Strong Coupling Limit Lattice QCD Approach to Nuclear Matter

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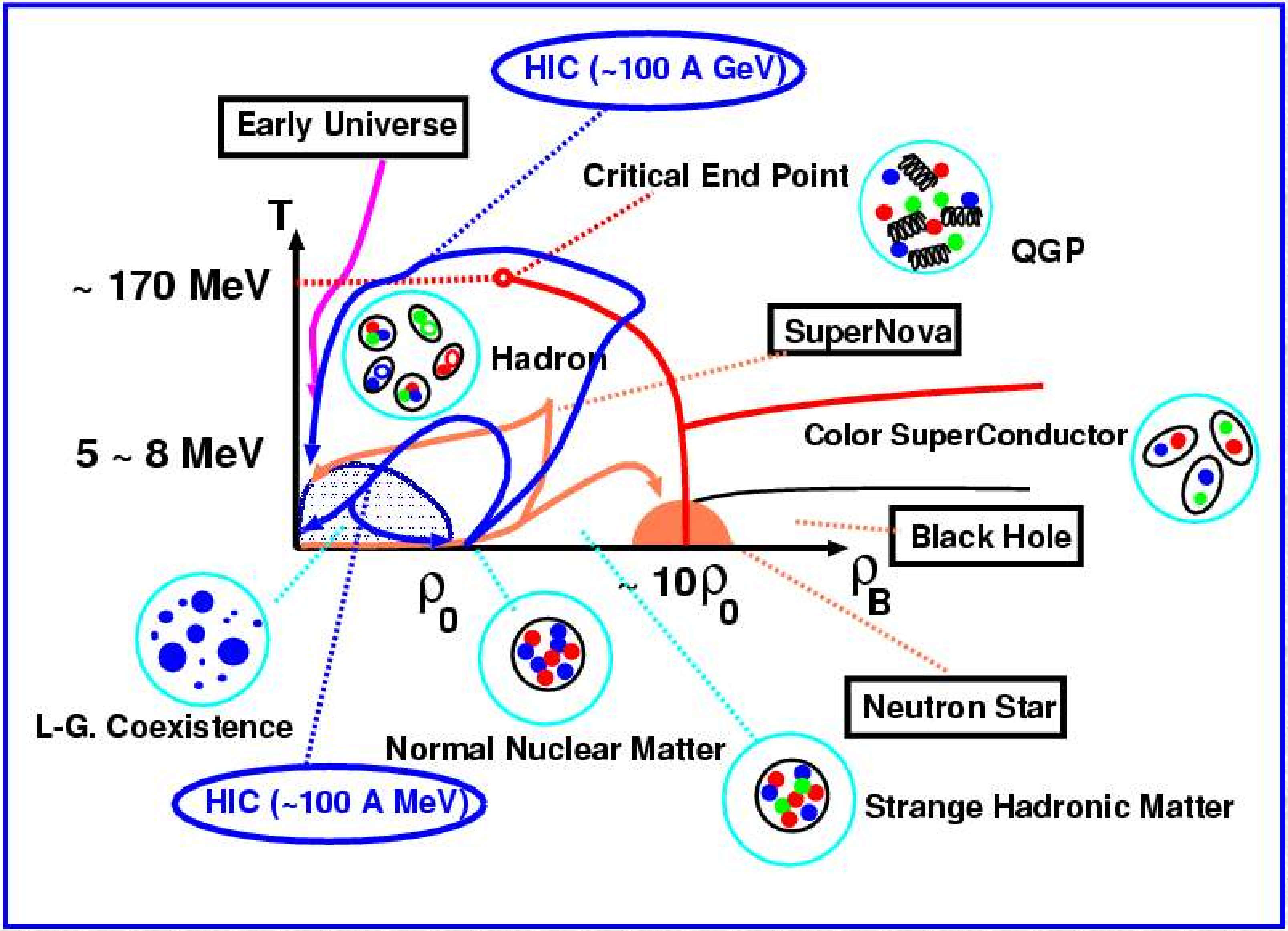
Hadronic Matter Phase Diagram



Phase Diagram in Strong Coupling Limit Lattice QCD with N_c=3

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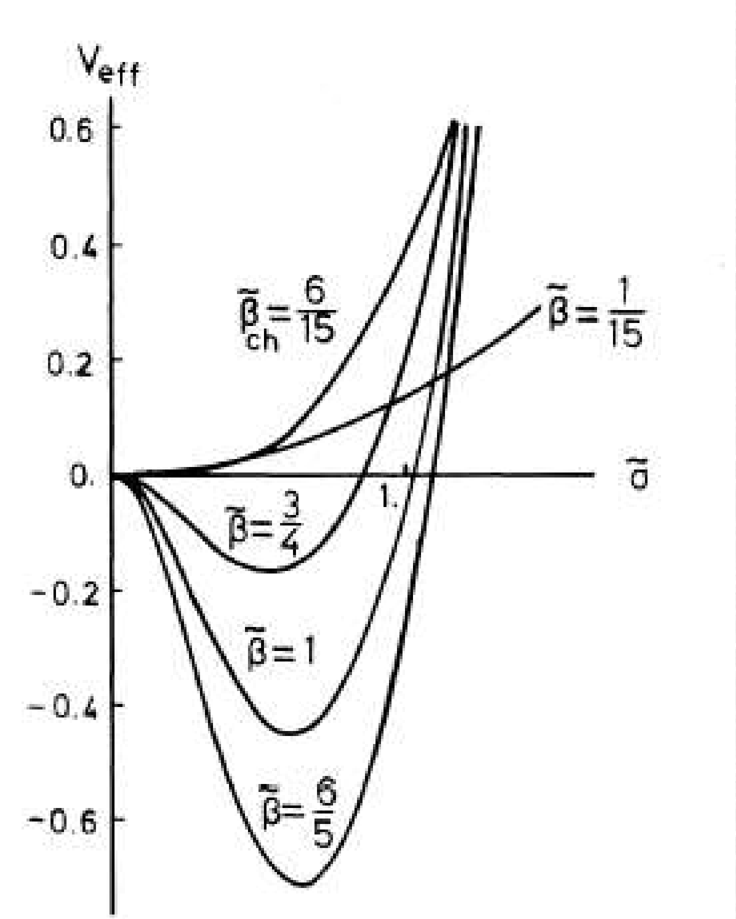
Strong Coupling Limit Lattice QCD (1)

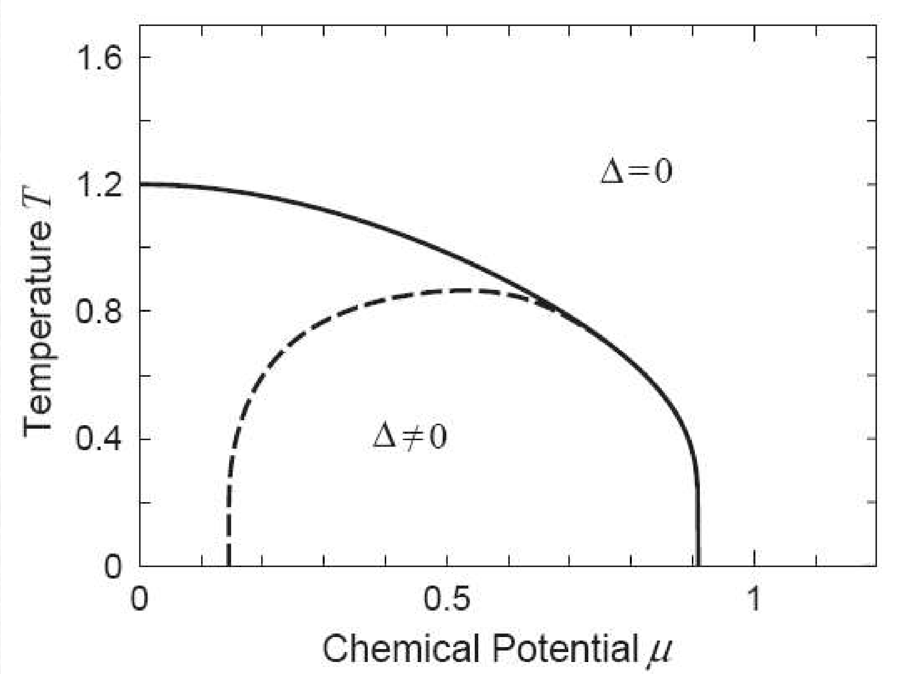
- **Full Lattice QCD at large μ and low T is not possible**
  - *Fermion Det. becomes complex → Monte-Carlo breaks down*
  - *Small μ → Re-Weighting / Expansion in μ*
- **Strong Coupling Limit: g → ∞**
  - *Semi-analytic analyses become possible.*
  - *At μ=0, Chiral Restoration at high T is explained.*
    - **Damgaard, Kawamoto, Shigemoto, PRL53(1984),2211**
  - *At $\mu \neq 0$ and $N_c = 2$, Phase diagram is drawn.*
    → **Baryon = Boson with $N_c = 2$**
    - **Nishida, Fukushima, Hatsuda, PRept 394(2004),281.**
  - *At $\mu \neq 0$, T=0 and $N_c = 3$, U_0 integral is done only approximately.*
    - **Azcoiti, Di. Carlo, Galante, Laliena, hep-lat/0307019.**

Strong Coupling Limit Lattice QCD (2)

- **Chiral Restoration at μ=0.**
  - *Damgaard, Kawamoto, Shigemoto, PRL53(1984),2211*



