Phase Diagram of Quark Matter at Finite Temperature and Density in the Strong Coupling of Limit Lattice QCD with Nc = 3

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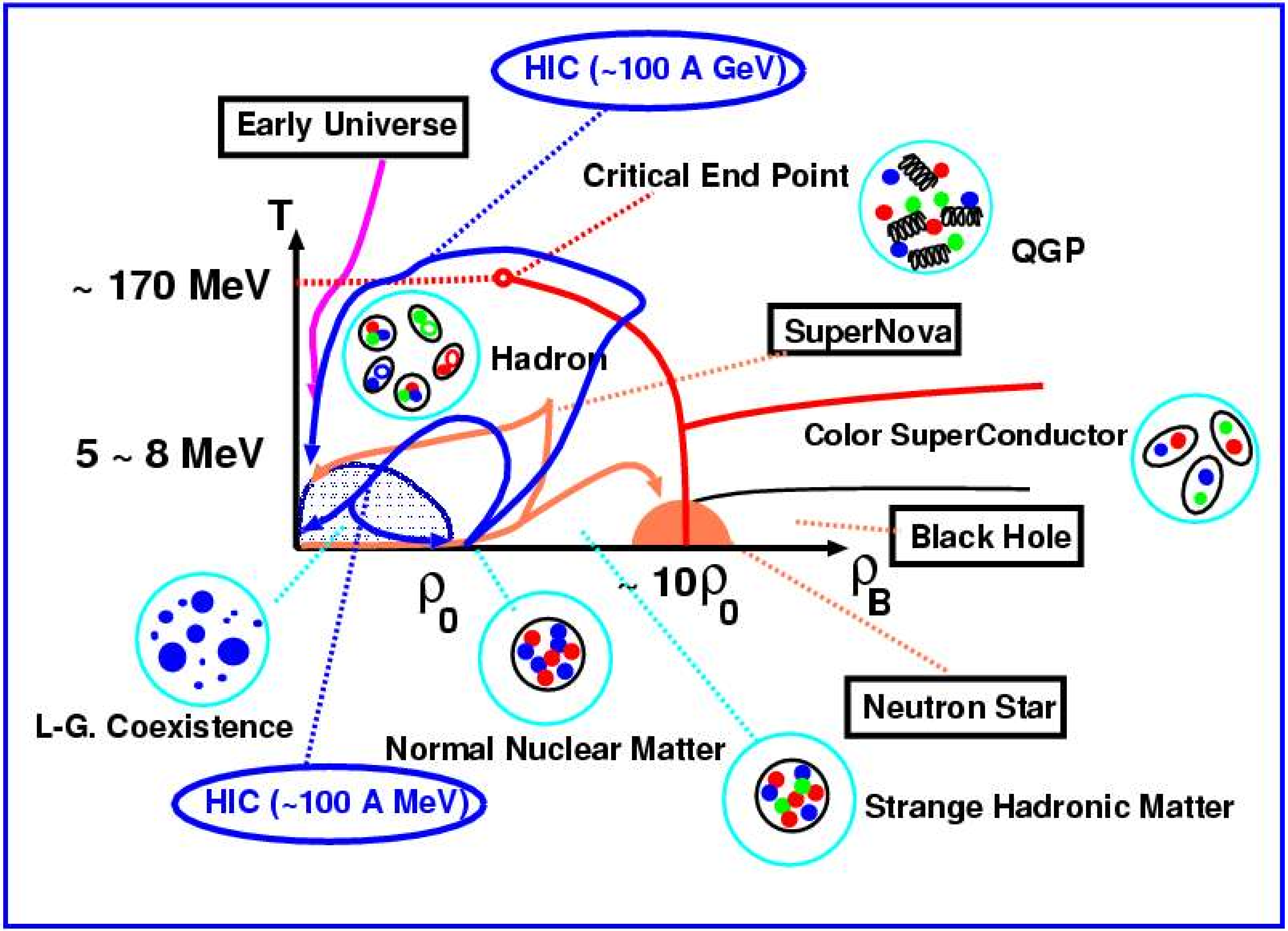
Hadronic Matter Phase Diagram

