原子核基礎論B

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1. 核力・特に非中心力や 3 体力 (1 回)
2. 原子核構造を記述するための種々の模型の最近の進展 (2 回)
3. 最近の中性子過剰核の物理の最近の進展 (2 回)
4. 原子核構造における異なる状態の混合や競合 (2 回)

板垣

5. 高温・高密度核物質概観 (1 回) (高エネルギー重イオン衝突、コンパクト天体現象)
   → 前期の Sec. 3 と重なりが大きいのでスキップ
6. 有限温度・密度における場の理論入門 (2 回)
7. QCD 有効模型における相転移と相図 (2 回)
8. 有限温度・密度格子 QCD と符号問題 (1 回)
9. 高エネルギー重イオン衝突における輸送理論 (1 回)

大西

Sec. 6 の復習 (1)

- **分配関数とユークリッド化**

$$\mathcal{Z}(T) = \int \mathcal{D}\phi\, e^{-S_E[\phi]},\ \ S_E[\phi] = \int_0^\beta d\tau \int d^3x \mathcal{L}_E(\phi, \partial_i\phi, \partial_\tau\phi)$$

$$\mathcal{L}_E(\phi, \partial_i\phi, \partial_\tau\phi) = -\mathcal{L}(\phi, \partial_i\phi, \partial_t\phi \to i\partial_\tau\phi)$$

- **スカラー場（2行目以降、自由場 (U=0) とする）**

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - U(\phi),\ \ \mathcal{L}_E = \frac{1}{2}\partial_\mu\phi\,\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + U(\phi)$$

$$S_E = \frac{1}{2}\sum_{n,\mathbf{k}} (\omega_n^2 + \mathbf{k}^2 + m^2)\,\phi_n^*(\mathbf{k})\phi_n(\mathbf{k})$$

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S_E} = N \prod_{n,\boldsymbol{k}} \sqrt{2\pi}(\omega_n^2 + \boldsymbol{k}^2 + m^2)^{-1/2}$$

$$\Omega = \frac{T}{2}\sum_{\boldsymbol{k}} \log[\sinh(E_{\boldsymbol{k}}/T)] \to V\int \frac{d^3k}{(2\pi)^3}\left[\frac{E_{\boldsymbol{k}}}{2} + T\log(1 - e^{E_{\boldsymbol{k}}/T})\right]$$

松原和

Sec. 6 の復習 (2)

- **松原和**

$$S = T\sum_{n} g(\omega_n) = \mp i \sum_{\omega_0} \frac{\mathrm{Res}\ g(\omega_0)}{e^{i\beta\omega_0} \mp 1} \quad (\omega_n = 2\pi n T, \pi(2n+1)T)$$

- **フェルミオン (ボソンについては平均場近似)**

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - M + \gamma^0\mu^*)\psi + \mathcal{L}_\Phi,\ \ \mathcal{L}_E = \overline{\psi}D\psi + \mathcal{L}_{\mathrm{E},\Phi}$$

$$\mathcal{Z} = \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}\Phi\ e^{-\int d^4x L_E} = \int \mathcal{D}\Phi\ e^{-S_{\mathrm{eff}}(\Phi;T,\mu)} \simeq e^{-S_{\mathrm{eff}}(\Phi_{\mathrm{eq}};T,\mu)}$$

$$S_{\mathrm{eff}} = S^{(F)}_{\mathrm{eff}} + S_{E,\Phi} = -\log\det\ D + \int d^4x\mathcal{L}_{E,\Phi}$$

$$(M = m + g_\sigma\sigma, \mu^* = \mu - V_0, \gamma_\tau = i\gamma_0, D = -i\gamma_\mu\partial_\mu - \mu^*\gamma^0 + M)$$

$$\Omega_F = -T\log\det D = -\sum_k\left[\frac{T}{2}\sum_n \log((E_k-\mu)^2+\omega_n^2)\right]$$

$$= -T\sum_k \log[\cosh((E_k-\mu)/T)]$$ **松原和**

$$= -\sum_k \frac{|E_k|}{2} - \sum_{k,E_k>0} T\log(1+e^{-\beta(E_k-\mu)}) - \sum_{k,E_k<0} T\log(1+e^{-\beta(|E_k|+\mu)})$$

Spontaneous Chiral Symmetry Breaking in NJL model

Chiral Symmetry in Quantum Chromodynamics

notation: Yagi, Hatsuda, Miake

- **QCD Lagrangian**

$$\mathcal{L} = \bar{q}(i\gamma^\mu D_\mu - m) - \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$$

- **Chiral symmetry: SU(N_f)$_L$ x SU(N_f)$_R$**
  - **Left- and Right-handed quarks can rotate independently**

$$q_L = (1-\gamma_5)q/2, q_R = (1+\gamma_5)q/2 \rightarrow V_L q_L, V_R q_R$$

$$\mathcal{L}_q = \underset{\textbf{invariant}}{\underline{\bar{q}_L(i\gamma^\mu D_\mu)q_L + \bar{q}_R(i\gamma^\mu D_\mu)q_R}} - \underset{\textbf{small (for u, d)}}{\boxed{m}}(\bar{q}_L q_R + \bar{q}_R q_L)$$