- **Phase Diagram with Nc=2**
  - *Nishida, Fukushima, Hatsuda, PRept 394(2004),281.*



$$F_{\rm eff}[\sigma,\Delta]=\frac{d}{2}\sigma^2+\frac{d}{2}|\Delta|^2-T\log\{1+4\cosh(E_+/T)\cosh(E_-/T)\}$$

$$E_\pm=\mathrm{arccosh}\left(\sqrt{(1+M^2)\cosh^2\mu+(d/2)^2|\Delta|^2}\pm M\sinh\mu\right)$$

Strong Coupling Limit Lattice QCD (3)

- ***Proper Understanding of QCD phase diagram with $N_c = 3$ is not achieved yet.***
  - *Nc=2: Diquark = Color Singlet Boson = Baryon*
    → No Fermi Energy for Baryons ?
  - *Nc=3: 1/d Expansion also for U_0 term.*
    → Conversion would be bad.
    - **Azcoiti, Di. Carlo, Galante, Laliena, hep-lat/0307019.**

- ***This work:***
  - *N_c=3: Baryon Integral is required*
  - *EXACT integral of U_0 term*
  - *Diquark condensate is tentatively ignored.*

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Useful Techniques in Lattice QCD

- **Fermion determinant** $\int D\chi D\bar{\chi} \exp(\bar{\chi} G \chi) = det\, G$
- **Group Integral**

$$\int \mathcal{D}[U] U_{ij} U^{\dagger}_{kl} = \frac{1}{N_c} \delta_{il}\delta_{jk} \, , \quad \int \mathcal{D}[U] U_{ij} U_{kl} U_{mn} = \frac{1}{N_c!} \varepsilon_{ikm}\varepsilon_{jln}$$

- **Polyakov Gauge and SU(3) Group Integral**

$$U_0(x) = diag(\exp(i\theta_1), \exp(i\theta_2), \exp(i\theta_3)), \theta_1 + \theta_2 + \theta_3 = 0$$

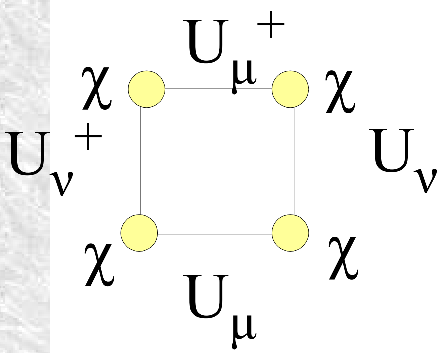
$$\int \mathcal{D}[U_0] = \prod_i \left[\int_{-\pi}^{\pi} \frac{d\theta_i}{2\pi} \right] \delta(\sum_j \theta_j) \Delta$$

$$\Delta = \prod_{i<j} \left| e^{i\theta_i} - e^{i\theta_j} \right|^2 = \prod_{i<j} 2\left[1 - \cos(\theta_i - \theta_j)\right]$$

- **Bosonization, Mean Field Approximation, ...**

Step 0: Lattice Action in SCL-LQCD

- ***Lattice Action with staggered Fermions***



$$S[U,\chi,\bar\chi] = S_G[U] + S_F[U,\chi,\bar\chi]\ ,$$

$$S_G[U] = \frac{2N_c}{g^2}\sum_{x,\mu,\nu}\left\{1-\frac{1}{N_c}\mathrm{ReTr}U_{\mu\nu}(x)\right\}$$

$g \rightarrow \infty$ → **0**

$$U_{\mu\nu}(x) = U_\nu^\dagger(x)U_\mu^\dagger(x+\hat\nu)U_\nu(x+\hat\mu)U_\mu(x)\ ,$$

Chem. Pot.

$$\begin{aligned}
S_F[U,\chi,\bar\chi] &= S_F^{(m)}[\chi,\bar\chi] + S_F^{(j)}[U_j,\chi,\bar\chi] + S_F^{(U0)}[U_0,\chi,\bar\chi]\ ,\\
S_F^{(m)}[\chi,\bar\chi] &= m\sum_x \bar\chi^a(x)\chi^a(x)\ ,\\
S_F^{(j)}[U_j,\chi,\bar\chi] &= \frac{1}{2}\sum_x\sum_{j=1}^{d}\eta_j(x)\left\{\bar\chi(x)U_j(x)\chi(x+\hat\jmath) - \bar\chi(x+\hat\jmath)U_j^\dagger(x)\chi(x)\right\}\ ,\\
S_F^{(U0)}[U_0,\chi,\bar\chi] &= \frac{1}{2}\sum_x \eta_0(x)\left\{\bar\chi(x)e^{\mu}U_0(x)\chi(x+\hat0) - \bar\chi(x+\hat0)U_0^\dagger(x)e^{-\mu}\chi(x)\right\}\ .
\end{aligned}$$

- ***In the Strong Coupling Limit ($g \rightarrow \infty$), we can ignore SG, and semi-analytic calculation becomes possible.***

Step 1: Integral over Uj: 1/d Expansion (1)

- **Group Integral**

$$\int \mathcal{D}[U] U_{ij} U^\dagger_{kl} = \frac{1}{N_c} \delta_{il} \delta_{jk} \ , \quad \int \mathcal{D}[U] U_{ij} U_{kl} U_{mn} = \frac{1}{N_c!} \varepsilon_{ikm} \varepsilon_{jln}$$

- **Expand $\exp(-S_F)$, and perform U_j integral.**

$$\begin{aligned} I_j &\equiv \int \mathcal{D}[U_j(x)] \exp\left[\frac{-\eta_j(x)}{2} \left\{ \bar{\chi}(x) U_j(x) \chi(x+\hat{j}) - \bar{\chi}(x+\hat{j}) U_j^\dagger(x) \chi(x) \right\}\right] \\ &= \int \mathcal{D}[U_j(x)] \ \{1 + \frac{1}{8} \left[\bar{\chi}(x) U_j(x) \chi(x+\hat{j}) - \bar{\chi}(x+\hat{j}) U_j^\dagger(x) \chi(x)\right]^2 \\ &\quad + \left(\frac{-\eta_j}{2}\right)^{N_c} \frac{1}{N_c!} \left[\bar{\chi}(x) U_j(x) \chi(x+\hat{j}) - \bar{\chi}(x+\hat{j}) U_j^\dagger(x) \chi(x)\right]^{N_c} + \ldots \} \end{aligned}$$

$$\begin{aligned} &= 1 - \frac{1}{4N_c} \bar{\chi}^a(x) \chi^b(x_j) \bar{\chi}^b(x_j) \chi^a(x) \\ &\quad + \left(\frac{-\eta_j}{2}\right)^{N_c} \frac{1}{(N_c!)^2} \varepsilon_{ace} \ \varepsilon_{bdf} \left[\bar{\chi}^a(x) \chi^b(x_j) \bar{\chi}^c(x) \chi^d(x_j) \bar{\chi}^e(x) \chi^f(x_j) \right. \\ &\quad \left. - \bar{\chi}^a(x_j) \chi^b(x) \bar{\chi}^c(x_j) \chi^d(x) \bar{\chi}^e(x_j) \chi^f(x)\right] + \ldots \end{aligned}$$

$$= \exp\left[\frac{1}{4N_c} M(x) M(x_j) + \left(\frac{-\eta_j}{2}\right)^{N_c} \left(\bar{B}(x) \ B(x_j) - \bar{B}(x_j) \ B(x)\right) + \ldots\right]$$

Step 1: Integral over Uj: 1/d Expansion (2)

- *Action in Meson and Baryon Fields made of Quarks*

$$S_F^{(j)}[\chi^a, \bar{\chi}^a] = -\frac{1}{2}(M, V_M M) - (\bar{B}, V_B B)$$

$$M(x) = \delta_{ab}\bar{\chi}^a(x)\chi^b(x) \ ,$$

$$B(x) = \frac{1}{6}\varepsilon_{abc}\chi^a(x)\chi^b(x)\chi^c(x) \ , \quad \bar{B}(x) = \frac{1}{N_c!}\varepsilon_{abc}\bar{\chi}^c(x)\bar{\chi}^b(x)\bar{\chi}^a(x)$$

$$V_M(x,y) = \frac{1}{4N_c}\sum_{j=1}^{3}\left(\delta_{y,x+\hat{j}} + \delta_{y,x-\hat{j}}\right)$$

$$V_B(x,y) = \sum_{j=1}^{3}\left(\frac{-\eta_j(x)}{2}\right)^{N_c}\left(\delta_{y,x+\hat{j}} - \delta_{y,x-\hat{j}}\right)$$

- *In SCL, spatial gluon components can be integrated out !*

Step 2: Auxiliary Fields and MFA (1)

- **Pfaffian Integral: Bi-Linear in Grassmann variables (χ and b) → Determinant**

$$G \equiv \int \mathcal{D}[\chi,\bar{\chi}] e^{\bar{\mathbf{X}}\mathbf{G}^{-1}\mathbf{X}} = \prod_{\mathbf{x}} \left\{\det\left[\mathbf{G}^{-1}_{ab}(m,n)\right]\right\}^{1/2} \qquad \mathbf{X} = (\chi,\bar{\chi})$$

- **Reduce the action to Bi-Linear form in χ → Auxiliary Fields**

  - *Baryon Field*

$$\exp\left[(\bar{B},V_B B)\right] = \det V_B \int \mathcal{D}[\bar{b},b] \exp\left[-(\bar{b},V_B^{-1}b) + (\bar{b},B) + (\bar{B},b)\right]$$