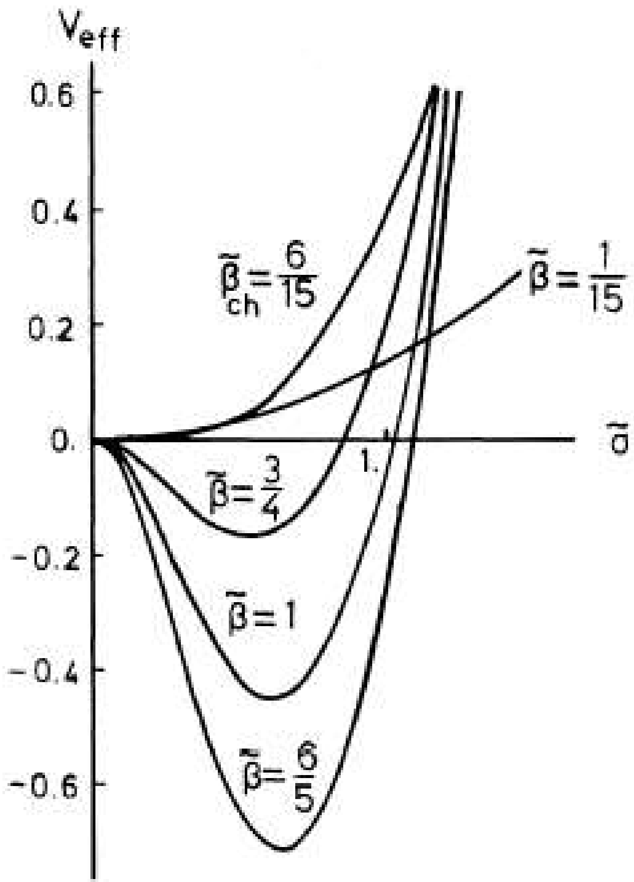


Phase Diagram and Dense Matter

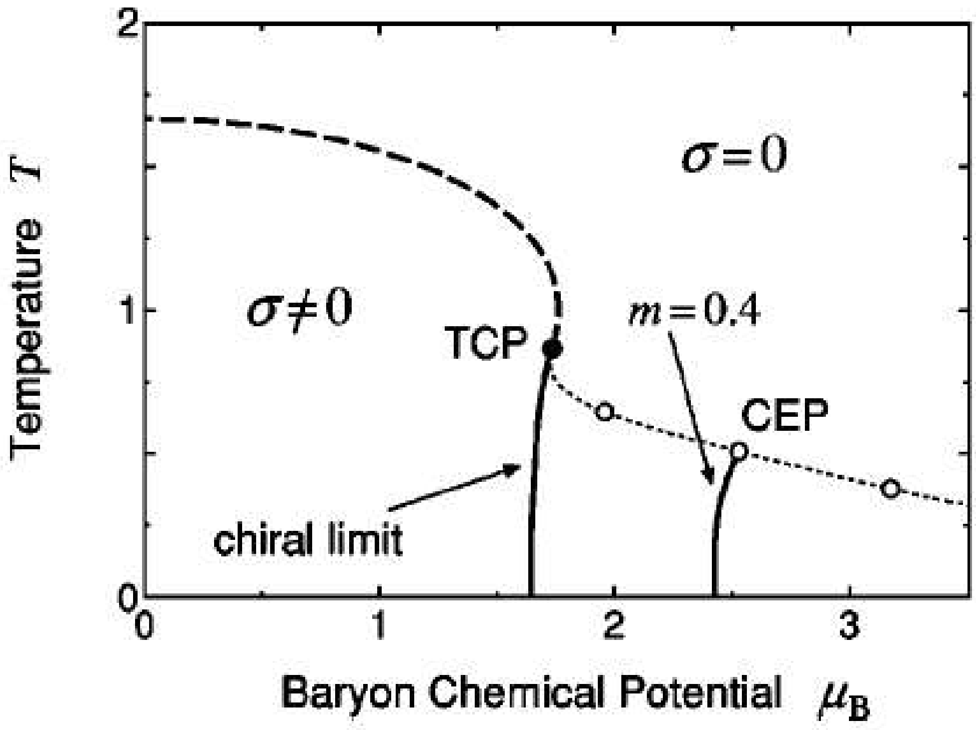
- High T → well studied theoretically and experimentally
  - Lattice QCD Monte-Carlo simulation / RHIC, SPS
- High Density Matter → Interesting but Difficult
  - Exp't: FAIR(GSI), SPS(20-80 AGeV), AGS (10 A GeV)
  - Theor.: Weight becomes complex at finite μ in Lattice QCD
    → ***Model/Approximate approaches are necessary !***
  - Monte-Carlo calc. of Lattice QCD:
    Improved ReWeighting Method (Fodor-Katz)
    Taylor Expansion in μ (Bielefeld U.)
    Analytic Continuation (de Forcrand-Philipssen), ...
  - Model / Phenomenological Approaches: QMC(Thomas), NJL (Hatsuda-Kunihiro, ...),
    HIC simulation (Isse-AO-Otuka-Sahu-Nara, Hirano-Isse-Nara-AO-Yoshino), RMF (Tsubakihara-AO JD6(9/22,10:30-))
  - ***Strong Coupling Limit of Lattice QCD***

Strong Coupling Limit of Lattice QCD (1)

- Chiral Restoration at μ=0.
  - Damgaard, Kawamoto, Shigemoto, PRL53(1984),2211



- Phase Diagram with Nc=3
  - Nishida, PRD69, 094501 (2004)



Strong Coupling Limit of Lattice QCD (2)

Strong Coupling

- Lattice Action (staggered fermion)

$$Z = \int \mathcal{D}[\chi,\bar{\chi},U_0,U_1,U_2,U_3] \exp\left[-S_F^{(U_0)} - \sum_{j=1}^{3} S_F^{(U_j)} - S_F^{(m)} \cancel{-S_G}\right]$$

- Spatial Link Integral

$$\simeq \int \mathcal{D}[\chi,\bar{\chi},U_0] \exp\left[\frac{1}{2}(M,V_M M) + \cancel{(\bar{B},V_B B)} - S_F^{(U_0)} - S_F^{(m)}\right]$$

1/d Expansion (1/√d)

- Bosonization (HS transf.)

$$\simeq \int \mathcal{D}[\chi,\bar{\chi},U_0,\sigma] \exp\left[-\frac{1}{2}(\sigma,V_M^{-1}\sigma)\underbrace{-(\sigma,M) - S_F^{(U_0)} - S_F^{(m)}}_{(\bar{\chi},G^{-1}(\sigma)\chi)}\right]$$

- Quark and U_0 Integral

$$\simeq \int \mathcal{D}[\sigma] \exp\left[-\frac{1}{2}a_\sigma\sigma^2\right] \underbrace{\prod_{x}\int dU_0 \ \mathrm{Det}\ \left[G^{-1}(\sigma)\right]}_{\exp\left[-L^3\beta F^q(\sigma)\right]}$$

- Effective Free Energy

$$\simeq \exp\left[-L^3\beta F_{\mathrm{eff}}(\sigma)\right]$$

- ***Baryonic Composite will be important at High Densities, but they have been ignored in finite T treatments !***
  → This work: Nc = 3, Baryonic Composite, Finite T and μ

- Lattice QCD action

$$S_F^{(U_j)} = \frac{1}{2}\sum_x \eta_j(x)\left[\bar{\chi}(x)U_\mu(x)\chi(x+\hat{\mu}) - \bar{\chi}(x+\hat{\mu})U_\mu^\dagger(x)\chi(x)\right]$$

$$S_F^{(U_0)} = \frac{1}{2}\sum_x \left[\bar{\chi}(x)e^{\mu}U_\mu(x)\chi(x+\hat{\mu}) - \bar{\chi}(x+\hat{\mu})e^{-\mu}U_\mu^\dagger(x)\chi(x)\right]$$

$$S_F^{(m)} = m_0\sum_x \bar{\chi}^a(x)\chi^a(x)\ ,$$

- Mesonic and Baryonic Composites

$$M(x) = \delta_{ab}\bar{\chi}^a(x)\chi^b(x)\ ,$$

$$B(x) = \frac{1}{6}\varepsilon_{abc}\chi^a(x)\chi^b(x)\chi^c(x)\ ,\quad \bar{B}(x) = \frac{1}{N_c!}\varepsilon_{abc}\bar{\chi}^c(x)\bar{\chi}^b(x)\bar{\chi}^a(x)$$

