Quarkyonic Phase in Lattice QCD at Strong Coupling

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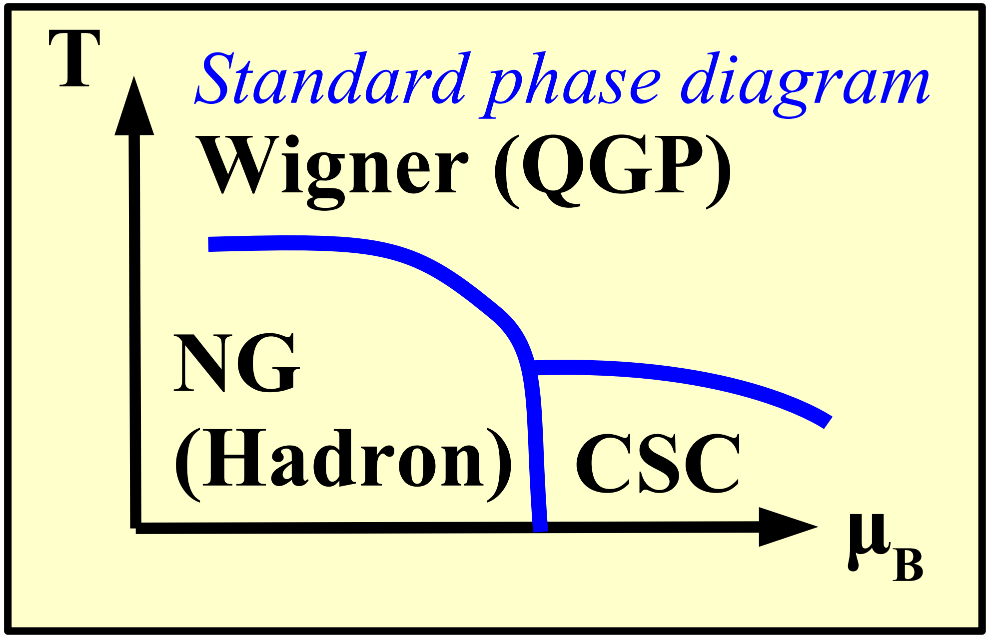
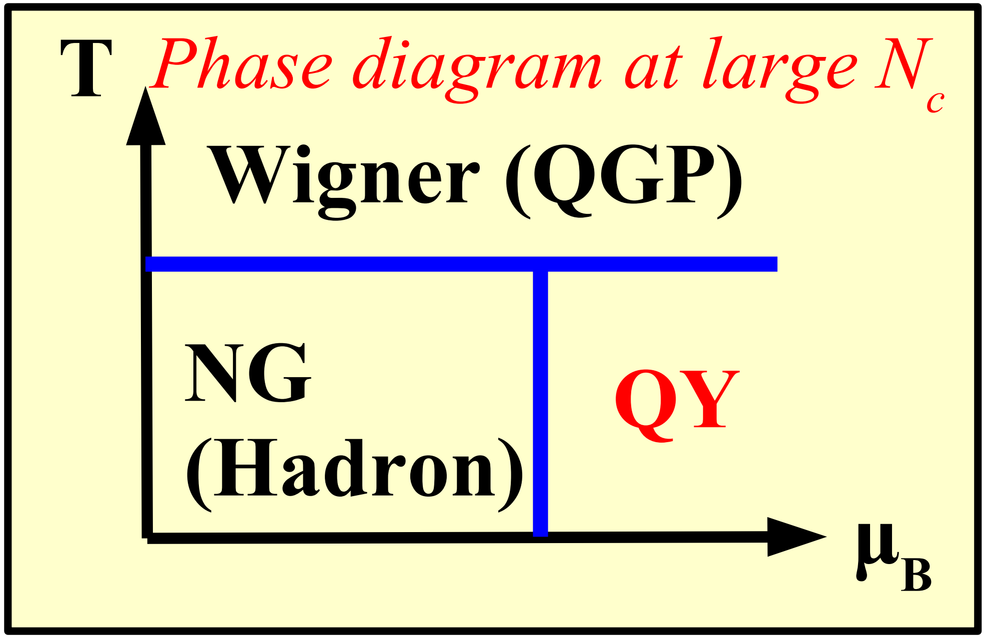
- **Introduction**
- **Strong Coupling Lattice QCD with $1/g^2$ Correction**
- **Phase diagram in SC-LQCD and Quarkyonic Phase**
- **Summary**