  - *Di-quark Field*

$$D_a = \gamma\varepsilon_{abc}\chi^b\chi^c + \frac{1}{6\gamma}\bar{\chi}^a b\ , \quad D_a^\dagger = -\gamma\varepsilon_{abc}\bar{\chi}^b\bar{\chi}^c + \frac{1}{6\gamma}\bar{b}\chi^a$$

$$D_a^\dagger D_a = 2\gamma^2 M^2 + \bar{B}b + \bar{b}B - \frac{1}{36\gamma^2} M\bar{b}b$$

$$\int \mathcal{D}[\phi_a,\phi_a^\dagger] \exp\left\{-(\phi_a^\dagger/2\gamma + D_a^\dagger)(\phi_a/2\gamma + D_a) + D_a^\dagger D_a\right\} = \exp\left\{D_a^\dagger D_a\right\}$$

Step 2: Auxiliary Fields and MFA (2)

- ***Auxiliary Fields (cont.)***
  - *Baryon Potenital (note:* $(\bar{b}b)^2=0$ *)*

$$\exp\left[\frac{1}{36\gamma^2}M\bar{b}b\right] = \int \mathcal{D}[\omega]\exp\left[-\frac{1}{2g_\omega^2}\omega^2 - \frac{\omega}{g_\omega}(\alpha M + g_\omega \bar{b}b) - \frac{1}{2}\alpha^2 M^2\right]$$

  - *Chiral Condensate* σ

$$\begin{aligned}
&\exp\left[\frac{1}{2}(M,\widetilde{V}_M M) - \frac{\alpha}{g_\omega}(\omega, M)\right] \\
&= \int \mathcal{D}[\sigma]\exp\left[-\frac{1}{2}(\sigma',\widetilde{V}_M^{-1}\sigma') + \frac{1}{2}(M,\widetilde{V}_M M) - \frac{\alpha}{g_\omega}(\omega, M)\right] \\
&= \int \mathcal{D}[\sigma]\exp\left[-\frac{1}{2}(\sigma - \alpha\omega/g_\omega, \widetilde{V}_M^{-1}(\sigma - \alpha\omega/g_\omega)) - (\sigma, M)\right]
\end{aligned}$$

$$\sigma'(x) = \sigma(x) - \alpha\omega(x)/g_\omega + \widetilde{V}_M(x,y)M(y)\ .$$

Step 2: Auxiliary Fields and MFA (3, Summary)

- *Action with Auxiliary Fields*
  - *Bi-Linear in χ and b → Determinant Technique*
  - *Local in position → Small Matrix Size*
  - *Contains coupling term of χ and b → Problematic ...*

$$S_F[U_0, \chi^a, \bar{\chi}^a, b, \bar{b}, \phi_a, \phi_a^\dagger, \omega, \sigma] = S_F^{(q)} + S_F^{(X)}$$

$$S_F^{(X)} = (\bar{b}, \widetilde{V}_B^{-1} b) + (\phi^\dagger, \phi)/4\gamma^2 + \frac{1}{2}\omega^2/g_\omega^2 + \frac{1}{2}(\sigma - \alpha\omega/g_\omega, \widetilde{V}_M^{-1}(\sigma - \alpha\omega/g_\omega))$$

$$\begin{aligned}
S_F^{(q)} &= S_F^{(m)} + S_F^{(jq)} + S_F^{(U0)} \\
&= (\sigma + m_0, M) + (\phi_a^\dagger, D_a)/2\gamma + (D_a^\dagger, \phi_a)/2\gamma + S_F^{(U0)} \\
&= (\bar{\chi}^a, (\sigma + m_0)\chi^a) + (\bar{\chi}^a, \phi_a^\dagger b/12\gamma^2) + (\bar{b}\phi_a/12\gamma^2, \chi^a) \\
&\quad + \frac{1}{2}\varepsilon_{cab}\left[(\phi_c^\dagger, \chi^a\chi^b) - (\bar{\chi}^a\bar{\chi}^b, \phi_c)\right] \\
&\quad + \frac{1}{2}\sum_x \left\{\bar{\chi}(x)e^\mu U_0(x)\chi(x+\hat{0}) - \bar{\chi}(x+\hat{0})U_0^\dagger(x)e^{-\mu}\chi(x)\right\}
\end{aligned}$$

Step 3: Fermion Determinant (1-1)

- ***Fourier Transformation***
  - *Anti-Periodic Boundary Condition is Satisfied*
  - *"Derivative" term becomes sin(kt)*

$$\psi(x) = \frac{1}{\sqrt{\beta}} \sum_{m=1}^{\beta} e^{ik_m\tau} \psi_m(\mathbf{x}) \ , \quad \bar{\psi}(x) = \frac{1}{\sqrt{\beta}} \sum_{m=1}^{\beta} e^{-ik_m\tau} \bar{\psi}_m(\mathbf{x})$$

$$S_F^{(q)} = \sum_{\mathbf{x}} \sum_{m=1}^{\beta} \left[B_m^a \bar{\chi}_m^a \chi_m^a + \bar{C}_m^a \chi_m^a + \bar{\chi}_m^a C_m^a + \frac{1}{2} \varepsilon_{abc} (\phi_c^\dagger \chi_m^a \chi_{m'}^b - \bar{\chi}_m^a \bar{\chi}_{m'}^b \phi_c) \right]$$

$$C_m^a(\mathbf{x}) = \frac{1}{12\gamma^2} \phi_a^\dagger(\mathbf{x}) b_m(\mathbf{x}) \ , \quad \bar{C}_m^a(\mathbf{x}) = \frac{1}{12\gamma^2} \bar{b}_m(\mathbf{x}) \phi_a(\mathbf{x})$$

$$B_m^a(\mathbf{x}) = m_0 + \sigma(\mathbf{x}) + i \sin(k_m + \theta^a(\mathbf{x})/\beta - i\mu)$$

Step 3: Fermion Determinant (1-2)

- ***Pfaffian Form***
  - *G: Size=β (Number of Time-Step) x 3 (color) x 2 (χ, χbar)*

$$
\begin{aligned}
S_F^{(q)} &= \frac{1}{2}\sum_{\mathbf{x},m,n,a,b}\Bigg[(\bar{\chi}_m^a, \chi_m^a)\begin{pmatrix} B_m^a\delta_{ab}\delta_{mn} & -\varepsilon_{cab}\phi_c\delta_{m,\beta-n+1} \\ \varepsilon_{cab}\phi_c^\dagger\delta_{m,\beta-n+1} & -B_m^a\delta_{ab}\delta_{mn}\end{pmatrix}\begin{pmatrix}\chi_n^b \\ \bar{\chi}_n^b\end{pmatrix} \\
&\qquad + (\bar{\chi}_m^a, \chi_m^a)\begin{pmatrix} C_m^a \\ -\bar{C}_m^a\end{pmatrix} + (\bar{C}_m^a, -C_m^a)\begin{pmatrix}\chi_m^a \\ \bar{\chi}_m^a\end{pmatrix}\Bigg] \\
&= \frac{1}{2}\sum_{\mathbf{x},m,n,a,b}\left[\bar{\mathbf{X}}_m^a(\mathbf{x})\mathbf{G}_{ab}^{-1}(m,n;\theta(\mathbf{x}))\mathbf{X}_n^b(\mathbf{x}) + \left(\bar{\mathbf{X}}_m^a(\mathbf{x})\mathbf{Y}_m^a(\mathbf{x}) + \bar{\mathbf{Y}}_m^a(\mathbf{x})\mathbf{X}_m^a(\mathbf{x})\right)\right] \\
&= \sum_{\mathbf{x},m,n,a,b}\left[\frac{1}{2}\left(\bar{\mathbf{X}} + \bar{\mathbf{Y}}\mathbf{G}\right)_m^a \mathbf{G}_{ab}^{-1}(m,n)\left(\mathbf{X} + \mathbf{G}\mathbf{Y}\right)_n^b - \frac{1}{2}\bar{\mathbf{Y}}_m^a\mathbf{G}_{ab}(m,n)\mathbf{Y}_n^b\right]
\end{aligned}
$$

$$m' = \beta - m + 1$$

Step 3: Fermion Determinant (1-3)

- ***Determinant of Big Matrix G***
  - *Block diagonal → 6 x 6 Matrix (g)*

$$\int \mathcal{D}[\chi,\bar{\chi}] \exp\left[-S_F^{(q)}\right] = \prod_{\mathbf{x}} \prod_{m=1}^{\beta} \left\{-\det\left[\mathbf{g}_{ab}(m)\right]\right\}^{1/2}$$

$$\begin{aligned} g(\mathbf{x},k_m) \equiv & -\det\left[\mathbf{g}_{ab}(m;\theta^a(\mathbf{x}))\right] \\ = & (B_1|\phi_1|^2 + B_2|\phi_2|^2 + B_3|\phi_3|^2)(B_1'|\phi_1|^2 + B_2'|\phi_2|^2 + B_3'|\phi_3|^2) \\ & + \sum_{(a,b,c)=cyc.} B_a B_a' |\phi_a|^2 (B_b B_c' + B_b' B_c) + B_1 B_2 B_3 B_1' B_2' B_3' \end{aligned}$$

  - *Matsubara Frequency Sum → Total Matrix Determinant*

$$G(\mathbf{x}) = \left[\prod_j (1 + \cos\beta z_j(\mathbf{x}))\right]^{1/2}$$

Step 3: Fermion Determinant (1-3b)