- Fermion Integral

$$\int \mathcal{D}[U_0,\chi,\bar{\chi}]\exp\left[-\sum_t \sigma M - S_F^{(U_0)}\right] = \int \mathcal{D}[U_0,\chi,\bar{\chi}]\prod_k \exp\left[-\bar{\chi}_k G(k)\chi_k/2\right]$$

$$= \cdots = C_\sigma^3 - \frac{1}{2}C_\sigma + \frac{1}{4}\cosh(3\beta\mu)$$

$$F_{\rm eff}^{(q)}(\sigma_q) = -T\log\left[\frac{4}{3}\left(C_\sigma^3 - \frac{1}{2}C_\sigma + \frac{1}{4}C_{3\mu}\right)\right] \qquad C_\sigma = \cosh\left[\beta \operatorname{arcsinh}\tilde{\sigma}\right]$$

Decomposition of Baryonic Composite Action

- Introducing Auxiliary Baryon Field

$$\exp(\bar{B}, V_B B) = \det V_B \int \mathcal{D}[\bar{b}, b] \exp\left[-(\bar{b}, V_B^{-1} b) + (\bar{b}, B) + (\bar{B}, b)\right]$$

- Diquark Composites (Azcoiti et al., JHEP 0309, 014 ('03))

$$\bar{b}B = \underbrace{\bar{b}\chi^a}_{\text{antibaryon-quark}} \times \underbrace{\chi^b\chi^c}_{\text{diquark}} \times \varepsilon_{abc}/6$$

$D^\dagger D$ makes $\bar{b}B$

$$D_a = \frac{\gamma}{2}\varepsilon_{abc}\chi^b\chi^c + \frac{1}{3\gamma}\bar{\chi}^a b\,, \quad D_a^\dagger = \frac{\gamma}{2}\varepsilon_{abc}\bar{\chi}^c\bar{\chi}^b + \frac{1}{3\gamma}\bar{b}\chi^a$$

- Decomposition of coupling of baryon and 3 quarks

$$\exp(\bar{b}B + \bar{B}b) = \int d[\phi_a, \phi_a^\dagger] \exp\Big[-\phi_a^\dagger\phi_a + (\phi_a^\dagger D_a + D_a^\dagger \phi_a) \underbrace{-\frac{\gamma^2}{2}M^2 + M\bar{b}b/9\gamma^2}_{\bar{B}b + \bar{b}B - D_a^\dagger D_a}\Big]$$

Effective Action is not yet bilinear in fermions

- ***four fermi interaction terms, M^2 and $M\bar{b}b$***
- ***diquark-quark-antibaryon coupling***

Bosonization of Four Fermi Interactions

- $M\bar{b}b$ term → Baryon potential auxiliary field ω

$$\exp(M\bar{b}b/9\gamma^2) = \int d[\omega] \exp\left[-\omega^2/2 - \omega(\alpha M + g_\omega \bar{b}b) - \alpha^2 M^2/2\right]$$

  - ☆ $(\bar{b}b)^2 = 0$ in One species of Staggered Fermion

- M^2 and $(M, V_M M)$ terms → Chiral Condensate σ

$$\frac{1}{2}(M, V_M M) - \frac{1}{2}(\gamma^2 + \alpha^2)M^2 = \frac{1}{2}(M, \tilde{V}_M M)$$

$$\exp\left[\frac{1}{2}(M, \tilde{V}_M M)\right] = \int \mathcal{D}[\sigma] \exp\left[-\frac{1}{2}(\sigma, \tilde{V}_M^{-1}\sigma) - (\sigma, M)\right]$$

  - ☆ By absorbing "Mass" in the Hopping Term,
    We can replace both of the terms simultaneously !

Effective Action in bilinear form of Fermions !

Effective Free Energy at Zero Diquark Condensate

- Effective Action

Zero Diquark Condensate

$$S_F = (\bar{b}, \widetilde{V}_B^{-1} b) + \frac{1}{2}(\omega, \omega) + \frac{1}{2}(\sigma, \widetilde{V}_M^{-1}\sigma) + (\sigma_q, M) + S_F^{(U_0)} + S_F^{(m)}$$

$$\cancel{+ (\phi^\dagger, \phi) + \frac{1}{3\gamma}\left[(\bar{\chi}^a, \phi_a^\dagger b) + (\bar{b}\phi_a, \chi^a)\right] + \frac{\gamma}{2}\varepsilon_{cab}\left[(\phi_c^\dagger, \chi^a\chi^b) + (\bar{\chi}^b\bar{\chi}^a, \phi_c)\right]}$$

- After Quark, U_0, Baryon Integral at zero diquark cond.