Miura and AO, arXiv:0806.3357
Our previous refs. on $1/g^2$ corr.
    AO et al. PTP Suppl. 168(07) 261 [arXiv:0704.2823]
    AO, Kawamoto, Miura, J.Phys.G34(07)S655 [hep-lat/0701024]

Introduction: Quark Matter Phase Diagram

- **What is the NEXT to the hadron phase ?**
  - **High T direction → (Strongly correlated) Quark Gluon Plasma**
  - **High μ direction ; Important for Dense Matter Physics**
    Baryon rich QGP
    or Color SuperConductor (CSC)
    or Quarkyonic (QY) matter *McLerran, Pisarski (07)*





Quarkyonic Phase & Strong Coupling Lattice QCD

- **Quarkyonic matter at large N_c** *McLerran, Pisarski (07)*
  - **Gluon contribution $O(N_c^2)$ >> Quark $O(N_c)$, Hadron $O(1)$**
    → Gluonic (deconf.) P.T. is independent of μ (as far as $\mu = O(1)$)
  - **At $N_c\mu > M_B$, baryon density rapidly grows, and soon reaches $O(N_c)$**
    → Existence of "Confined High Baryon Density Matter"
    made of *quark*s but with bar*yonic* excitation (Quarkyonic Matter)

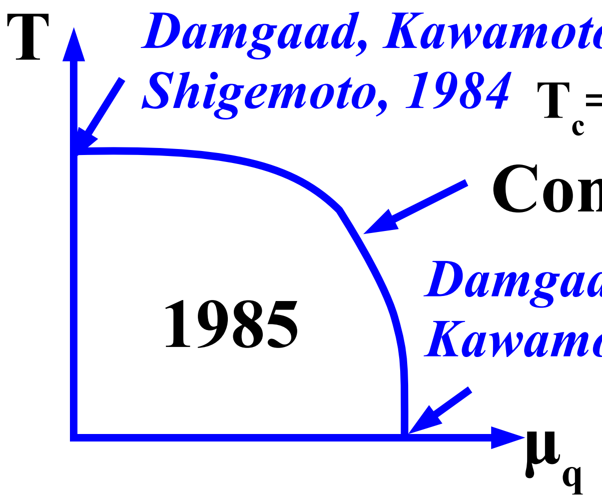
Do we have Quarkyonic phase at N_c=3 ?

- **Strong Coupling Limit ($g \to \infty$)**
  - **Plaquette action disappears**
  - **In MFA, Eff. Pot. is analytically obtained at finite T and μ.**
  - **Problem: Too small $R = N_c\mu_c/T_c$ value**

Does the finite coupling effects ($1/g^2$) helps ?

"Evolution" of Phase Diagram

- **Phase Diagram "Shape" becomes closer to that of Real World.**
  - **Real world R=3 μ_c/T_c ~ (6-12)**
  - **1985 → R=0.79 (Zero T / Finite T)**
  - **1992 → R=0.83 (Finite T & μ)**
  - **2004 → R= 0.99 (Finite T& μ)**
  - **2007 → R=1.34 (Baryon)**
    but it is still too small .....



Damgaad, Kawamoto, Shigemoto, 1984 **Finite T**

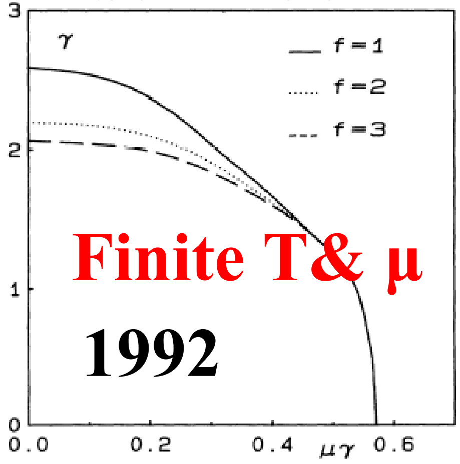
T_c=1.1 GeV

Conjecture !