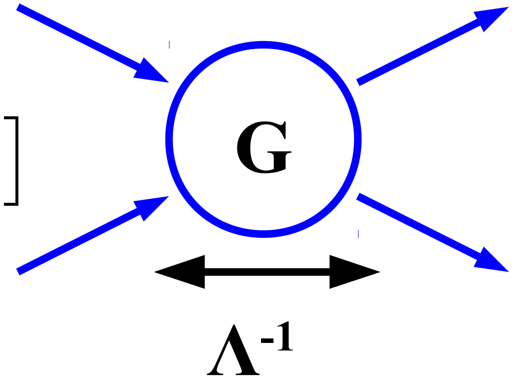
- **Chiral transf. of hadrons**

$$\sigma = \bar{q}q, \pi^a = \bar{q}i\gamma_5\tau^a q \rightarrow \begin{pmatrix} \sigma' \\ \pi' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix}$$

  - **σ (J^π=0^+) and π (J^π=0^-) mix via chiral transf. but have diff. masses.**
    → Spontaneous breaking of chiral symmetry.

Nambu-Jona-Lasinio (NJL) model

- **NJL Lagrangian**

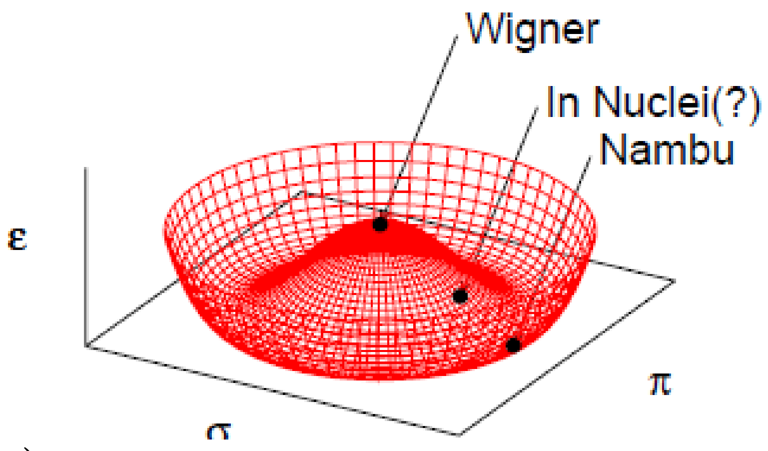


$$\mathcal{L} = \bar{q}(i\gamma^\mu \partial_\mu - m)q + \frac{G^2}{2\Lambda^2}\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5\boldsymbol{\tau}q)^2\right]$$

- **Integrating out gluons and hard quarks in QCD**
  → Effective theory of quarks
  with the same symmetry as QCD

$$S = \bar{q}q, \boldsymbol{P} = \bar{q}i\gamma_5\boldsymbol{\tau}q$$

$$\to S^2 + \boldsymbol{P}^2 = \text{inv. under chiral transf.}$$



- **Euclidean action**

$$(x_\mu)_E = (\tau = it), (\gamma_\mu)_E = (\gamma_4 = i\gamma^0, \boldsymbol{\gamma})$$

$$\mathcal{L} = \bar{q}(-i\gamma_\mu \partial_\mu + m)q - \frac{G^2}{2\Lambda^2}\left[(\bar{q}q)^2 + (\bar{q}i\gamma_5\boldsymbol{\tau}q)^2\right]$$

Nambu, Jona-Lasinio ('61), Hatsuda, Kunihiro ('94)

Partition Function in NJL

- **Bosonization (Hubbard-Stratonovich transf.)**

$$-\frac{G^2}{2\Lambda^2}\left[(\bar{q}q)^2+(\bar{q}i\gamma_5\boldsymbol{\tau}q)^2\right]\to\frac{\Lambda^2}{2}(\sigma^2+\boldsymbol{\pi}^2)+G\bar{q}\underbrace{(\sigma+i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi})}_{\Sigma}q$$

- **Partition Function**

$$G = G_0 - G_0\Sigma G$$

$$\mathcal{Z}_{\rm NJL}=\int\mathcal{D}q\mathcal{D}\bar{q}\exp\left[-\int d^4x\mathcal{L}_{\rm NJL}\right]$$

$$=\int\mathcal{D}q\mathcal{D}\bar{q}\mathcal{D}\Sigma\exp[-\int d^4x\left[\bar{q}\underbrace{(-i\gamma\partial+m+G\Sigma)}_{D}q+\frac{\Lambda^2}{2}(\sigma^2+\boldsymbol{\pi}^2)\right]$$

$$=\int\mathcal{D}\Sigma\exp\left[-S_{\rm eff}(\Sigma;T)\right]$$

- **Effective Action**

$$S_{\rm eff}(\Sigma;T)=-\log\det D+\int d^4x\frac{\Lambda^2}{2}\left[\sigma^2(x)+\boldsymbol{\pi}^2\right]$$