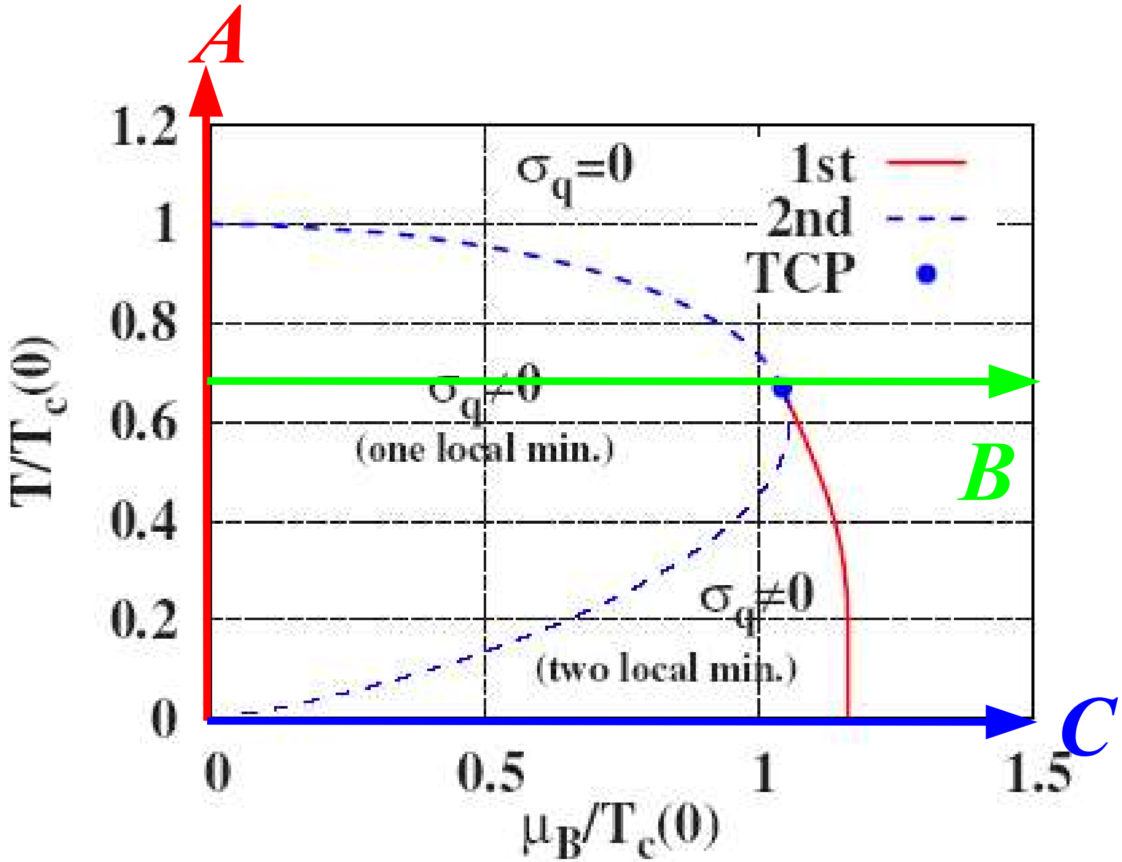
$$\mathcal{F}_{\rm eff} = \frac{1}{2}a_\sigma \sigma^2 + \frac{1}{2}\omega^2 + F_{\rm eff}^{(b)}(g_\omega \omega) + F_{\rm eff}^{(q)}(\sigma_q) \qquad a_\sigma = \left[\frac{d}{2N_c} - (\gamma^2 + \alpha^2)\right]^{-1}$$

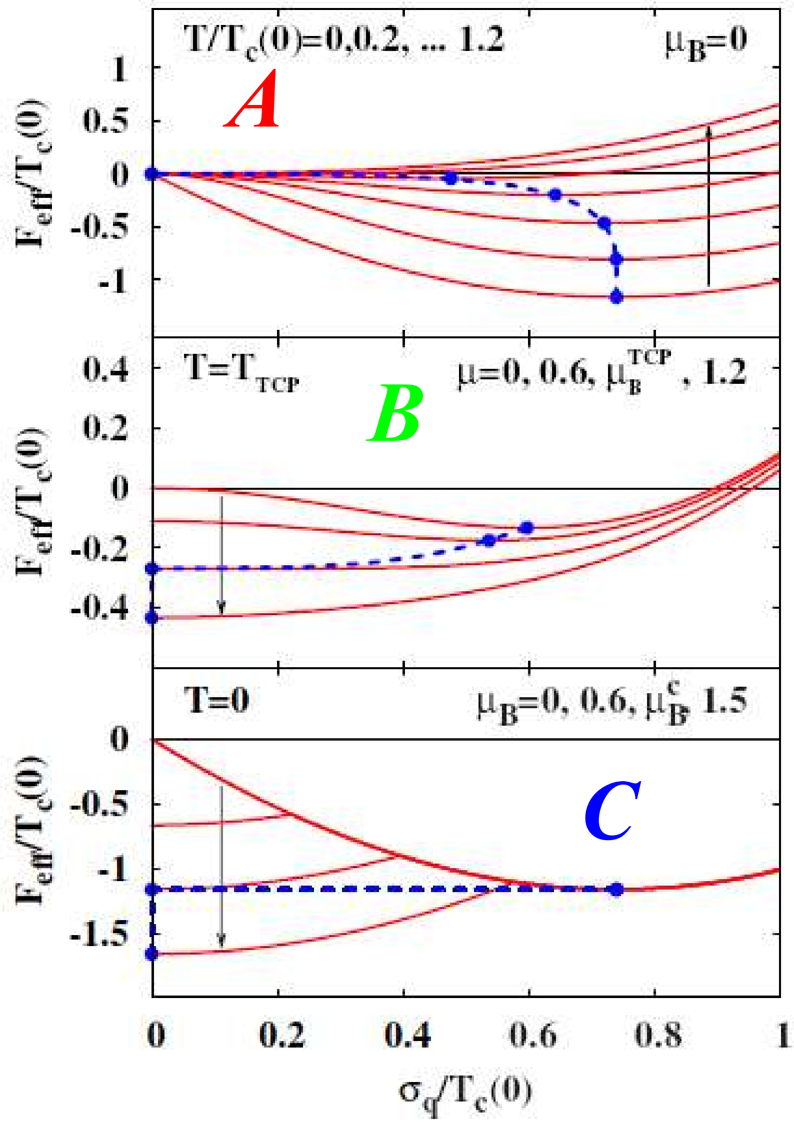
and Setting convenient parameters (γ and ω are removed), we get an analytical expression of Effective Free Energy

$$\boxed{\mathcal{F}_{\rm eff}(\sigma_q) = \frac{\sigma_q^2}{2\alpha^2} + F_{\rm eff}^{(b)}(g_\sigma \sigma_q) + F_{\rm eff}^{(q)}(\sigma_q; T, \mu)}$$

Free Energy Surface and Phase Diagram

- At $\mu \neq 0$, quark can gain Free Energy even at $\sigma = 0$
  - $\rightarrow$ Two Min. Structure
  - $\rightarrow$ First Order

