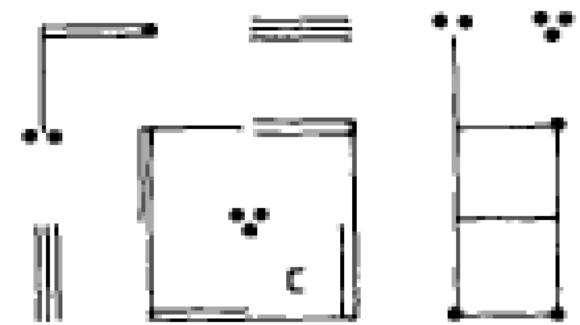
Fukushima ('04)



AO, Miura, Nakano, Kawamoto ('09)

SC-LQCD with Fluctuations

- **Monomer-Dimer-Polymer (MDP) simulation**
  Mutter, Karsch ('89), de Forcrand, Fromm ('10), de Forcrand, Unger ('11)



  - **Integrating out all links**
    → Z= weight sumof monomer, dimer, polymer configurations

$$Z(m,\mu)=\sum_{\{n_x,n_b,C_B\}}\prod_b\frac{(N_c-n_b)!}{N_c!n_b!}\prod_x\frac{N_c!}{n_x!}(2m)^{n_x}\prod_{C_B}w(C_B)\qquad w(C_B,\pm)=\varepsilon(C_B)\exp(\pm 3\ell L_t\mu)$$

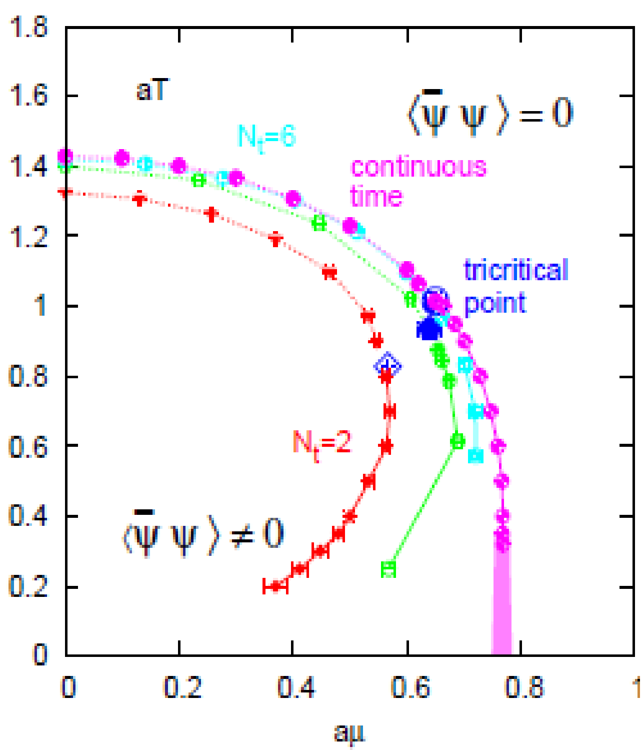
- **Auxiliary Field Monte-Carlo (AFMC) method**
  Ichihara, AO, Nakano ('14)
  - **Bosonize the effective action, and MC integral over aux. field.**

$$S_{\mathrm{eff}}=S_F^{(t)}+\sum_x m_x M_x+\frac{L^3}{4N_c}\sum_{\boldsymbol{k},\tau}f(\boldsymbol{k})\left[|\sigma_{\boldsymbol{k},\tau}|^2+|\pi_{\boldsymbol{k},\tau}|^2\right]$$