Damgaad, Hochberg, Kawamoto, 1985

Finite μ

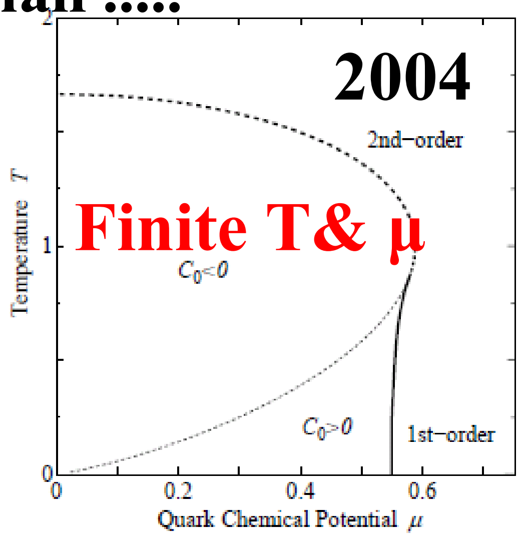
μ_c=290 MeV



Finite T& μ

1992

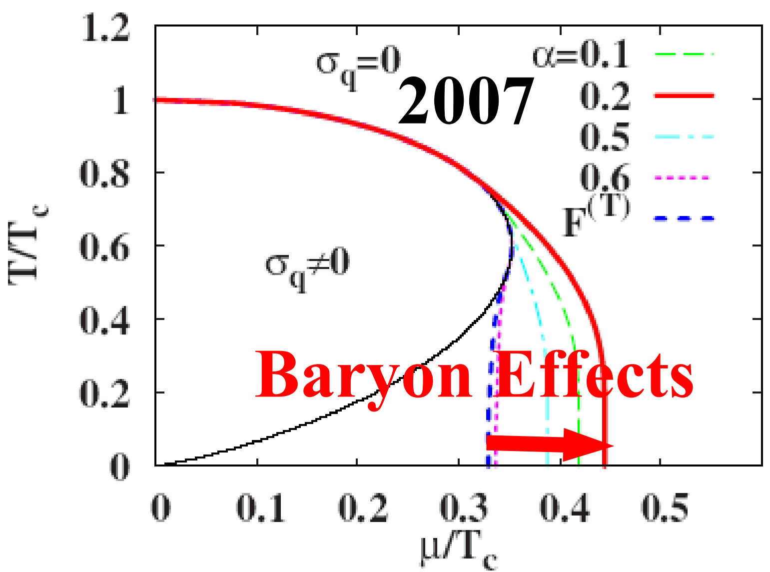
Bilic, Karsch, Redlich, 1992



2004

Finite T& μ

Fukushima, 2004



2007

Baryon Effects

Kawamoto, Miura, AO, Ohnuma, 2007

Quarkyonic phase in SC-LQCD

- **We study the phase diagram**
  in Strong Coupling Lattice QCD (SC-LQCD)
  with $1/g^2$ correction,
  and examine the existence of the Quarkyonic (QY) phase.

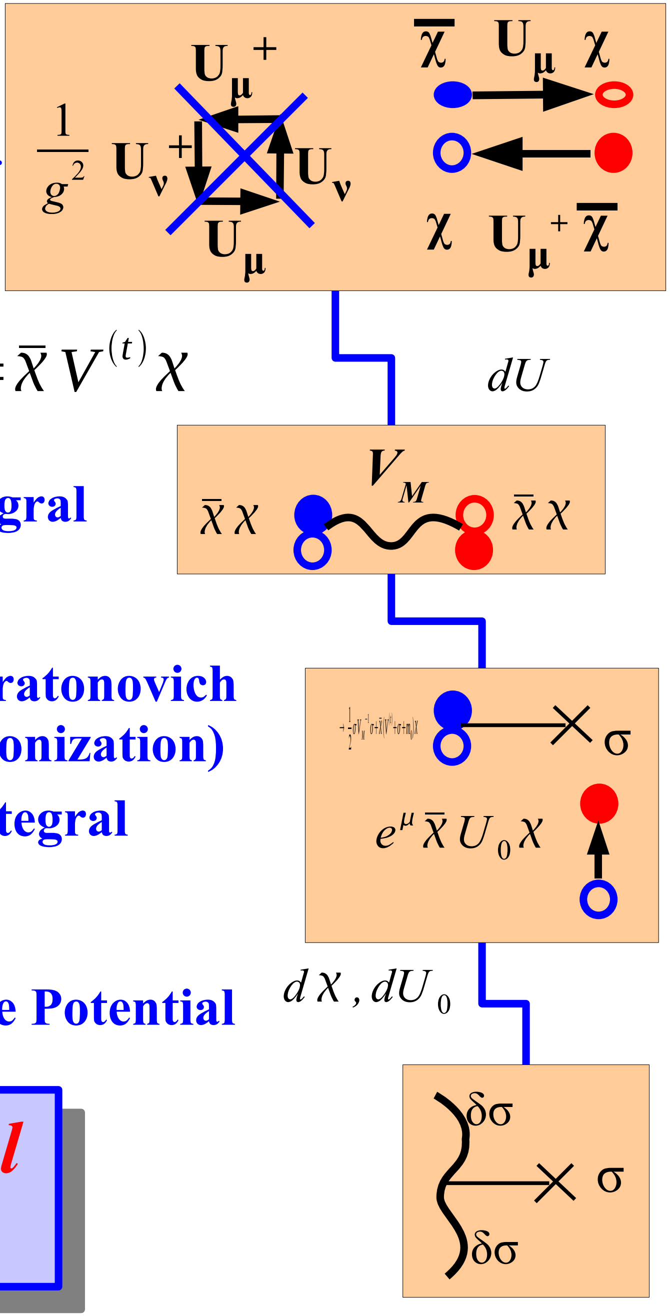
- **Reservations:** ***It is still a "Toy"***
  - **One species of staggered fermion without quarter/square root**
    → $N_f = 4$
  - **Leading order in 1/d (d=spatial dim.)**
    → No baryon effects (cf. *Par-Tue, Miura***)**
  - **Mean Field treatment**
  - **No Diqaurk condensate**
  - **NLO in $1/g^2$ expansion, ...**