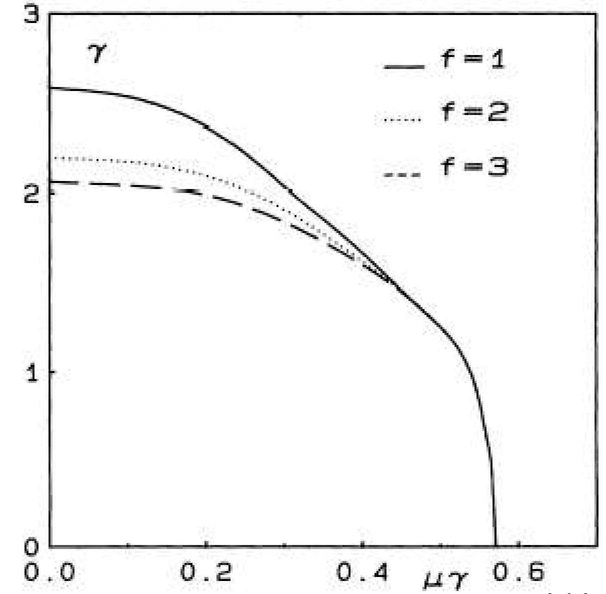




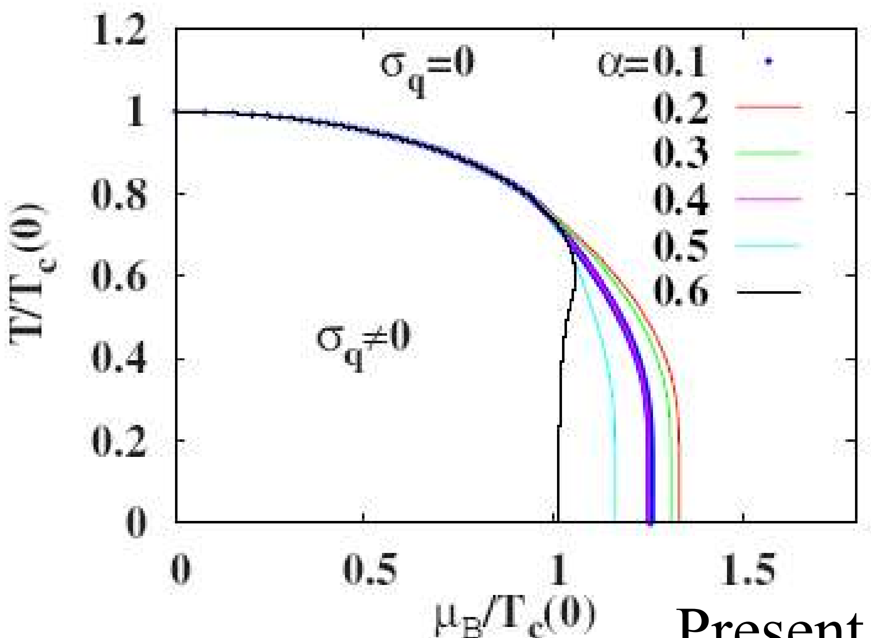
$\alpha = 1/2$

Small Critical μ : Common in SCL-LQCD ?

- Strong Coupling Limit
  - Damgaard,Hochberg,NK ('85): $\mu_B^c(0)/T_c(0) \sim 1.6$ (T=0, T≠0)
  - T≠0, No B: $\mu_B^c(0)/T_c(0) \sim 1.0$ (Nishida2004, Bilic et al 1992 (Bielefeld), ...)
  - Present: $\mu_B^c(0)/T_c(0) < 1.5$ (Parameter dep.)
- Monte-Carlo: $\mu_B^c(0)/T_c(0) >> 1$
  - Fodor-Katz, Bielefeld, de Forcrand-Philipsen, ...
- Real World: $\mu_B^c(0)/T_c(0) \sim 7$



Bilic et al.



Present

Color Angle Average

- Problem: Diquark Condensates induce quark-baryon coupling, and Baryon integral becomes difficult.
  $\rightarrow$ Solution: Color Angle Average

  - Integral of "Color Angle Variables"

$$\int \mathcal{D}[\phi_a, \phi_a^\dagger] \exp\left\{\phi_a^\dagger D_a + D_a^\dagger \phi_a\right\} = \int \mathcal{D}[v] \exp\left\{\frac{v^2}{3} D_a^\dagger D_a + \frac{v^4}{162} M^3 \bar{b} b\right\}$$

  - Three-Quark and Baryon Coupling is ReBorn !

$$D_a^\dagger D_a = Y + \bar{b}B + \bar{B}b\ , \quad Y = \frac{\gamma^2}{2} M^2 - \frac{1}{9\gamma^2} M \bar{b} b$$

  - Solve "Self-Consistent" Equaton

$$\exp(\bar{b}B + \bar{B}b) \simeq \exp\left[-v^2 - Y + \frac{v^2}{3}(\bar{b}B + \bar{B}b + Y) + \frac{v^4}{162} M^3 \bar{b} b\right]$$

$$\simeq \exp\left[-\frac{v^2}{R_v} + \frac{v^4 M^3 \bar{b} b}{162 R_v} - Y\right] \quad (R_v = 1 - v^2/3)$$

Effective Free Energy with Diquark Condensate

- Bosonization of $M^k \bar{b} b \rightarrow$ Introduce k bosons

$$\exp M^k \bar{b} b = \int d\omega_k \exp[-\frac{1}{2}(\omega_k + \alpha_k M + 1/\alpha_k M^{k-1} \bar{b} b)^2 + M^k \bar{b} b]$$

$$= \int d\omega_k \exp[-\omega_k^2/2 - \omega(\alpha_k M + 1/\alpha_k M^{k-1} \bar{b} b) - \alpha_k^2 M^2/2]$$

- Effective Free Energy

$$\mathcal{F}_{\rm eff}^{(Tbv)} = F_X(\sigma, v, \omega_i) + F_{\rm eff}^{(b)}(g_\omega \omega) + F_{\rm eff}^{(q)}(\sigma_q)$$

$$F_X = \frac{1}{2}(a_\sigma \sigma^2 + \omega^2 \boxed{+ \omega_1^2 + \omega_2^2) + \frac{v^2}{R_v}} \qquad g_\omega = \frac{1}{9\alpha\gamma^2}\left[1 \boxed{+ \frac{\gamma^2 v^4 \omega_1 \omega_2}{18\alpha_1 \alpha_2 R_v}}\right]$$

$$a_\sigma = \left(\frac{1}{2} - \gamma^2 - \alpha^2 \boxed{- \alpha_1^2 - \alpha_2^2}\right)^{-1} \qquad \sigma_q = \sigma + \alpha\omega \boxed{+ \alpha_1 \omega_1 + \alpha_2 \omega_2}$$

Similar form to the previous one at v=0.
Diquark Effects in interaction start from v^4.