- *Matsubara Sum Technique*
  - *Using z (the solution of g(k)=0)*

$$\log G(\mathbf{x}) = \log\left[\prod_{m=1}^{\beta} g(\mathbf{x},k_m)\right] = \sum_m \log\left[\prod_{j=1}^{6}\left(\cos k_m - rY_j(\mathbf{x})\right)\right]^{1/2} \qquad (2\cdot21)$$

$$\begin{aligned}
\frac{d\log G(\mathbf{x})}{dr} &= \frac{1}{2}\sum_{m,j}\frac{-Y_j}{\cos k_m - rY_j} \\
&= \frac{1}{2\Omega}\sum_j\left[\oint\frac{dz}{2\pi i}\frac{-Y_j}{\cos z - rY_j}\frac{-i\beta}{1+e^{i\beta z}} - \sum_{z_j^r}\frac{-Y_j}{-\sin z_j^r}\frac{-i\beta}{1+e^{i\beta z_j^r}}\right] \\
&= \frac{i\beta}{2\Omega}\sum_{j,z_j^r}\frac{Y_j}{\sin z_j^r}\frac{1}{1+\exp(i\beta z_j^r)} = \frac{-i\beta}{2\Omega}\sum_{j,z_j^r}\frac{dz_j^r}{dr}\frac{1}{1+\exp(i\beta z_j^r)} \\
&= \frac{d}{dr}\frac{1}{2\Omega}\sum_{j,z_j^r}\log\left(1+\exp(-i\beta z_j^r)\right) \qquad (2\cdot22)
\end{aligned}$$

$$\log G(\mathbf{x}) = \frac{1}{2\Omega}\sum_{j,z_j}\log\left(1+\exp(-i\beta z_j)\right) + \text{const.} \qquad (2\cdot23)$$

Step 3: Fermion Determinant (2-1)

- ***Baryon Integral***

$$\int \mathcal{D}[b,\bar{b}] \exp\left[-\sum_{m,n}\left(\bar{b}_m, \widetilde{V}_B^{-1} b_n\right)\right]$$

$$\widetilde{V}_B^{-1}(x,y) = V_B^{-1}(x,y) + \omega\delta_{x,y}$$

$$V_B(x,y) = \sum_{j=1}^{3}\left(\frac{-\eta_j(x)}{2}\right)^{N_c}\left(\delta_{y,x+\hat{j}} - \delta_{y,x-\hat{j}}\right)$$

  - *Different Spatial Points are connected → Spatial Fourier Transf.*

$$b_m(\mathbf{x}) = \frac{1}{\sqrt{L^3}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} b_{m\mathbf{k}}\ , \quad \mathbf{k} = \frac{2\pi}{L}(k_1, k_2, k_3)$$

  - *Momentum Repr. of V_B:*

$$\begin{aligned} V_B(m\mathbf{k}, n\mathbf{k}') &= \frac{-i}{4}\delta^{\beta}_{m,n}\sum_{j=1,2,3}\prod_{1\le i<j}\delta^L_{k_i,k'_i}\prod_{l\ge j}\delta_{k_l,k'_l}\sin k_j \\ &= \frac{-i}{4}\delta^{\beta}_{m,n}\delta_{k_3,k'_3}\left[\delta_{k_1,k'_1}\delta_{k_2,k'_2}\sin k_1\right. \\ &\quad \left.+\delta^L_{k_1,k'_1}\delta_{k_2,k'_2}\sin k_2 + \delta^L_{k_1,k'_1}\delta^L_{k_2,k'_2}\sin k_3\right]\ , \end{aligned}$$

$$\delta^{\beta}_{m,n} = \delta_{m,n+\beta/2} + \delta_{m,n-\beta/2}\ , \quad \delta^L_{a,b} = \delta_{a,b+\pi} + \delta_{a,b-\pi}$$

Step 3: Fermion Determinant (2-2)

- ***Baryon Matrix in Momentum Repr. and Determinant***

$$\sum_{m,n} \left(\bar{b}_m, \widetilde{V}_B^{-1} b_n\right)$$

$$= \sum_{k_1=1}^{L/2} \sum_{k_2=1}^{L/2} \sum_{k_3} \sum_{m=1}^{\beta/2} \begin{pmatrix}\bar{\mathbf{b}}_m & \bar{\mathbf{b}}'_m\end{pmatrix} \begin{pmatrix} 0 & -\frac{i}{4}\mathbf{S}(\mathbf{k}) \\ -\frac{i}{4}\mathbf{S}(\mathbf{k}) & 0 \end{pmatrix} \begin{pmatrix}\mathbf{b}_m \\ \mathbf{b}'_m\end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} \sin k_1 & \sin k_2 & \sin k_3 & 0 \\ \sin k_2 & -\sin k_1 & 0 & \sin k_3 \\ \sin k_3 & 0 & -\sin k_1 & -\sin k_2 \\ 0 & \sin k_3 & -\sin k_2 & \sin k_1 \end{pmatrix}$$

$$\mathbf{b}_m = (b_{m\mathbf{k}^{(1)}},\ b_{m\mathbf{k}^{(2)}},\ b_{m\mathbf{k}^{(3)}},\ b_{m\mathbf{k}^{(4)}})$$

$$\mathbf{b}'_m = \left(b_{m+\beta/2,\mathbf{k}^{(1)}},\ b_{m+\beta/2,\mathbf{k}^{(2)}},\ b_{m+\beta/2,\mathbf{k}^{(3)}},\ b_{m+\beta/2,\mathbf{k}^{(4)}}\right)$$

$$\mathbf{k}^{(1)} = (k_1, k_2, k_3), \quad \mathbf{k}^{(2)} = (k_1+\pi, k_2, k_3)\ ,$$

$$\mathbf{k}^{(3)} = (k_1+\pi, k_2+\pi, k_3), \quad \mathbf{k}^{(4)} = (k_1, k_2+\pi, k_3)\ ,$$

Step 3: Fermion Determinant (2-3)

- *Effective Potential from Baryon Loop*

$$\exp(-\beta L^3 F_{\mathrm{eff}}^{(b)}) \equiv \mathrm{Det} V_B \int \mathcal{D}[b,\bar{b}] \exp\left[-\sum_{m,n}\left(\bar{b}_m, \widetilde{V}_B^{-1} b_n\right)\right]$$

$$= \mathrm{Det} V_B \mathrm{Det}\left[V_B^{-1} + \omega \mathbf{1}\right] = \mathrm{Det}\left[1 + \omega V_B\right]$$

$$= \prod_{k_1=1}^{L/2} \prod_{k_2=1}^{L/2} \prod_{k_3=1}^{L} \prod_{m=1}^{\beta/2} \det\left[\begin{pmatrix} 1 & -i\omega \mathbf{S}/4 \\ -i\omega \mathbf{S}/4 & 1 \end{pmatrix}\right]$$

$$= \prod_{k_1=1}^{L/2} \prod_{k_2=1}^{L/2} \prod_{k_3=1}^{L} \prod_{m=1}^{\beta/2} \left(1 + \omega^2 s^2/16\right)^4 = \prod_{k_1=1}^{L/2} \prod_{k_2=1}^{L/2} \prod_{k_3=1}^{L} \left(1 + \omega^2 s^2/16\right)^{2\beta}$$

$$= \prod_{\mathbf{k}} \left(1 + \omega^2 s^2/16\right)^{\beta/2}$$

$$F_{\mathrm{eff}}^{(b)} = -\frac{1}{2L^3} \sum_{\mathbf{k}} \log\left[1 + \frac{\omega^2 s^2}{16}\right]$$

$$\simeq -\frac{a_0^{(b)}}{2} \left(\frac{4\pi}{3}\Lambda^3\right)^{-1} \int_0^{\Lambda} 4\pi k^2 dk \log\left[1 + c^2 k^2\right]$$

Step 4: Gluon Integral (1)

- ***Link Variable in Polyakov gauge***

$$U_0(x) = diag(\exp(i\theta_1), \exp(i\theta_2), \exp(i\theta_3)), \theta_1 + \theta_2 + \theta_3 = 0$$

- ***SU(3) Haar Measure***

$$\int \mathcal{D}[U_0] = \prod_i \left[\int_{-\pi}^{\pi} \frac{d\theta_i}{2\pi}\right] \delta(\sum_j \theta_j)\Delta$$

$$\Delta = \prod_{i<j} \left| e^{i\theta_i} - e^{i\theta_j} \right|^2 = \prod_{i<j} 2\left[1 - \cos(\theta_i - \theta_j)\right]$$

Step 4: Gluon Integral (2)

- ***Quark Integral at φ=0 (without diquark condensate)***

$$\beta z_i^{\pm} = \theta_i - i\beta\mu \pm i\beta \sinh^{-1}\widetilde{\sigma}\ , \quad \widetilde{\sigma} = m_0 + \sigma$$

$$\sinh^{-1} x = \log(\sqrt{1+x^2} + x)$$

$$G = \prod_i \left[(1+\cos\beta z_i^+)(1+\cos\beta z_i^-)\right]^{1/2} = F_1 F_2 F_3$$

$$F_i = C_\sigma + C_\mu \cos\theta_i - i S_\mu \sin\theta_i$$

$$C_\sigma = \cosh\left[\beta \sinh^{-1}\widetilde{\sigma}\right]$$

$$C_\mu = \cosh\beta\mu\ , \quad S_\mu = \sinh\beta\mu$$

- ***Gluon Integral***

$$\int dU_0 G = C_\sigma^3 - \frac{1}{6} C_\sigma C_\mu^2 - \frac{1}{3} C_\sigma + \frac{3}{4} C_\mu^3 - \frac{1}{2} C_\mu$$

Effective Potential at Zero Quark Condensate

- ***Fermion Integral, Matsubara Freq. Sum, and U_0 Integral***
  → Effective Action at Zero Diquark Condensate

$$F_{\rm eff} = \frac{\sigma^2}{2\alpha^2} + F_{\rm eff}^{(q)} + F_{\rm eff}^{(b)}$$

$$F_{\rm eff}^{(q)} = -T \log \left(C_\sigma^3 - \frac{1}{6} C_\sigma C_\mu^2 - \frac{1}{3} C_\sigma + \frac{3}{4} C_\mu^3 - \frac{1}{2} C_\mu \right)$$

$$C_\sigma = \cosh \left[\beta \sinh^{-1} \sigma \right] \ , \quad C_\mu = \cosh \beta\mu$$