Effective Potential in Strong Coupling lattice QCD with $1/g^2$ Correction

Effective Potential in SCL-LQCD

- **QCD Lattice Action (Finite T treatment)**

Damgaard, Kawamoto, Shigemoto, 1984; Bilic et al., 1992; Nishida, '04; Fukushima, '04; Kawamoto, Miura, AO, Ohnuma, '07; ...



$$S = \cancel{S_G} + S_F^{(s)} + S_F^{(t)} + m_0 \bar\chi \chi$$

Strong Coupling Limit

$$S_F^{(t)} = \frac{1}{2} \sum_x \left(e^\mu \bar\chi_x U_0(x) \chi_{x+\hat{0}} - e^{-\mu} \bar\chi_{x+\hat{0}} U_0^+(x) \chi_x \right) = \bar\chi V^{(t)} \chi$$

$$\to -\frac{1}{2} (\bar\chi \chi) V_M (\bar\chi \chi) + \bar\chi (V^{(t)} + m_0) \chi$$

Spatial-link integral (1/d expansion)

$$\to \frac{1}{2} \sigma V_M \sigma + \bar\chi (V^{(t)} + V_M \sigma + m_0) \chi$$

Hubbard-Stratonovich Transf. (Bosonization)

Fermion and Temporal-link Integral

$$\to L^d N_\tau \boxed{\left[\frac{d}{4 N_c} \bar\sigma^2 + V_q(b_\sigma \bar\sigma, T, \mu) \right]}$$

SCL Effective Potential

We can obtain the Effective Potential analytically at finite T and μ

Effective Potential with $1/g^2$ (1)

- **1/d expansion of Plaquette action (Spatial One-Link Integral)**

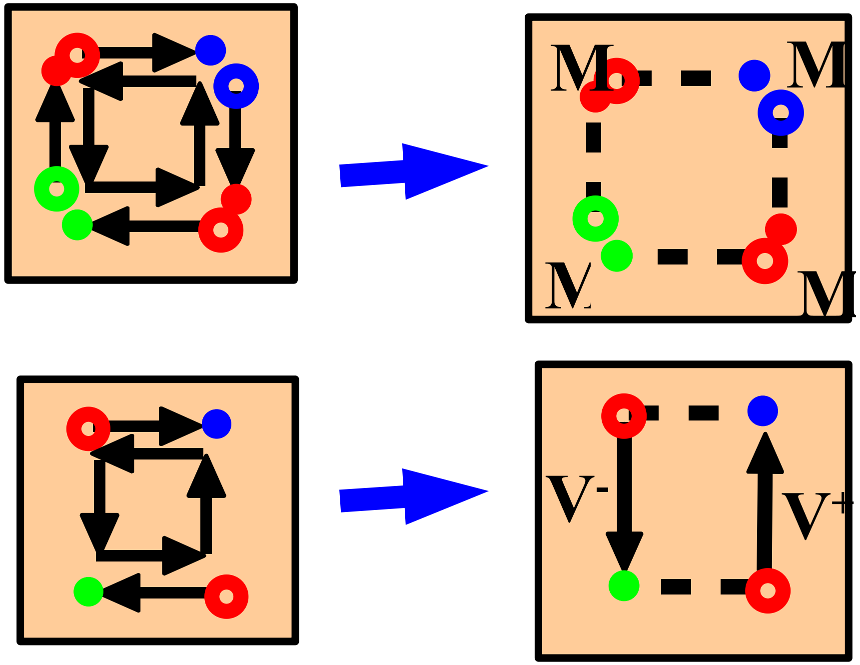
Faldt, Petersson (86); Bilic, Karsch, Redlich (92)