Summary

- We have obtained an analytical expression of effective free energy ***at finite T and finite μ*** with ***baryonic composite action*** effects in the strong coupling limit of lattice QCD.
- In order to achive above, three techniques are developed.
  - Auxiliary ***baryon potential ω*** is introduced, using $(\bar{b}b)^2 = 0$
  - Mesonic propagator is modified ***to absorb M^2*** terms.
  - ***Color angle average*** and solving self-consistent condition
- Baryonic composite action is found to result in ***mesonic propagator modification*** and ***auxiliary baryon determinant***
- Obtained phase diagram @ zero diquark cond.
  → qualitatively reasonable.
  - 1st (low T) and 2nd (high T) phase transition is separated by TCP.
  - too small μ_B^{crit}, parameter dependence, one "flavor", diquarks, ...

Comparison with Other Treatments

- T=0, without or with baryons
  (e.g., NK-Smit1981, Damgaard-Hochberg-NK 1985)

$$\mathcal{F}^{(0)}_{\rm eff} = \frac{N_c\sigma^2}{d+1} - N_c \log\widetilde{\sigma} \qquad \mathcal{F}^{(0\rm b)}_{\rm eff} = \frac{N_c\sigma^2}{d+1} + F^{(b\mu)}_{\rm eff}(4\widetilde{\sigma}^3; T,\mu)$$

- T=0, with b and diquark (ACGL2002)

$$\Theta = \frac{1}{3}\left(R_v^2 - \frac{R_v\widetilde{\sigma}^2}{\gamma^2} + \frac{2}{9}v^2\right) ,$$

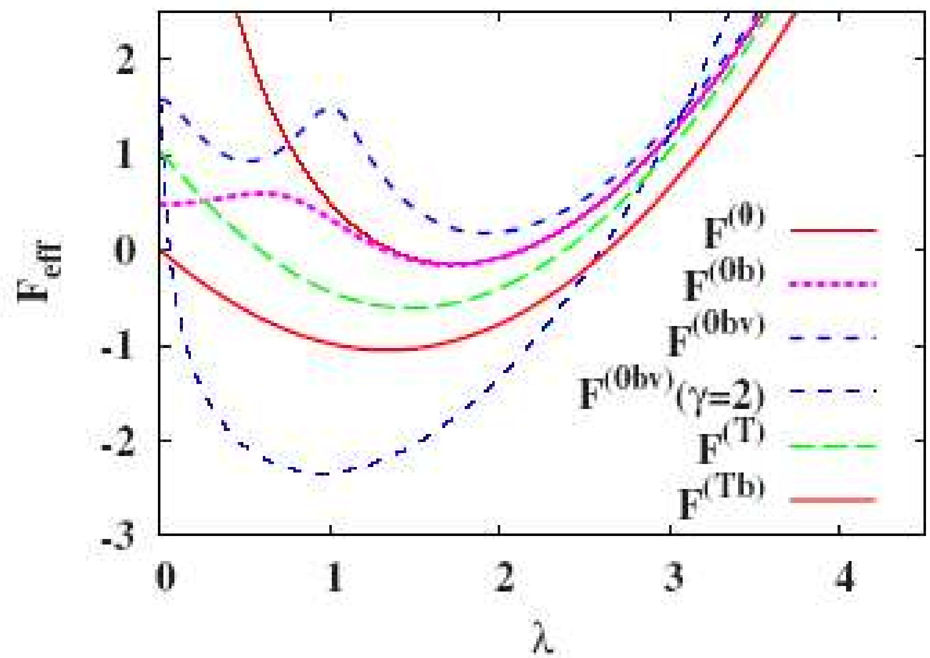
$$\mathcal{F}^{(0\rm bv)}_{\rm eff} = \frac{N_c\sigma^2}{d+1} + v^2 - \log\Theta + F^{(b\mu)}_{\rm eff}(m; T,\mu) \qquad m = \frac{4\widetilde{\sigma}\left(3\gamma^2 R_v - \widetilde{\sigma}^2\right)}{\Theta}$$

- $T \neq 0$, no baryons (e.g., Nishida2004)

$$\mathcal{F}^{(T)}_{\rm eff} = \frac{N_c\sigma^2}{d} + F^{(q)}_{\rm eff}(\widetilde{\sigma})$$

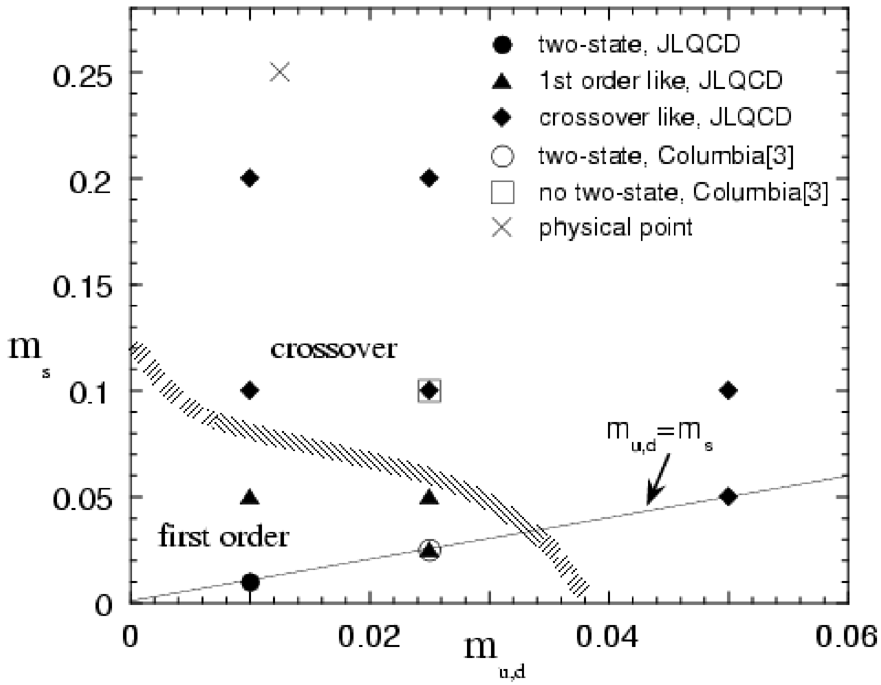
Fixing asymptotic behavior
→ $F^{(Tb)}$ is smaller

$$\mathcal{F}^{(Tb)}_{\rm eff}(\lambda = \sigma_q/\alpha) \to \frac{\lambda^2}{2} - N_c \log\lambda + F^{(b)}_{\rm eff}(g_\omega \alpha\lambda)$$