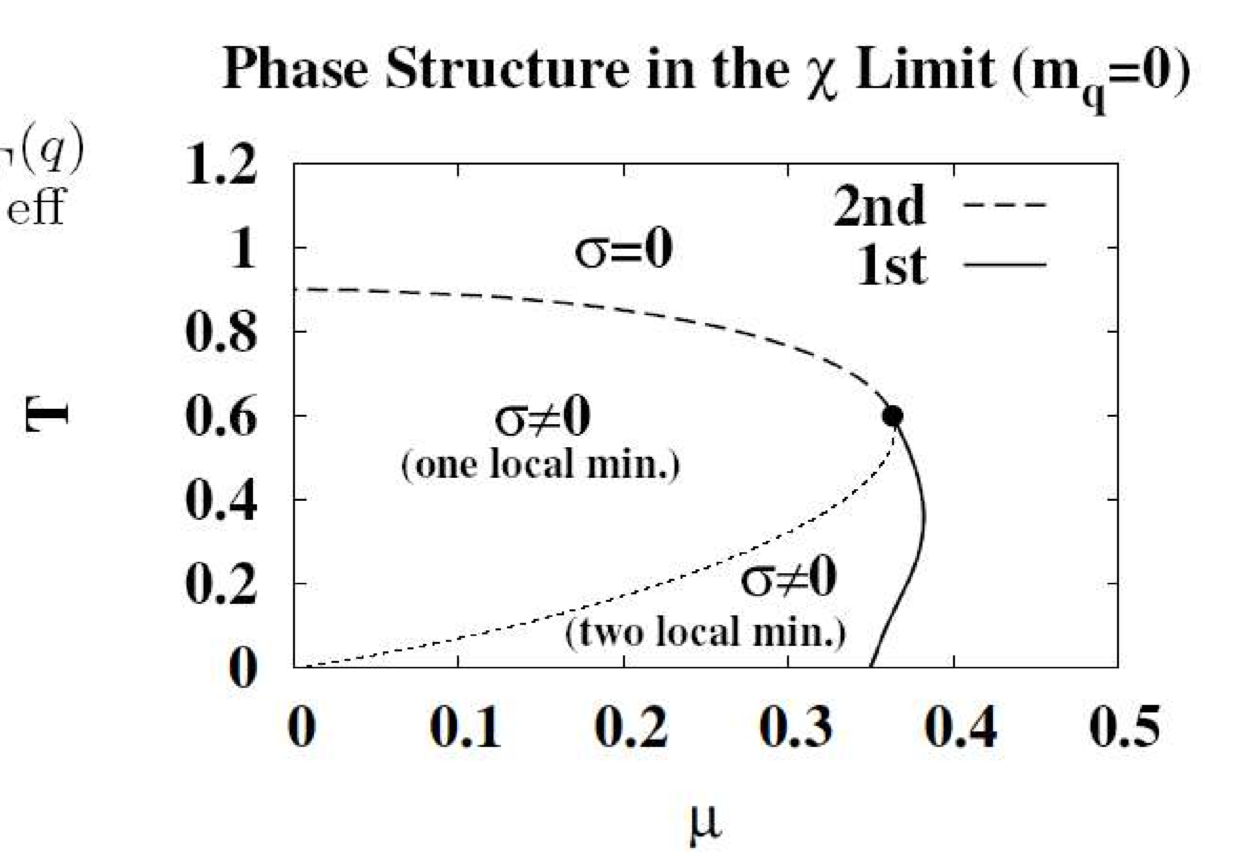
$$F_{\rm eff}^{(b)} \simeq -a_0^{(b)} f^{(b)}(c\Lambda) \ , \quad f^{(b)}(x) = \frac{3}{2x^3} \int_0^x k^2 dk \log(1+k^2)$$

- ***Two Types of Fermion Log(Det) Terms !***

Phase Diagram

- ***Minimum of Effective Action → Phase Diagram***
  - *Kawamoto, Miura, AO, Ohnuma, in preparation.*

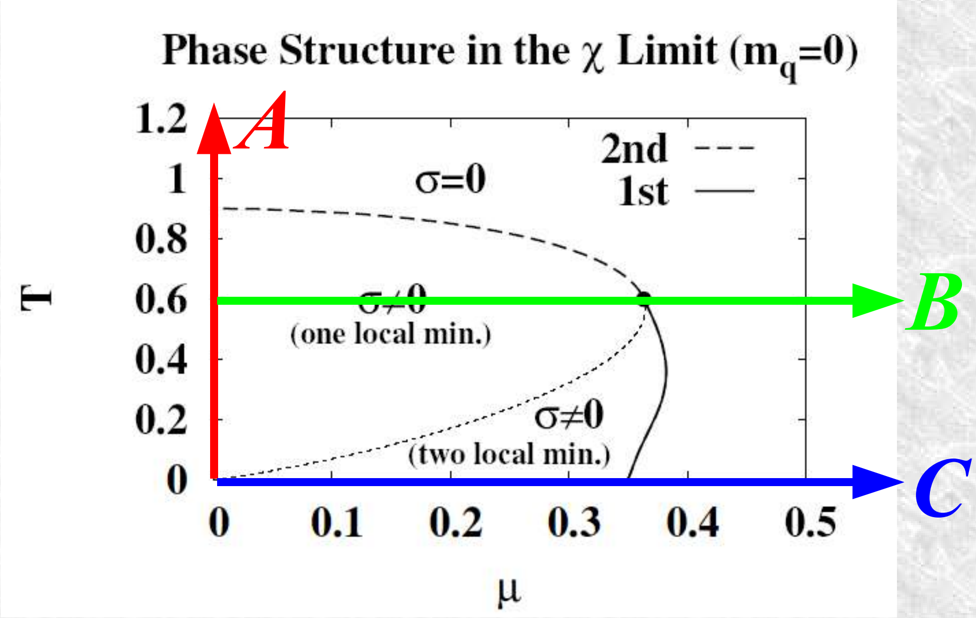
$$F_{\rm eff} = \frac{\sigma^2}{2\alpha^2} + F_{\rm eff}^{(b)} + F_{\rm eff}^{(q)}$$

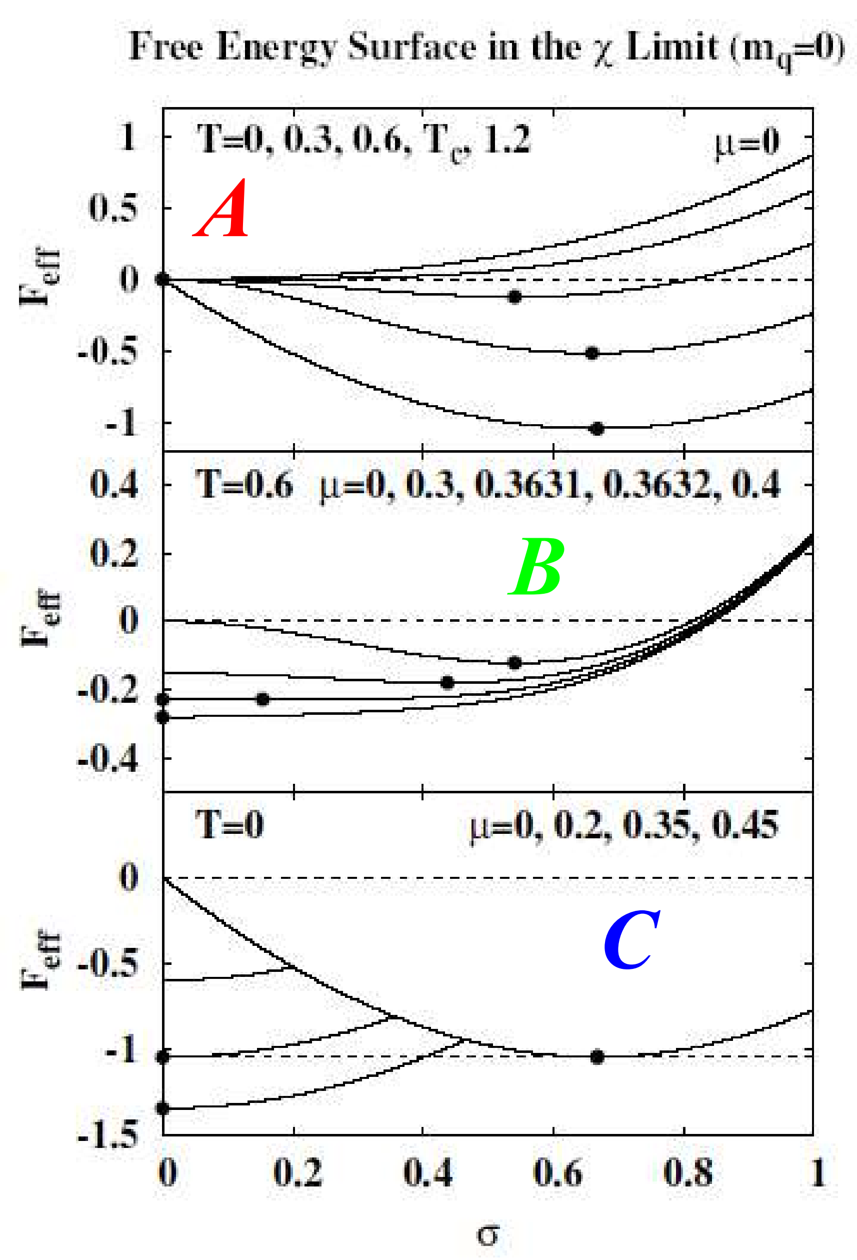


- ***Change from 2nd order to 1st order at Finite μ***

Free Energy Surface

- *At $\mu \neq 0$, quark can gain Free Energy even at $\sigma = 0$*
  → Two Min. Structure
  → First Order