$$\int dU\, U_{ab} U^{+}_{cd} = \frac{1}{N_c}\delta_{ad}\delta_{bc}$$

  - **Spatial plaquett → MMMM**
  - **Temporal Link → $V^{+}V^{-}$**

$$V_x^{+} = e^{\mu}\bar{\chi}_x U_0(x)\chi_{x+\hat{0}}$$

$$V_x^{-} = e^{-\mu}\bar{\chi}_{x+\hat{0}} U_0^{\dagger}(x)\chi_x$$



- **Effective Action**

$$\Delta S_\beta^{(\tau)} = \frac{1}{4N_c^2 g^2}\sum_{x,j>0}\left(V_x^{+}V_{x+\hat{j}}^{-} + V_x^{+}V_{x-\hat{j}}^{-}\right)$$

$$\Delta S_\beta^{(s)} = -\frac{1}{8N_c^4 g^2}\sum_{x,k>j>0} M_x M_{x+\hat{j}} M_{x+\hat{k}} M_{x+\hat{k}+\hat{j}}$$

Effective Potential with 1/g² (2)

- **Extended Hubbard-Stratonovich Transf.**

$$\boxed{\begin{aligned} e^{\alpha AB} &= \int d\varphi\, d\phi\, e^{-\alpha\{\varphi^2-(A+B)\varphi+\phi^2-i(A-B)\phi\}} \\ &\approx e^{-\alpha\{\varphi^2-(A+B)\varphi-\phi^2+(A-B)\phi\}} \;. \end{aligned}}$$

  - **Mean field approx. φ , Saddle point approx. for ϕ**
  - **E.g. Temporal Plaquette action becomes,**

$$\Delta S_\beta^{(\tau)} \approx \frac{1}{4N_c^2 g^2}\sum_{x,j>0}\left[\varphi_\tau^2 + (V_x^+ - V_{x+\hat{j}}^-)\varphi_\tau - \phi_\tau^2 - (V_x^+ + V_{x+\hat{j}}^-)\phi_\tau\right] + (j \leftrightarrow -j)$$

  - **Effective Action becomes similar to the SCL action,**

$$S_{\rm eff} = \frac{1}{2}\boxed{(1+\beta_\tau\varphi_\tau)}\sum_x \left(\boxed{e^{-\beta_\tau\phi_\tau}}V_x^+ - \boxed{e^{\beta_\tau\phi_\tau}}V_x^-\right) + m_0\sum_x M_x \qquad \boxed{\mathbf{1/g^2}}$$

$$-\left(\frac{1}{4N_c}\boxed{+\beta_s\varphi_s}\right)\sum_{x,j>0} M_x M_{x+\hat{j}}\boxed{+N_\tau L^d\left[\frac{\beta_\tau}{2}(\varphi_\tau^2-\phi_\tau^2)+\frac{d\beta_s}{2}\varphi_s^2\right]}$$

Effective Potential with $1/g^2$ (2)

- **Extended Hubbard-Stratonovich Transf.**

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  - **Effective Action becomes similar to the SCL action,**

WF Renormalization

μ mod.