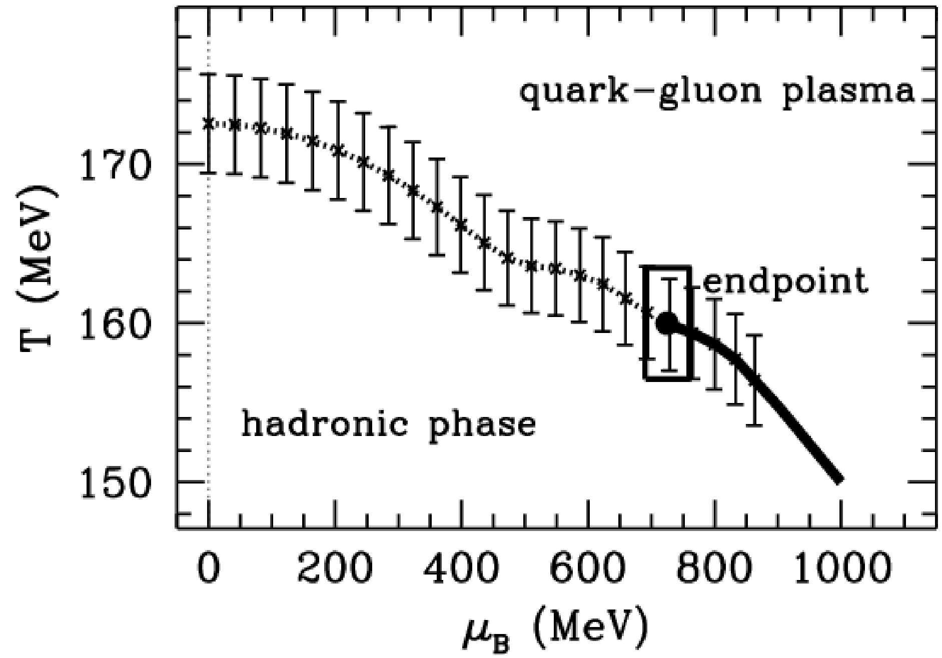
QCD Phase Diagram from Lattice QCD

- Zero Chem. Pot.



☆ JLQCD Collab. (S. Aoki et al.), Nucl. Phys. Proc. Suppl. 73 (1999), 459.

- Finite Chem. Pot.



☆ Fodor & Katz, JHEP 0203 (2002), 014.

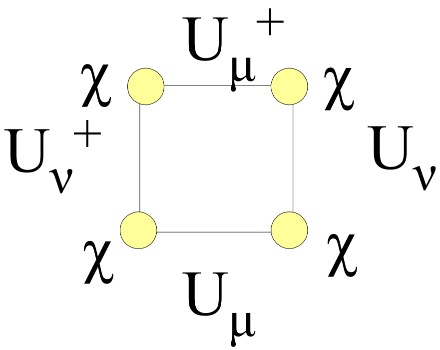
Zero Chem. Pot. : Cross Over
Finite Chem. Pot.: Critical End Point

Strong Coupling Limit of Lattice QCD

- Full Lattice QCD at large μ and low T is not possible
  - Fermion Det. becomes complex → Monte-Carlo breaks down
  - Small μ → Re-Weighting / Expansion in μ
- Strong Coupling Limit: $g \to \infty$
  - Semi-analytic analyses become possible.
  - At μ=0, Chiral Restoration at high T is explained.
    - Damgaard, Kawamoto, Shigemoto, PRL53(1984),2211
  - At $\mu \neq 0$ and $N_c = 2$, Phase diagram is drawn.
    → ***Baryon = Boson with $N_c = 2$***
    - Nishida, Fukushima, Hatsuda, PRept 394(2004),281.
  - At $\mu \neq 0$ and $N_c = 3$, U_0 integral is done only approximately.
    - Azcoiti, Di. Carlo, Galante, Laliena, hep-lat/0307019.

Lattice Action in SCL-LQCD (1)

- Lattice Action with staggered Fermions



$$S[U,\chi,\bar{\chi}] = S_G[U] + S_F[U,\chi,\bar{\chi}] \;,$$

$$\boxed{S_G[U] = \frac{2N_c}{g^2}\sum_{x,\mu,\nu}\left\{1 - \frac{1}{N_c}\mathrm{ReTr}U_{\mu\nu}(x)\right\}} \xrightarrow{g\to\infty} \mathbf{0}$$

$$U_{\mu\nu}(x) = U_\nu^\dagger(x)U_\mu^\dagger(x+\hat{\nu})U_\nu(x+\hat{\mu})U_\mu(x) \;,$$

Chem. Pot.

$$\begin{aligned}
S_F[U,\chi,\bar{\chi}] &= S_F^{(m)}[\chi,\bar{\chi}] + S_F^{(j)}[U_j,\chi,\bar{\chi}] + S_F^{(U0)}[U_0,\chi,\bar{\chi}] \;, \\
S_F^{(m)}[\chi,\bar{\chi}] &= m\sum_x \bar{\chi}^a(x)\chi^a(x) \;, \\
S_F^{(j)}[U_j,\chi,\bar{\chi}] &= \frac{1}{2}\sum_x\sum_{j=1}^{d}\eta_j(x)\left\{\bar{\chi}(x)U_j(x)\chi(x+\hat{j}) - \bar{\chi}(x+\hat{j})U_j^\dagger(x)\chi(x)\right\} \;, \\
S_F^{(U0)}[U_0,\chi,\bar{\chi}] &= \frac{1}{2}\sum_x \eta_0(x)\left\{\bar{\chi}(x)e^{\mu}U_0(x)\chi(x+\hat{0}) - \bar{\chi}(x+\hat{0})U_0^\dagger(x)e^{-\mu}\chi(x)\right\} \;.
\end{aligned}$$

- In the Strong Coupling Limit ($g \to \infty$), we can ignore SG, and semi-analytic calculation becomes possible.