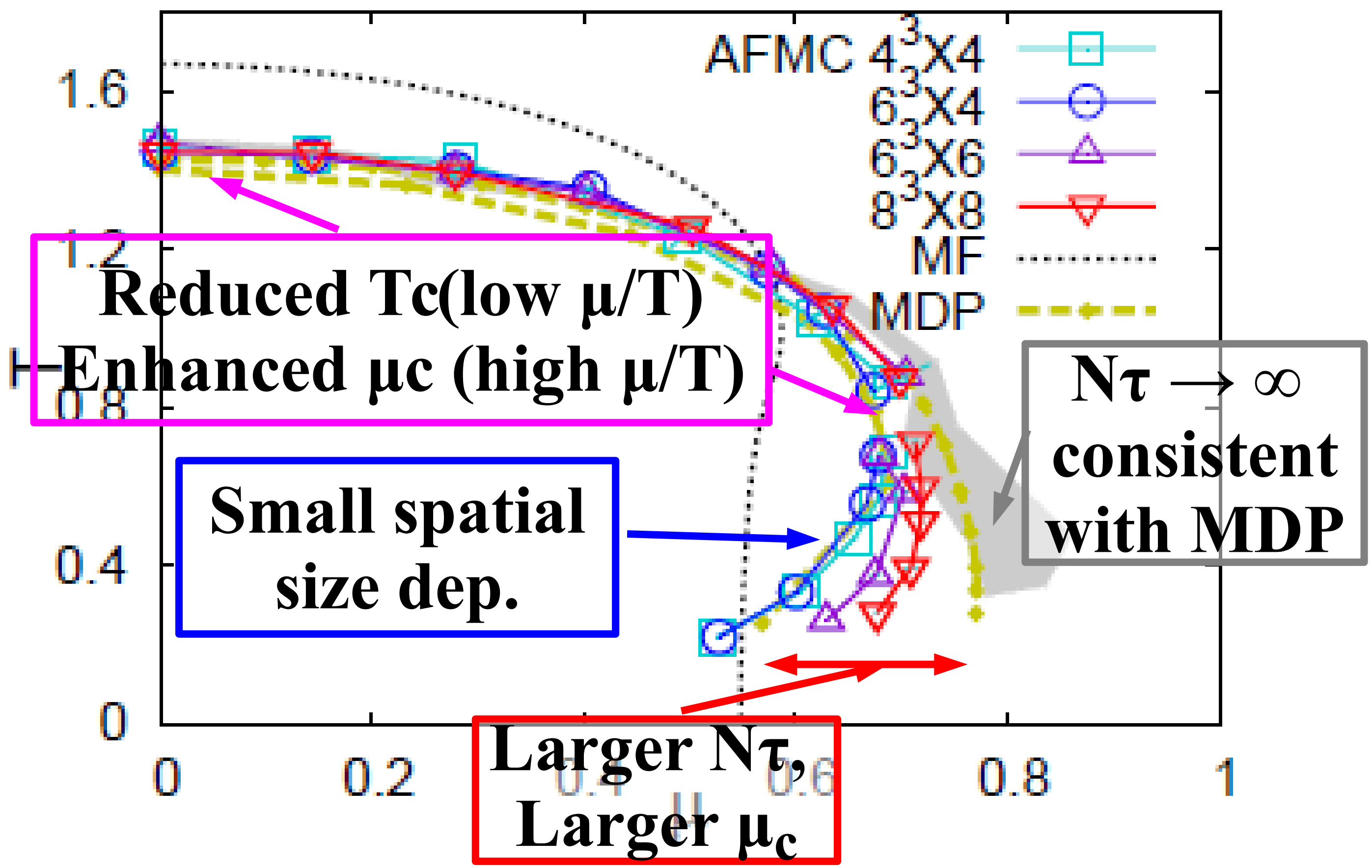
$$m_x=m_0+\frac{1}{4N_c}\sum_j(\sigma+i\varepsilon\pi)_{x\pm\hat{j}},\quad f(\boldsymbol{k})=\sum_j\cos k_j,\quad \varepsilon=(-1)^{x_0+x_1+x_2+x_3}$$

Phase diagram

- **Phase diagrams in two independent methods (MDP & AFMC) agree with each other in the strong coupling limit.**
  → SCL phase diagram is determined !



de Forcrand, Fromm ('10), de Forcrand, Langelage, Philipsen, Unger ('14)



Ichihara, AO, Nakano ('14)

レポート問題

- あと2問だします。4問中、2問以上解いて提出。〆切は1月末。
  - ボソン化した NJL 模型の作用から出発して、ゼロ温度（ $T=0$ ）での有効ポテンシャルを求めよ。
    余裕があれば、有限温度・有限密度 (有限化学ポテンシャル) での有効ポテンシャルを構成子クォーク質量で 2 次まで展開し、カイラル極限で 2 次相転移線が $T^2+\mu_B{}^2/3\pi^2 = T_c{}^2$ で与えられることを示せ。
  - リンク積分を利用して、Wilson ループの期待値を強結合領域で求めよ。
    余裕があれば、強結合極限での結果に加えて、$1/g^2$ 補正がどのように与えられるか評価せよ。