Aux. Terms

$$S_{\rm eff} = \frac{1}{2}\boxed{(1+\beta_\tau\varphi_\tau)}\sum_x\left(\boxed{e^{-\beta_\tau\phi_\tau}}V_x^+ - \boxed{e^{\beta_\tau\phi_\tau}}V_x^-\right) + m_0\sum_x M_x$$

$$-\left(\frac{1}{4N_c}\boxed{+\beta_s\varphi_s}\right)\sum_{x,j>0} M_x M_{x+\hat{j}} \boxed{+ N_\tau L^d\left[\frac{\beta_\tau}{2}(\varphi_\tau^2-\phi_\tau^2) + \frac{d\beta_s}{2}\varphi_s^2\right]}$$

Effective Potential with $1/g^2$ (3)

- **Effective Potential (after subst. equil. value for φ_τ and φ_s)**

$$\mathcal{F}_{\rm eff} = \mathcal{F}_{\rm X}(\sigma, \phi_\tau) + \mathcal{V}_{\rm q}(m_q(\sigma), \tilde{\mu}(\phi_\tau), T)$$

$$\mathcal{V}_{\rm q} = -T \log\left[X_{N_c}(E_q/T) + 2\cosh(N_c\tilde{\mu}/T)\right]$$

Same as SCL

$$\mathcal{F}_{\rm X} = \frac{1}{2} b_\sigma \sigma^2 \boxed{+ \frac{\beta_\tau}{2}\sigma^2 (m_q^{\rm SCL})^2 + \frac{3d\beta_s}{2}\sigma^4 - \frac{\beta_\tau}{2}\phi_\tau^2}$$

$$m_q = m_q^{\rm SCL} \boxed{(1 - N_c\beta_\tau) + \beta_\tau\sigma(m_q^{\rm SCL})^2 + 2d\beta_s\sigma^3}$$

$$\tilde{\mu} = \mu \boxed{- \beta_\tau\phi_\tau}$$

from Plaq.

- **W.F.Renormalization factor $(1+\beta_\tau\varphi_\tau)$ in the Eff. Action**
  → suppr. of quark mass m_q
- **Higher order terms $M^4 \to \sigma^4$ (Self-energy of σ)**
- **Aux. Field $\phi_\tau = \rho_q$ (equil.) → μ is shifted by baryon density**

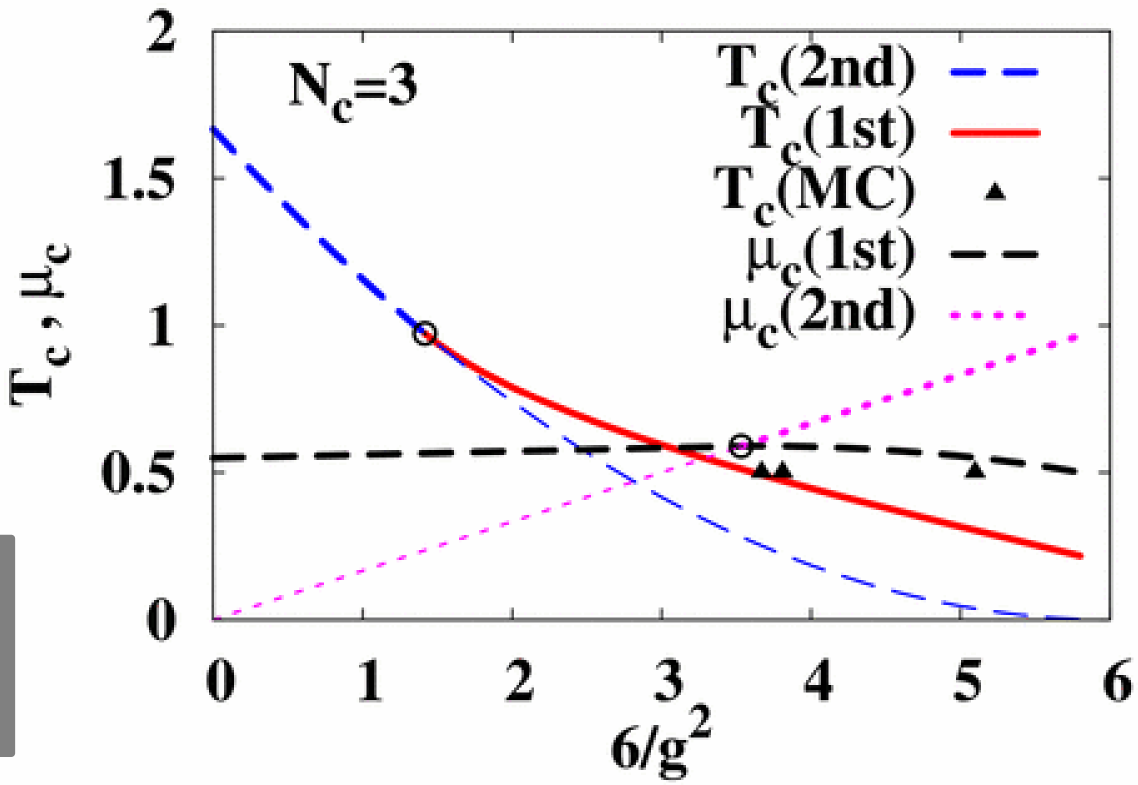
Let us examine the phase diagram with this F_{eff} !