Treatment of Diquark Condensate (1)

- ***Diquark Condensate with Nc=3***

$$D_a = \gamma \varepsilon_{abc} \chi^b \chi^c + \frac{1}{6\gamma} \bar{\chi}^a b$$

  - *Have Color → Expectation Value=0 → Cannot be Order Par.*

- ***Color Singlet Combination*** $v^2 = \phi_a^* \phi_a$

  - *Idea: Leave v^2, and integrate other "angle" variables*

$$Y = \frac{\gamma^2}{2} M^2 - \frac{1}{9\gamma^2} M \bar{b} b = D_a^\dagger D_a - (\bar{b} B + \bar{B} b)$$

$$e^{\bar{b} B + \bar{B} b} = \int \mathcal{D}[\phi_a, \phi_a^\dagger] e^{-\phi_a^\dagger \phi_a + \phi_a^\dagger D_a + D_a^\dagger \phi_a - Y}$$

$$\int \mathcal{D}[\phi_a, \phi_a^\dagger] \exp\left\{\phi_a^\dagger D_a + D_a^\dagger \phi_a\right\} = \int \mathcal{D}[v] \exp\left\{\frac{v^2}{3} D_a^\dagger D_a + \frac{v^4}{162} M^3 \bar{b} b\right\}$$

Treatment of Diquark Condensate (2)

- **Self-Consistent Replacement of quark-baryon coupling**

$$
\begin{aligned}
&\exp(\bar{b}B + \bar{B}b) \\
&\simeq \exp\left[-v^2 - Y + \frac{v^2}{3}\left(\bar{b}B + \bar{B}b + Y\right) + \frac{v^4}{162}M^3\bar{b}b\right] \\
&\simeq \exp\left[\frac{-v^2}{1-v^2/3} + E(v)M^3\bar{b}b - Y\right] \\
&E(v) = \frac{v^4}{162(1-v^2/3)}
\end{aligned}
$$

- **Reduction of $M^n\bar{b}b$ term**

$$
e^{EM^3\bar{b}b} = \int \mathcal{D}[\omega_2] e^{-\omega_2^2/2 - \omega_2(g_2 M + EM^2\bar{b}b/g_2) - g_2 M^2/2}
$$

Treatment of Diquark Condensate (3)

- *Effective Potential with Chiral and Color Condensates*

$$F_{\text{eff}}(v,\sigma,\omega_i) = F_X + F_b + F_q$$

$$F_X = \frac{1}{2}(a_\sigma \sigma'^2 + \omega_0^2 + \omega_1^2 + \omega_2^2) + \frac{v^2}{1 - v^2/3} ,$$

$$\sigma' = \sigma - (g_0\omega_0 + g_1\omega_1 + g_2\omega_2) ,$$

$$a_\sigma = 2\left(1 - \gamma^2 - g_0^2 - g_1^2 - g_2^2\right)^{-1} ,$$

$$F_q = -T \log\left(C_\sigma^3 - \frac{1}{6}C_\sigma C_\mu^2 - \frac{1}{3}C_\sigma + \frac{3}{4}C_\mu^3 - \frac{1}{2}C_\mu\right)$$

$$C_\mu = \cosh\beta\mu ,$$

$$C_\sigma = \cosh\left[\beta \sinh^{-1}\widetilde{\sigma}\right] ,$$

$$\widetilde{\sigma} = m_0 + \sigma .$$

$$F_b = \frac{1}{2L^3}\sum_{\mathbf{k}} \log\left[1 + \frac{c^2 s^2(\mathbf{k})}{16}\right] \simeq -a_0^{(b)} f^{(b)}\left(\frac{c\Lambda}{4}\right)$$

$$f^{(b)} = \frac{3}{2x^3}\int_0^x k^2 dk \log(1+k^2) ,$$

$$s^2(\mathbf{k}) = \sin^2 k_1 + \sin^2 k_2 + \sin^2 k_3 ,$$

$$c_b = \frac{\omega_0}{g_0}\left[\frac{1}{9\gamma^2} + \frac{\omega_1\omega_2}{g_1 g_2}E(v)\right] ,$$

Summary

- ***While full Lattice QCD is not (yet?) applicable to study low T and high ρ matter, we can obtain qualitative feature of the Phase Diagram with the Strong Coupling Limit of LQCD with Nc=3.***
  - *With Nc = 3, Two Fermion Integrals would give different results from Nc=2 case.*
  - *2nd order → 1st order as μ increases. (Chiral Limit)*
- ***With Diquark Condensate, we have developed "angle average" technique for colored condensate.***
  - *Consistent with the results with previous one when v=0*
  - *Diquark condensate can grow when σ is small.*
  - *Phase diagram → To be investigated later*

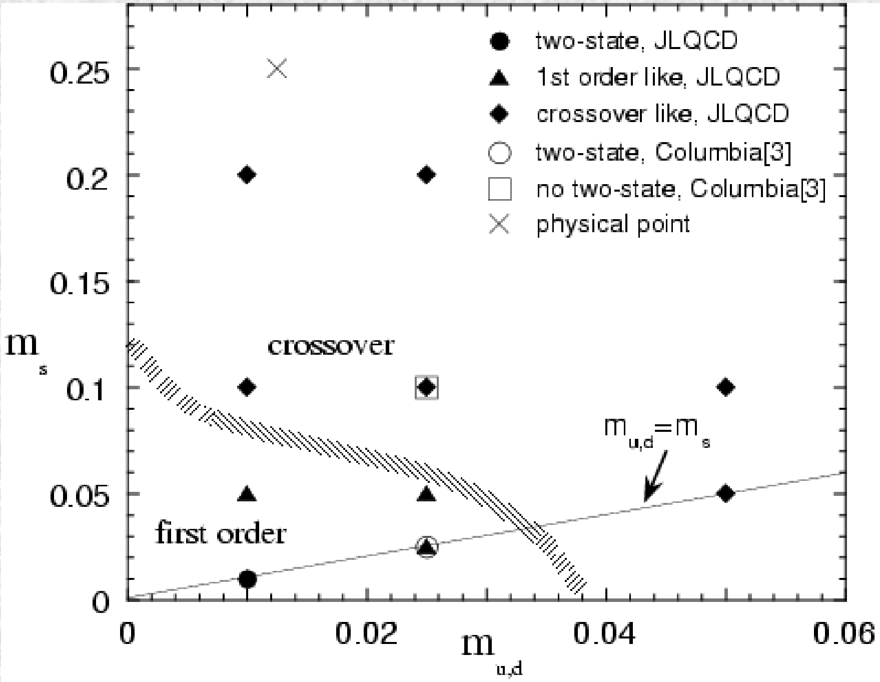
Collaborators

- ***T. Ohnuma (M1)***
- ***N. Kawamoto (Hokkaido U.)***
- ***K. Miura (M2)***

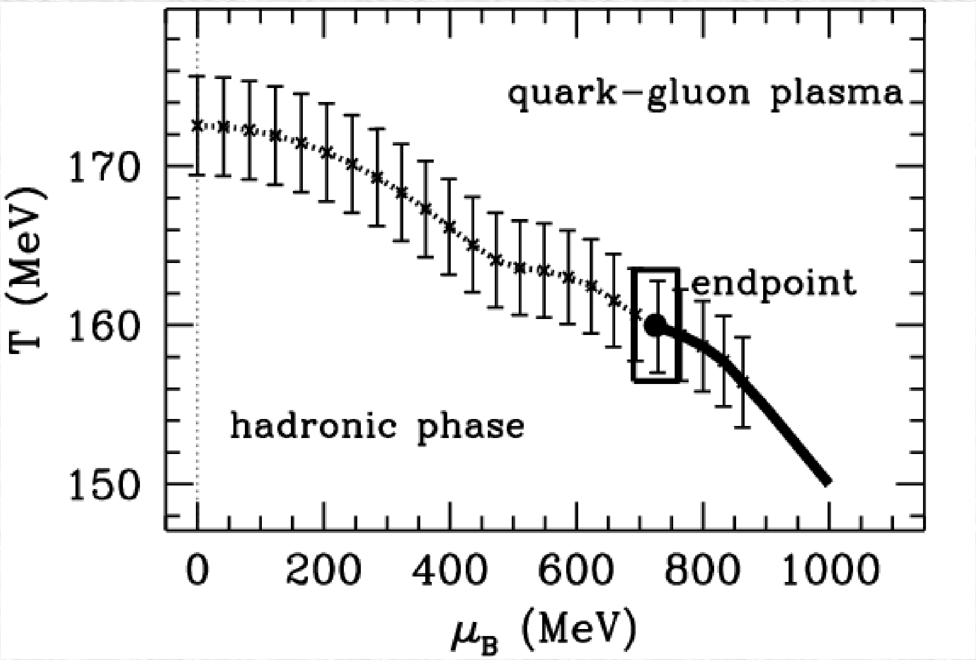
Thank You !