Bosonization & Grassman Integral

- **Bosonization (Hubbard-Stratonovich transf.)**

$$\exp\left[\frac{G^2S^2}{2\Lambda^2}\right] = \int d\sigma \, \exp\left[-\frac{\Lambda^2}{2}\left(\sigma - \frac{GS}{\Lambda^2}\right)^2 + \frac{G^2S^2}{2\Lambda^2}\right]$$

$$\exp\left[\frac{G^2(P^a)^2}{2\Lambda^2}\right] = \int d\pi^a \, \exp\left[-\frac{\Lambda^2}{2}\left(\pi^a - \frac{GP^a}{\Lambda^2}\right)^2 + \frac{G^2(P^a)^2}{2\Lambda^2}\right]$$

- **Grassman number**

$$\int d\chi \cdot 1 = \text{anti-comm. const.} = 0 \, , \int d\chi \cdot \chi = \text{comm. const.} = 1$$

$$\int d\chi d\bar{\chi} \, \exp\left[\bar{\chi} A \chi\right] = \int d\chi d\bar{\chi} \, \frac{1}{N!} (\bar{\chi} A \chi)^N = \cdots = \det A$$

$$= \exp\left[-(-\log \det A)\right]$$

Bi-linear Fermion action leads to -log(det A) effective action

Fermion Determinant in Mean Field Approximation

- **Mean Field approx.+Fourier transf.→ Diagonal Fermion matrix**

$$D_{n,\mathbf{k}} = -i\omega_n\gamma^0 + \gamma\cdot\mathbf{k} + m + g_\sigma\sigma, E^* = \sqrt{\mathbf{k}^2 + M^{*2}}\,, M^* = m + g_\sigma\sigma$$

$$\to \det D_{n,\mathbf{k}} = \left[\omega_n^2 + E_{\mathbf{k}}^{*2}\right]^2$$

$$\to \det D = \prod_{n,\mathbf{k}} (\omega_n^2 + E_{\mathbf{k}}^2)^{d/2} (d_f = 4N_c N_f = \text{Fermion dof})$$

- **Effective Potential**

$$F_{\text{eff}} = \Omega/V = -\frac{T}{V}\log\mathcal{Z} = \frac{\Lambda^2}{2}\sigma^2 - \frac{T}{V}\sum_{n,\mathbf{k}}\log(\omega_n^2 + \mathbf{k}^2 + M^2)^{d_f/2}$$

$$= \frac{\Lambda^2}{2}\sigma^2 - d_f\int\frac{d^3k}{(2\pi)^3}\left[\underline{\frac{E_k}{2}} + \underline{\frac{k^2}{3E_k}\frac{1}{e^{E_k/T}+1}}\right]$$

Matsubara sum

Fermion det. → Zero point energy (ħ ω/2)+ Thermal pressure

Effective potential of NJL model

- **Effective potential (Grand pot. density)**

$$F_{\rm eff} = \frac{\Lambda^2}{2}\sigma^2 - d_f \int \frac{d^3k}{(2\pi)^3}\left[\frac{E_k}{2} + \frac{k^2}{3E_k}\frac{1}{e^{E_k/T}+1}\right]$$

Zero point energy + Thermal (particle) excitation + Aux. Fields

- **Effective potential in vacuum (T=0, μ=0) in the chiral limit (m=0)**

$$F_{\rm eff} = \frac{\Lambda^2}{2}\sigma^2 - \frac{d_f}{2}\underbrace{\int^{\Lambda}\frac{d^3k}{(2\pi)^3}E_k}_{\Lambda^4 I(x)} = \Lambda^4\left[-\frac{d_f}{2}I(x) + \frac{x^2}{2G^2}\right] \quad (x = M/\Lambda)$$

$$\frac{F_{\rm eff}}{\Lambda^4} = -\frac{d_f}{16\pi^2} + \frac{x^2}{2}\boxed{\left[\frac{1}{G^2} - \frac{1}{G_c^2}\right]} + \mathcal{O}(x^4\log x)(G_c^2 = 8\pi^2/d_f)$$

$G>G_c$ → 2nd coef. < 0 →Spontaneous Chiral Sym. Breaking

$$I(x) = \frac{1}{16\pi^2}\left[\sqrt{1+x^2}(2+x^2) - x^4\log\frac{1+\sqrt{1+x^2}}{x}\right] \simeq \frac{1}{8\pi^2}\left[1 + x^2 + \frac{x^4}{8}\left(1 + 4\log\frac{x}{2} + \mathcal{O}(x^6)\right)\right]$$