Details of Functions

$$\sigma_q = \sigma + \alpha\omega$$

$$\widetilde{V}_B^{-1}(x,y) = V_B^{-1}(x,y) + g_\omega \omega \delta_{x,y} \ , \quad g_\omega = \frac{1}{9\alpha\gamma^2}$$

$$F_{\rm eff}^{(b)}(g_\omega\omega) \ = \ \frac{1}{\beta L^3}\log {\rm Det}\left[1 + g_\omega\omega V_B\right] = -\frac{1}{2L^3}\sum_{\mathbf{k}} \log\left[1 + \frac{g_\omega^2\omega^2 s^2}{16}\right] \simeq -a_0^{(b)} f^{(b)}\left(\frac{g_\omega\omega\Lambda}{4}\right)$$

$$f^{(b)}(x) = \frac{3}{2x^3}\int_0^x k^2 dk \log(1+k^2)$$

$$F_{\rm eff}^{(q)}(\sigma_q) = -T\log\left[\frac{4}{3}\left(C_\sigma^3 - \frac{1}{2}C_\sigma + \frac{1}{4}C_{3\mu}\right)\right]$$

$$F_{\rm eff}^{(q)} = -T\log\left\{\frac{\sinh[(N_c+1)E/T]}{\sinh[E/T]} + 2\cosh N_c\mu\right\}$$

$$\mathcal{F}_{\rm eff} \ = \ \frac{1}{2}a_\sigma\sigma^2 + \frac{1}{2}\omega^2 + F_{\rm eff}^{(b)}(g_\omega\omega) + F_{\rm eff}^{(q)}(\sigma_q)$$

$$\gamma^2 + \alpha^2 = \frac{1}{2} - \epsilon \quad (\epsilon \to +0 \ , \quad a_\sigma \to +\infty)$$

$$\mathcal{F}_{\rm eff}(\sigma_q) \ = \ \frac{\sigma_q^2}{2\alpha^2} + F_{\rm eff}^{(b)}(g_\sigma\sigma_q) + F_{\rm eff}^{(q)}(\sigma_q; T, \mu)$$

Several Analytic Results / Comparison

- Critical Temperature

$$T_c(0) = T_c^{(2\mathrm{nd})}(\mu=0) = \frac{5}{3b_\sigma}$$

  - ☆ b_σ = curvature of $\frac{\sigma_q^2}{2\alpha^2} + F_{\mathrm{eff}}^{(b)}(g_\sigma\sigma_q)$

- 2nd order critical μ

$$\mu_c^{(2\mathrm{nd})}(T) = \frac{T}{3}\cosh^{-1}\left(\frac{3T_c(0)}{T} - 2\right)$$

  - ☆ same as Nishida, 2004

- TriCritical Point

$$\frac{T_{\mathrm{TCP}}}{T_c(0)} = \frac{41}{25}\left[1+\sqrt{1+\frac{164}{625}T_c^2(0)\left(5+9T_c(0)c_4^{(b)}\right)}\right]^{-1}$$

  - ☆ $C_4^{(b)}$= coef. of σ^4 in $\frac{\sigma_q^2}{2\alpha^2} + F_{\mathrm{eff}}^{(b)}(g_\sigma\sigma_q)$

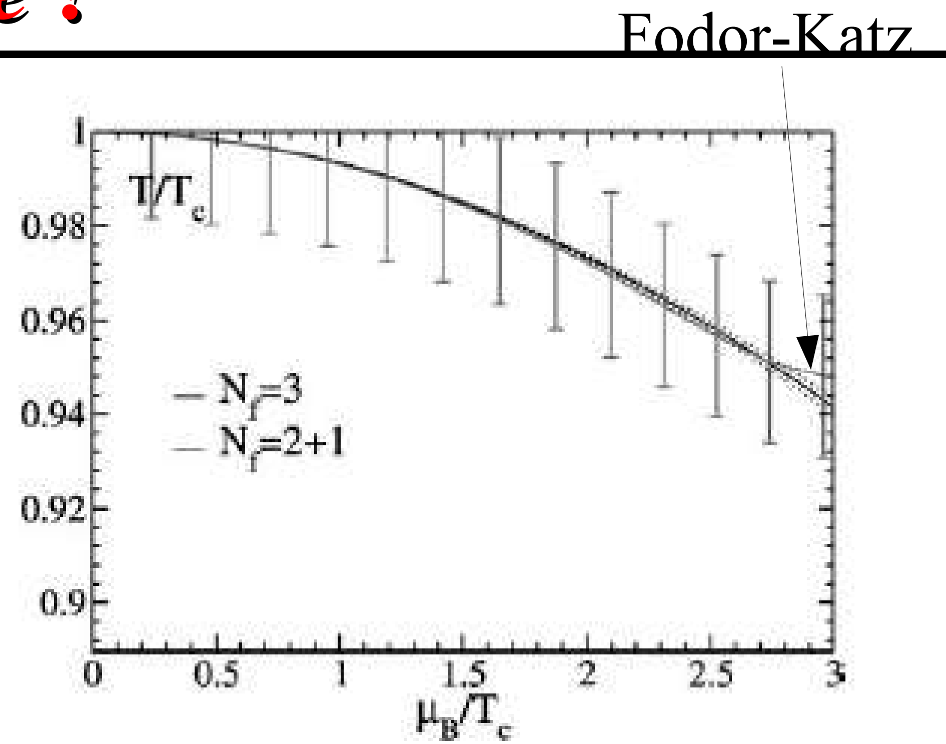
DKS1984
Tc=5/2 (U(3))
Nishida2004
Tc=5/3 (SU(3))
Bilic et al.
Tc~ 2.5 (f=1), 2.0 (f=3)

- ***Why we have $d\mu_c^{(1st)}/dT_c = d\mu_c^{(2nd)}/dT_c$***

$$F_{eff} = c_2\sigma^2 + c_4\sigma^4 + c_6\sigma^6$$
$$\rightarrow 4c_2c_6 = c_4^2$$
$$\rightarrow c_2 \simeq 0 \quad \text{around} \quad T_{tcp}$$

Debate ?

- Tc(μ) smoothly goes down or not.
  - de Forcrand-Philipsen → Analytic Continuation predicts smooth decrease.
  - Fodor-Katz → at TCP, the phase boundary seems to have a kink
- ***Why we have $d\mu_c^{(1st)}/dT_c = d\mu_c^{(2nd)}/dT_c$ → kink at TCP may suggest the exsitence of other order parameter(s)***



de Forcrand et al., NPB673,170(2003)

$$F_{eff} = c_2\sigma^2 + c_4\sigma^4 + c_6\sigma^6$$
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