QCD Phase Diagram from Lattice QCD

- ***Zero Chem. Pot.***



- ***Finite Chem. Pot.***



✯ *JLQCD Collab. (S. Aoki et al.), Nucl. Phys. Proc. Suppl. 73 (1999), 459.*

✯ *Fodor & Katz, JHEP 0203 (2002), 014.*

Zero Chem. Pot. : Cross Over
Finite Chem. Pot.: Critical End Point

Step 2: Auxiliary Fields and MFA (2b)

- ***Auxiliary Fields***

$$\exp\left[(\bar{B}, V_B B)\right] = \det V_B \int \mathcal{D}[\bar{b}, b] \exp\left[-(\bar{b}, V_B^{-1} b) + (\bar{b}, B) + (\bar{B}, b)\right]$$

$$\exp\left(\bar{b}B + \bar{B}b\right)$$

$$= \int \mathcal{D}[\phi_a, \phi_a^\dagger] \exp\left\{-\frac{1}{4\gamma^2}\phi_a^\dagger\phi_a - \frac{1}{2\gamma}(\phi_a^\dagger D_a + D_a^\dagger\phi_a) + \frac{1}{36\gamma^2}M\bar{b}b - 2\gamma^2 M^2\right\}$$

$$D_a = \gamma\varepsilon_{abc}\chi^b\chi^c + \frac{1}{6\gamma}\bar{\chi}^a b\,, \quad D_a^\dagger = -\gamma\varepsilon_{abc}\bar{\chi}^b\bar{\chi}^c + \frac{1}{6\gamma}\bar{b}\chi^a$$

$$\exp\left[\frac{1}{36\gamma^2}M\bar{b}b\right] = \int \mathcal{D}[\omega] \exp\left[-\frac{1}{2g_\omega^2}\omega^2 - \frac{\omega}{g_\omega}(\alpha M + g_\omega\bar{b}b) - \frac{1}{2}\alpha^2 M^2\right]$$

$$\exp\left[\frac{1}{2}(M, \widetilde{V}_M M) - \frac{\alpha}{g_\omega}(\omega, M)\right] = \int \mathcal{D}[\sigma] \exp\left[-\frac{1}{2}(\sigma', \widetilde{V}_M^{-1}\sigma') - (\sigma, M)\right]$$

$$\sigma'(x) = \sigma(x) - \alpha\omega(x)/g_\omega\ .$$

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RMF with σ Self Energy from Strong Coupling Limit Lattice QCD

RMF with Chiral Symmetry (1)

- **Good (approximate) Symmetry in QCD**
  - *Only the current quark mass terms break chiral sym.*
  - *Spontaneously Broken, and* $\langle \bar{q} q \rangle$ *determines hadron masses*
- **Schematic model: Linear σ model**

$$L=\frac{1}{2}\left(\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \pi \partial^\mu \pi\right) - \frac{\lambda}{4}\left(\sigma^2+\pi^2\right)^2 + \frac{\mu^2}{2}\left(\sigma^2+\pi^2\right) + c\sigma$$
$$+\bar{N} i \partial_\mu \gamma^\mu N - g_\sigma \bar{N}\left(\sigma + i\pi\tau\gamma_5\right) N$$

- **Problem: χ Sym. is restored at a very small density.**
  - *Smaller Nucleon Mass Energies are preferred*
  - *σω Coupling stabilizes normal vacuum, but gives Too Stiff EOS*
    - **J. Boguta, PLB120,34/PLB128,19.**
    - **Ogawa et al. PTP 111 (2004)75.**

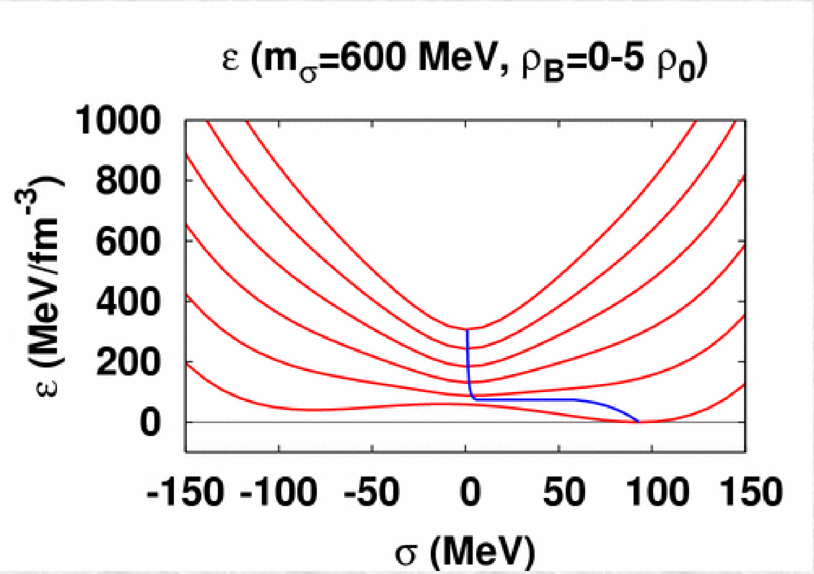
RMF with Chiral Symmetry (2)

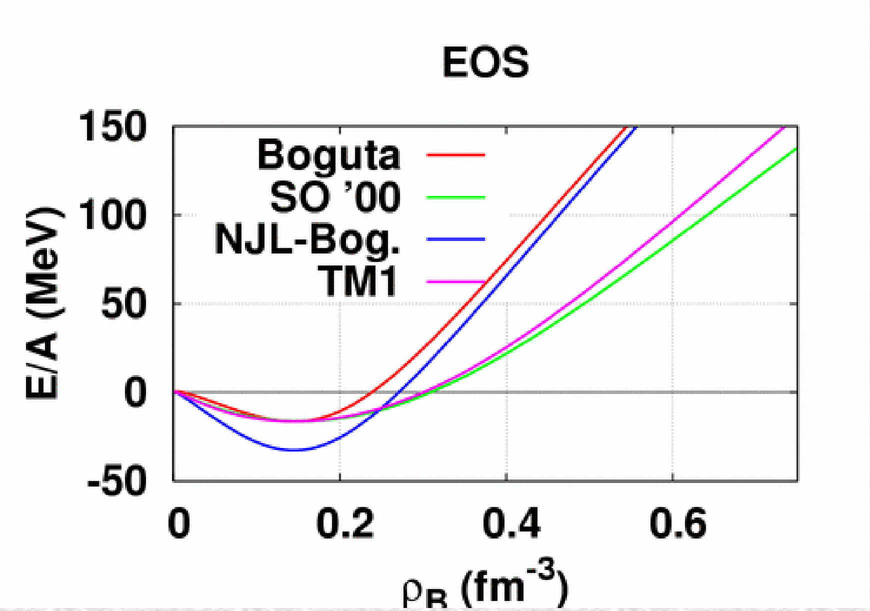
- ***Sudden Change of <σ>***
- ***σ ω Coupling***

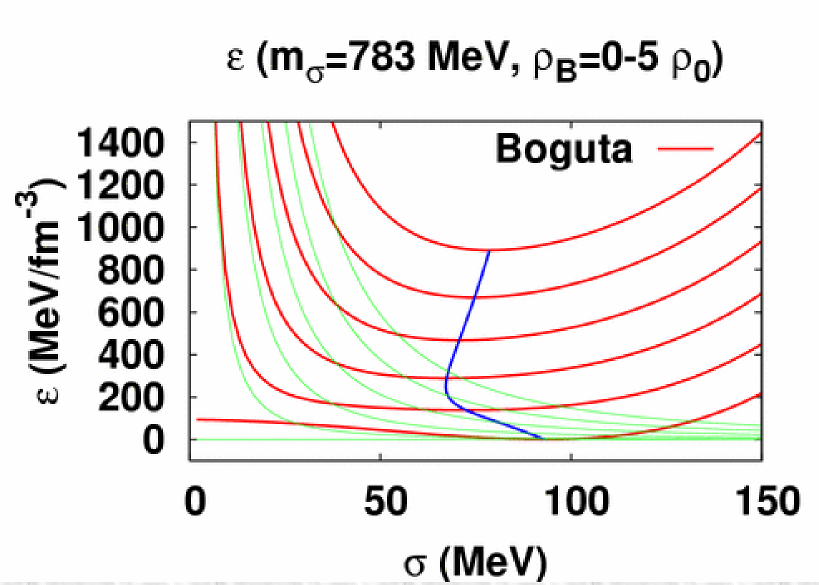
$$L_{\omega\sigma} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} C_{\sigma\omega} \sigma^2 \omega^2 - g_\omega \overline{N} \gamma_\mu \omega^\mu N$$

$$\omega = g_\omega \rho_B / C_{\sigma\omega} \sigma^2 \quad \rightarrow \quad V_{\sigma\omega} = \frac{g_\omega^2 \rho_B^2}{2 C_{\sigma\omega} \sigma^2}$$

- ***Stiff EOS***







RMF with σ Self Energy from SCL-LQCD

- **σ Self Energy from simple Strong Coupling Limit LQCD**

$$
\begin{aligned}
S &\to -\tfrac{1}{2}(M, V_M M) && (1/d \text{ expansion}) \\
&\to b\sigma^2 + (\bar{\chi}\ \sigma\chi) && (\text{auxiliary field}) \\
&\to b\sigma^2 - a\log\sigma^2 && (\text{Fermion Integral})
\end{aligned}
$$

- **RMF Lagrangian**
  - *σ is shifted by f_π, and small explicit χ breaking term is added.*