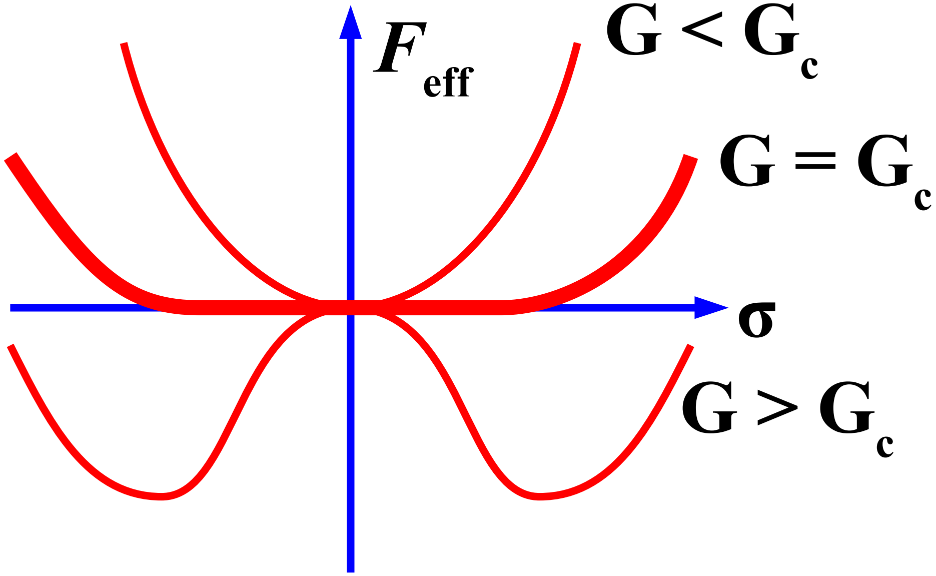
Spontaneous breaking of chiral symmetry

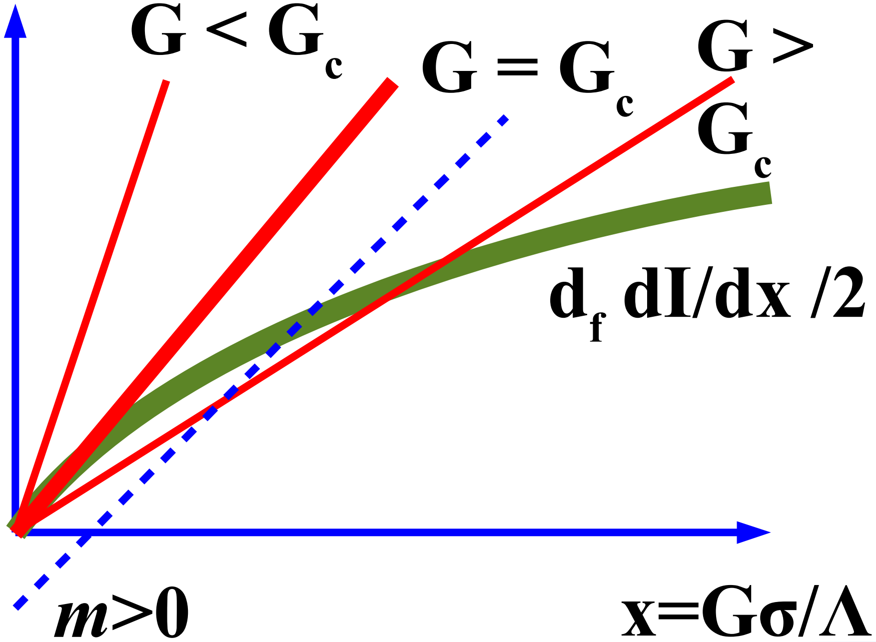
- **σ is chosen to minimize $F_{\rm eff}$ (Gap equation)**

$$\frac{1}{\Lambda^4}\frac{\partial F_{\rm eff}}{\partial x} = -\frac{d_f}{2}\frac{dI(x)}{dx} + \frac{x}{G^2} = 0$$

For $G>G_c$ → finite σ(~ $q^{\rm bar}q$) solution gives min. energy state.

If the interaction is strong enough, σ(~$q^{bar}q$) condensates and quark mass is generate.(Nambu, Jona-Lasinio ('61))





Chiral phase transition at finite T and μ (Chiral Limit)

NJL model with μ

- **NJL Lagrangian**

$$\mathcal{L} = \bar{q}(i\gamma^\mu \partial_\mu - m + \gamma^0 \mu) q + \frac{G^2}{2\Lambda^2}\left[(\bar{q}q)^2 + (\bar{q} i\gamma_5 \boldsymbol{\tau} q)^2\right]$$

$$\mathcal{L}_E = \bar{q}(-i\gamma_\mu \partial_\mu + m - \gamma_0 \mu) q - \frac{G^2}{2\Lambda^2}\left[(\bar{q}q)^2 + (\bar{q} i\gamma_5 \boldsymbol{\tau} q)^2\right]$$

- **Effective Action**

$$S_{\text{eff}} = -\log\det D + \int d^4x \frac{\Lambda^2}{2}\left[\sigma^2(x) + \boldsymbol{\pi}^2(x)\right]$$

$$D = -i\gamma_\mu \partial_m u - \gamma_0 \mu + M (M = m + G\Sigma)$$

$$\text{MF + Fourier} \rightarrow D = -\gamma_0(i\omega + \mu) + \boldsymbol{\gamma}\cdot\mathbf{k} + M (M = m + G\sigma)$$

- **Free energy density**

$$F_{\text{eff}} = \frac{\Lambda^2}{2}\sigma^2 - \frac{T}{V}\sum_{n,\mathbf{k}} \log((\omega_n - i\mu)^2 + \mathbf{k}^2 + M^2)^{d_f/2}$$

$$= \frac{\Lambda^2}{2}\sigma^2 - d_f \int \frac{d^3k}{(2\pi)^3}\left[\frac{E_k}{2} + \frac{k^2}{3E_k}\frac{1}{2}\left(\frac{1}{e^{(E_k-\mu)/T}+1} + \frac{1}{e^{(E_k+\mu)/T}+1}\right)\right]$$

T, μ and m dependence of thermal pressure

- **Thermal pressure as a function of T, μ, and m (Fermions)**
  Kapusta ('89), Kapusta, Gale (2006)

$$P^F/d_F = \frac{7}{8}\frac{\pi^2}{90}T^4 + \frac{1}{24}\mu^2T^2 + \frac{\mu^4}{48\pi^2}$$

Stefan-Boltzmann (m=0)

$$- \frac{m^2}{16\pi^2}\left[\frac{\pi^2}{3}T^2 + \mu^2\right]$$

m² term → phase transition

$$- \frac{m^4}{32\pi^2}\left[\log\left(\frac{m}{\pi T}\right) - \frac{3}{4} + \gamma_{\rm E} \boxed{- H^\nu\left(\frac{\mu}{T}\right)}\right] + \mathcal{O}\left(m^6\right)$$

m⁴ term → critical point

New

$$H^\nu(\nu) = \frac{7}{4}\zeta(3)\left(\frac{\nu}{\pi}\right)^2 - \frac{31}{16}\zeta(5)\left(\frac{\nu}{\pi}\right)^4 + \frac{127}{64}\zeta(7)\left(\frac{\nu}{\pi}\right)^6 + \cdots$$

Mass reduces pressure (enh. Feff) → phase transition ?

Chiral Transition at Finite T

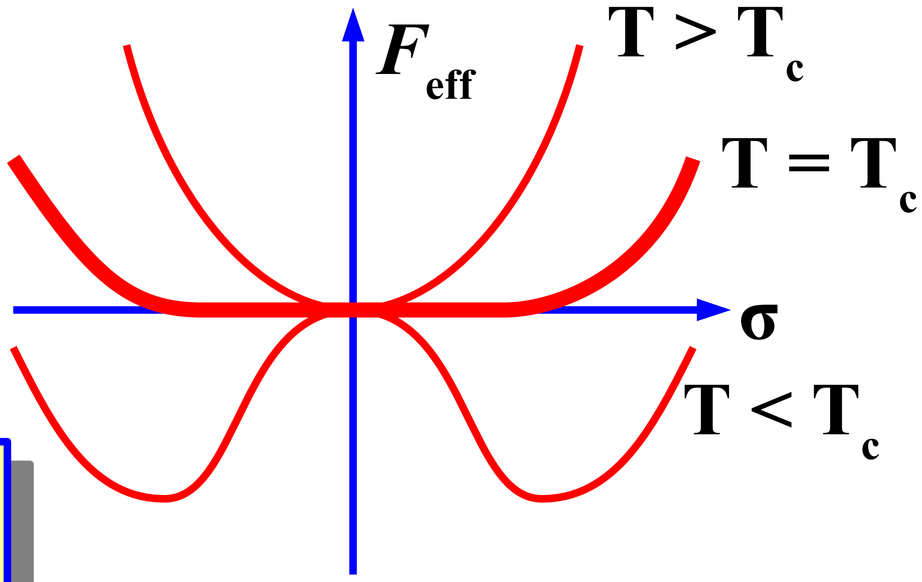
- **Effective potential at finite *T* in NJL**

$$\begin{aligned}\frac{F_{\rm eff}}{\Lambda^4} &= -\frac{d_f}{2}I(x) + \frac{x^2}{2G^2} - \frac{P^F}{\Lambda^4}\\ &= -\frac{d_f}{16\pi^2} - \underline{\frac{d_f\pi^2}{90}\frac{7}{8}\left(\frac{T}{\Lambda}\right)^4} + \frac{x^2}{2}\left[\frac{1}{G^2} - \frac{1}{G_c^2}\left(1 - \underline{\frac{\pi^2}{3}\left(\frac{T}{\Lambda}\right)^2}\right)\right]\end{aligned}$$

Stefan-Boltzmann

Correction from T

  - **Chiral transition should occur at T < $3^{1/2}$ Λ/π.**



Chiral Transition at finite T is suggested by NJL !