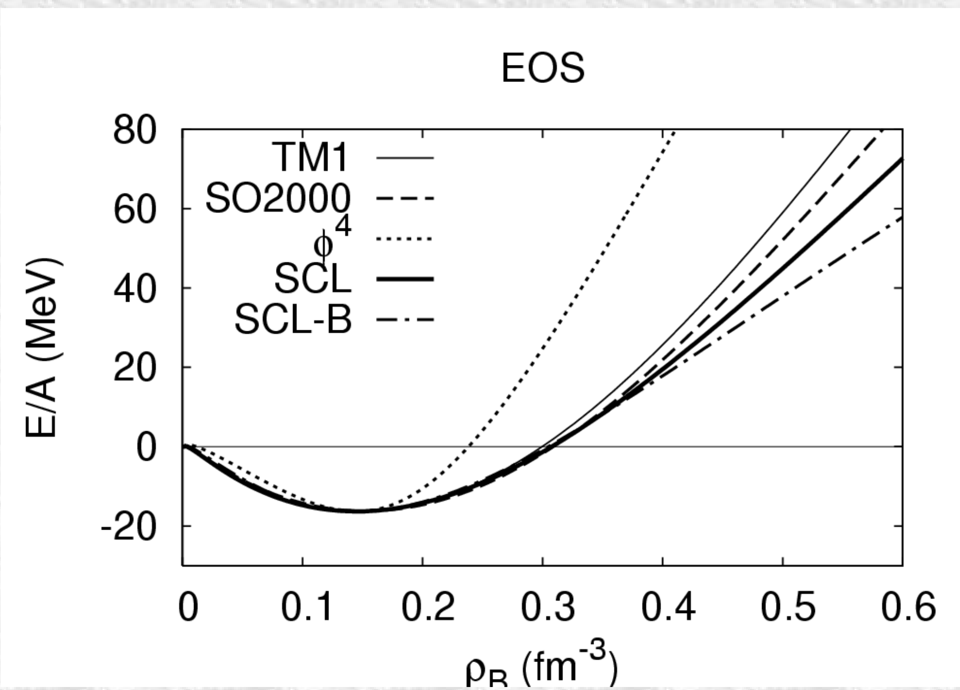
$$
\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - \gamma^\mu V_\mu - M + g_\sigma\sigma\right)\psi + \mathcal{L}_\sigma^{(0)} + \mathcal{L}_\omega^{(0)} + \mathcal{L}_\rho^{(0)} - U_\sigma + \frac{\lambda}{4}(\omega_\mu\omega^\mu)^2
$$

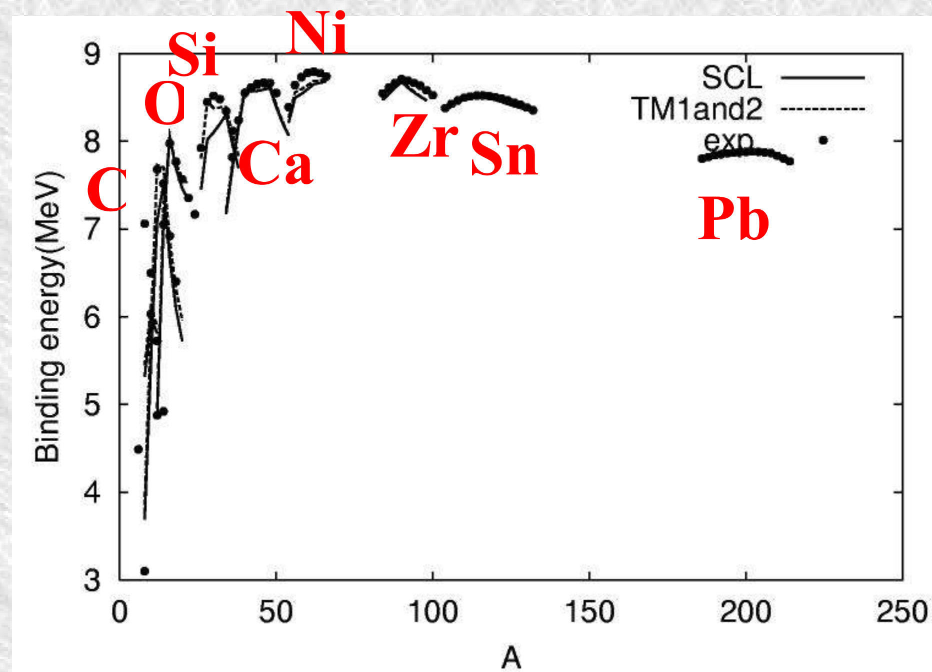
$$
U_\sigma = -af\left(\frac{\sigma}{f_\pi}\right) , \quad f(x) = 2\log(1+x) - 2x + x^2 , \quad a = \frac{f_\pi^2}{4}(m_\sigma^2 - m_\pi^2)
$$

Nuclear Matter and Finite Nuclei

- ***Nuclear Matter***
  - *By tuning λ, $g_{\omega N}$, m_{σ},* ***Soft EOS*** *can be obtained in Chirally Symmetric*

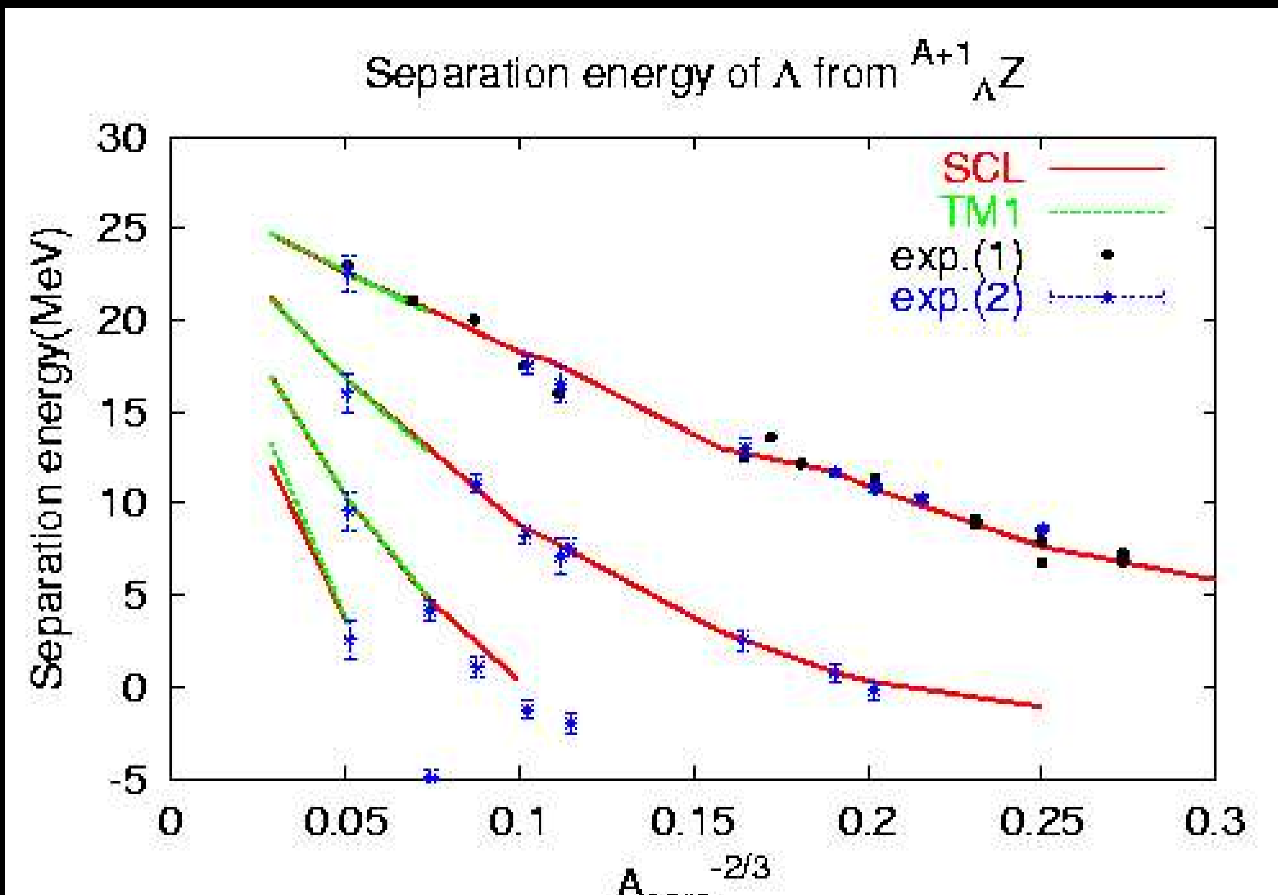
- ***Finite Nuclei***
  - *By tuning gρN, Global behavior of Nuclear B.E. is reproduced, except for j-j closed nuclei. (C, Si, Ni)*

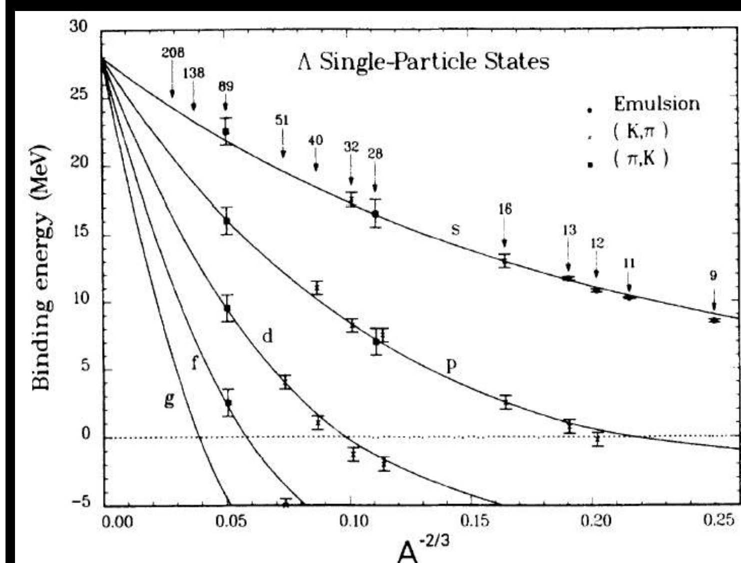




Single Λ Hypernucleus

- Λ の分離エネルギー





•各一粒子状態の分離エネルギーを良く再現する事が出来ている