High-Temperature Expansion (1)

- **Thermal pressure (Fermions)**

$$P^F = \frac{d_F}{2} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3\omega} \left[\frac{1}{e^{(\omega-\mu)/T}+1} + \frac{1}{e^{(\omega+\mu)/T}+1} \right]$$

$$\omega = \sqrt{p^2+m^2}$$

- **High-Temperature Expansion = Expansion in *m/T***
  - **Important to discuss chiral transition (*m = Gσ*)**
  - **Naive expansion does not work (non-analytic term in *m*)**
- **Kapusta method**
  - **Recursion formula: simpler integral → pressure**

$$P^F = \frac{4T^4 d_F}{\pi^2} h_5^F \left(y = \frac{m}{T}, \nu = \frac{\mu}{T} \right) \ , \quad \frac{dh_{n+1}}{dy} = -\frac{y}{n} h_{n-1}$$

  - **Replace integrand**

$$\frac{1}{2\omega} \left[\frac{1}{e^{\omega-\nu}+1} + \frac{1}{e^{\omega+\nu}+1} \right] = \frac{1}{2\omega} - \sum_{l=-\infty}^{\infty} \frac{1}{\omega^2 + [\pi(2l-1) - i\nu]^2}$$

High-Temperature Expansion (2)

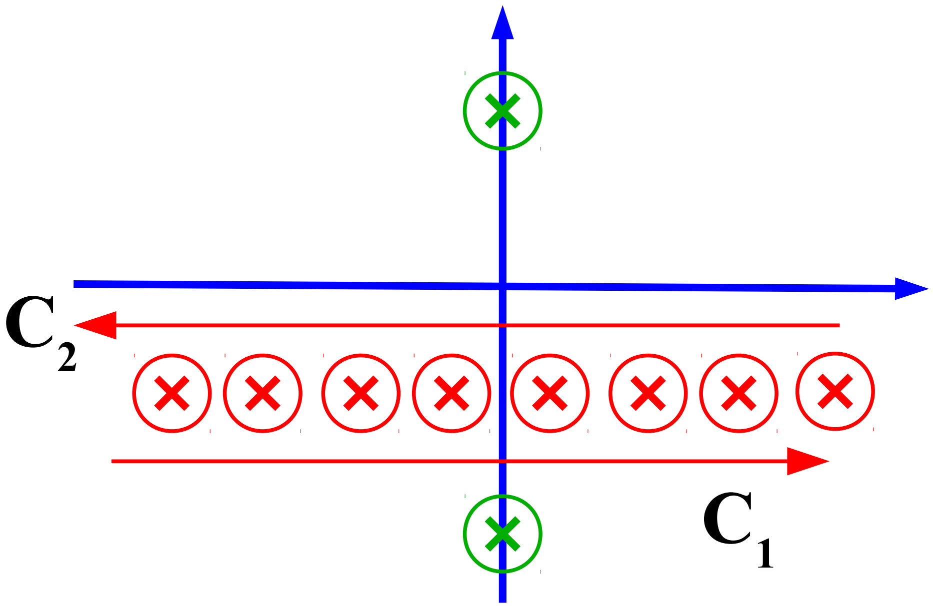
- **Following identity is obtained from contour integral.**

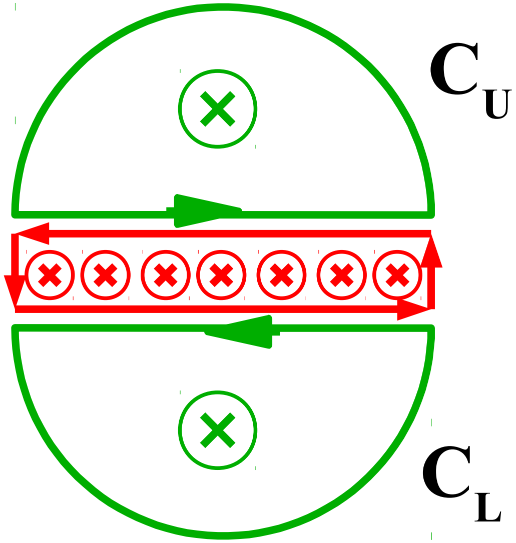
$$\frac{1}{2\omega}\left[\frac{1}{e^{\omega-\nu}+1}+\frac{1}{e^{\omega+\nu}+1}\right]=\frac{1}{2\omega}-\sum_{l=-\infty}^{\infty}\frac{1}{\omega^2+[\pi(2l-1)-i\nu]^2}$$

$$\oint_{\mathcal{C}_{\mathrm{U}}+\mathcal{C}_{\mathrm{L}}}\frac{dz}{2\pi}\,\frac{1}{e^{iz-\nu}+1}\,\frac{1}{z^2+\omega^2}=-\oint_{\mathcal{C}}\frac{dz}{2\pi}\,\frac{1}{e^{iz-\nu}+1}\,\frac{1}{z^2+\omega^2}$$

pole at $z=\pm i\,\omega$

pole at $z=\pi(2\,l-1)-i\,\nu$





$C_1+C_2+C_U+C_L=0$

High-Temperature Expansion (3)

- **Recursion relation of h-functions**

$$h_n^F(y,\nu) = \frac{1}{2(n-1)!}\int_0^\infty \frac{x^{n-1}dx}{\omega}\left\{\frac{1}{e^{\omega-\nu}+1}+\frac{1}{e^{\omega+\nu}+1}\right\}$$

$$\frac{dh_{n+1}}{dy} = -\frac{y}{n}h_{n-1}$$

  - **From $h_1(y, \nu)$, $h_3(0, \nu)$, $h_5(0, \nu)$, we obtain $h_5(y, \nu)$ and pressure.**

- **Key function= $h_1(y, \nu)$**

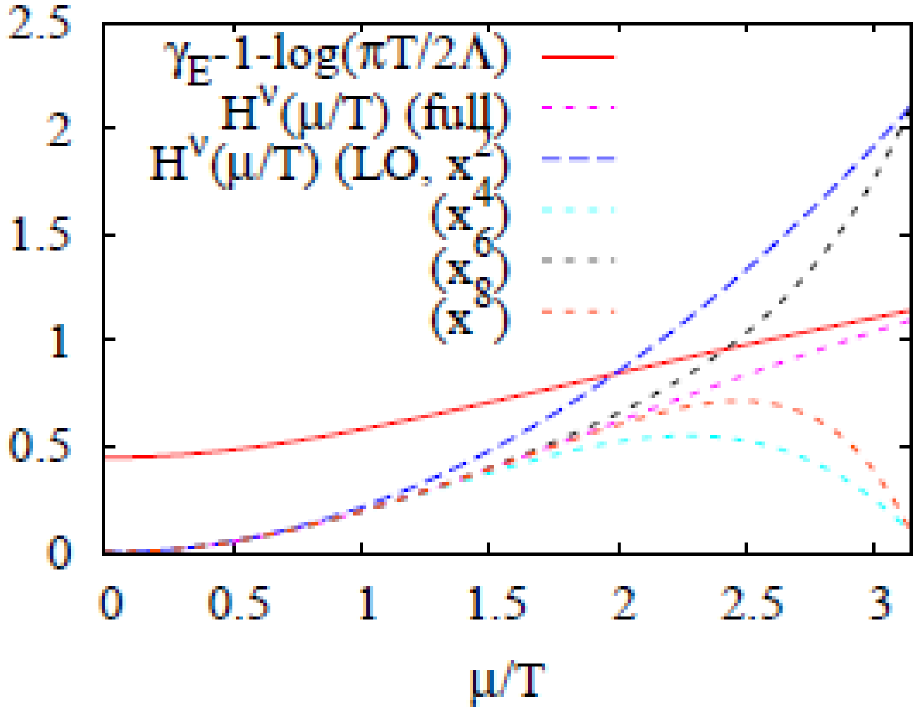
$$h_1^F(y,\nu) = \lim_{L\to\infty}\int_0^{2\pi L} dx\left[\frac{1}{2\omega} - \sum_{l=-\infty}^{\infty}\frac{1}{\omega^2+[\pi(2l-1)-i\nu]^2}\right]$$

$$= -\frac{1}{2}\log\frac{y}{\pi} - \frac{1}{2}\gamma_{\mathrm{E}} - \frac{1}{2}\sum_{l=1}^{\infty}\left[\frac{\pi}{\omega_l}+\frac{\pi}{\omega_l^*}-\frac{2}{2l-1}\right]$$

$$(\omega_l = \sqrt{y^2+[\pi(2l-1)-i\nu]^2})$$

(Tri)Critical Point

- **Do we expect the existence of (Tri)Critical Point in NJL ?**
  - **Yes, as first shown by Asakawa, Yazaki ('89)**
  - **TCP in the chiral limit → CP at finite bare quark mass**
- **Estimate from high-temperature expansion**
  - **TCP: $c_2 = 0$ and $c_4 = 0$ simultaneously.**
  - **c_4 decreases as μ/T increases.**
  - **Existence is probable,**
Position is sensitive
to parameters and treatment.



Chiral Transition at Finite μ

- **Effective potential at finite μ in NJL**

$$F_{\rm eff}(m;T,\mu) = F_{\rm eff}(0;T,\mu) + \frac{c_2(T,\mu)}{2} m^2 + \frac{c_4(T,\mu)}{24} m^4 + \mathcal{O}(m^6)$$

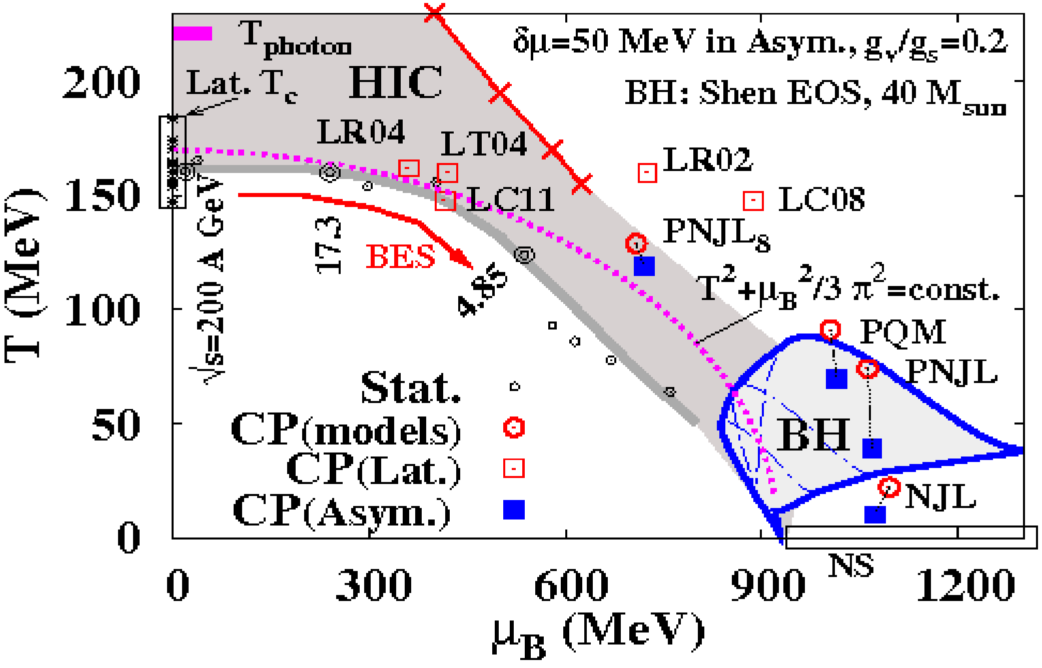
$$c_2(T,\mu) = -\frac{d_F}{24}\left[\boxed{\frac{3}{\pi^2}\Lambda^2\left(1-\frac{8\pi^2}{d_F G^2}\right)} - \left(T^2+\frac{3}{\pi^2}\mu^2\right)\right]$$

$T_c^2(\mu=0)$

- **2nd order phase boundary**

$$T^2 + \frac{3}{\pi^2}\mu^2 = T_c^2(\mu=0)$$

Roughly matches chem. freeze-out line.



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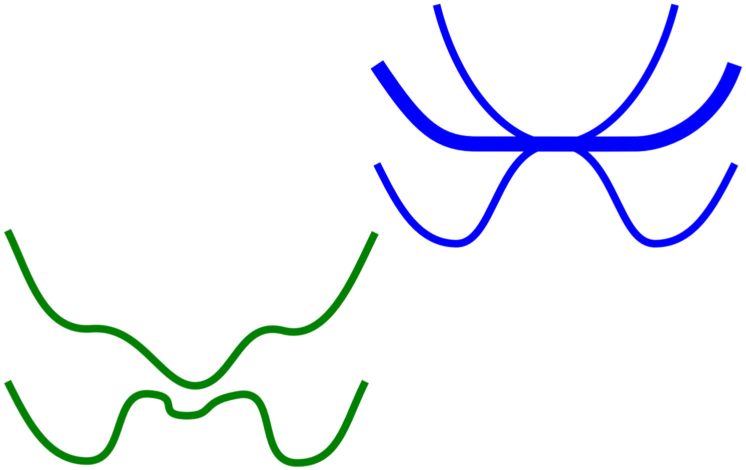
$$c_4(T,\mu) = \frac{3d_F}{4\pi^2}\Big[\underbrace{\gamma_E - 1 - \log\left(\frac{\pi T}{2\Lambda}\right)}_{\mu=0} \underbrace{- H^\nu(\mu/T)}_{\text{negative at finite } \mu}\Big]$$

- **$c_2 = 0$ and $c_4 > 0$ → 2nd order**
- **$c_2 \geq 0$ and $c_4 < 0$ → 1st order**
- **$c_2 = 0$ and $c_4 = 0$ → tricritical point**



Short Summary

- フェルミオンを含む有限温度･密度の場の理論 (NJL 模型) でボソンについて平均場近似を行うことにより、自由エネルギーを導出した。
  - ゼロ点エネルギー (の変化) は場の理論からみれば必要。
  - 負のエネルギーが現れても (ゼロ点エネルギー以外では) 松原和の発散は起こらない。
- NJL 模型 (Nf=2) では高密度において QCD(1 次) 相転移と臨界点の存在が期待される。
  - フェルミオンのゼロ点エネルギーが対称性を自発的に破る起源
  - 高温･高密度では粒子圧力の寄与によりカイラル対称性が回復
  - NJL 模型から期待される 2 次相転移線の「楕円」は実験から示唆される化学凍結線とほぼ一致

レポート(2)

- 全部で 5-7 問程度出します。3 問程度以上レポートを出してください。レポート(2) の〆切は 12/25(火) 。)
- (Report 3) ボソン化した NJL 模型の作用から平均場近似の下で有効ポテンシャルを求めよ。
  余裕があれば、有限温度・有限密度 (有限化学ポテンシャル) での有効ポテンシャルを構成子クォーク質量で 2 次まで展開し、カイラル極限で 2 次相転移線が

$$T^2 + \frac{3}{\pi^2}\mu^2 = T_c^2(\mu = 0)$$

で与えられることを示せ。
(前半は復習。後半はかなり面倒。無理しないように。)