Phase Diagram in Strong Coupling lattice QCD with $1/g^2$ Correction

Evolution of T_c

- T_c (μ=0) rapidly decreases with $\beta = 6/g^2$ increases.
  - WF Renormalization → $T_c^{(2nd)} = T_c^{(2nd)}(\mathrm{SCL}) \times (1-N_c\beta_\tau)^2$
  - Higher order terms of σ → P.T. becomes the first order at $6/g^2 \sim 1$
  - Comparison with MC results (Critical behavior with N_τ =2)

Quench β_c=5.097(1)
(Kennedy et al, 1985)
m_0=0.05 → β_c=3.81(2)
m_0=0.025 → β_c=3.67(2)
(de Forcrand,
private comm.)



MC results with small m_0 agrees with SC-LQCD !

Evolution of μ_c

- **$\mu_c^{(2nd)} > \mu_c^{(1st)}$ at $6/g^2 > 3.53$**

  → Appearance of

  weakly but spontaneously chiral broken high density matter

  - **Key: Effective chem. pot.**

$$\mu_{\text{eff}} = \mu - \beta_\tau \phi_\tau = \mu - \beta_\tau \rho_q$$

Smaller σ

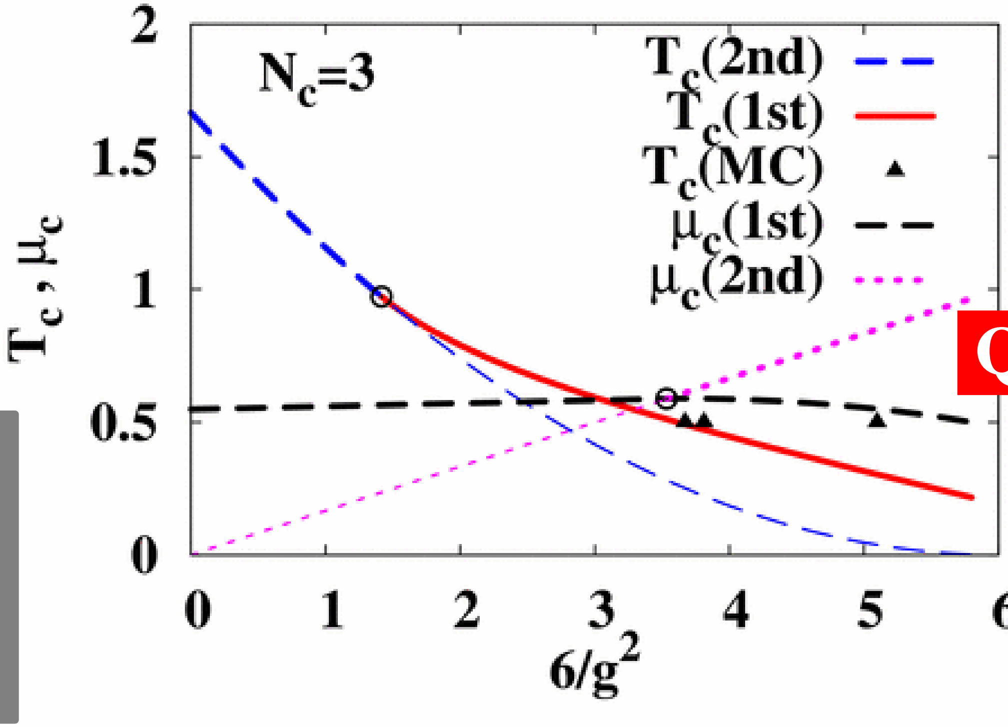
→ Smaller const. quark mass

→ Larger ρ_q

→ Smaller μ_{eff}

→ P.T. to Wigner phase (σ=0) is postponed

Auxiliary field ϕ_τ is regarded as "vector" field for quarks, which shifts μ effectively.



QY

Why do we have QY ? Which Explanation do you like ?

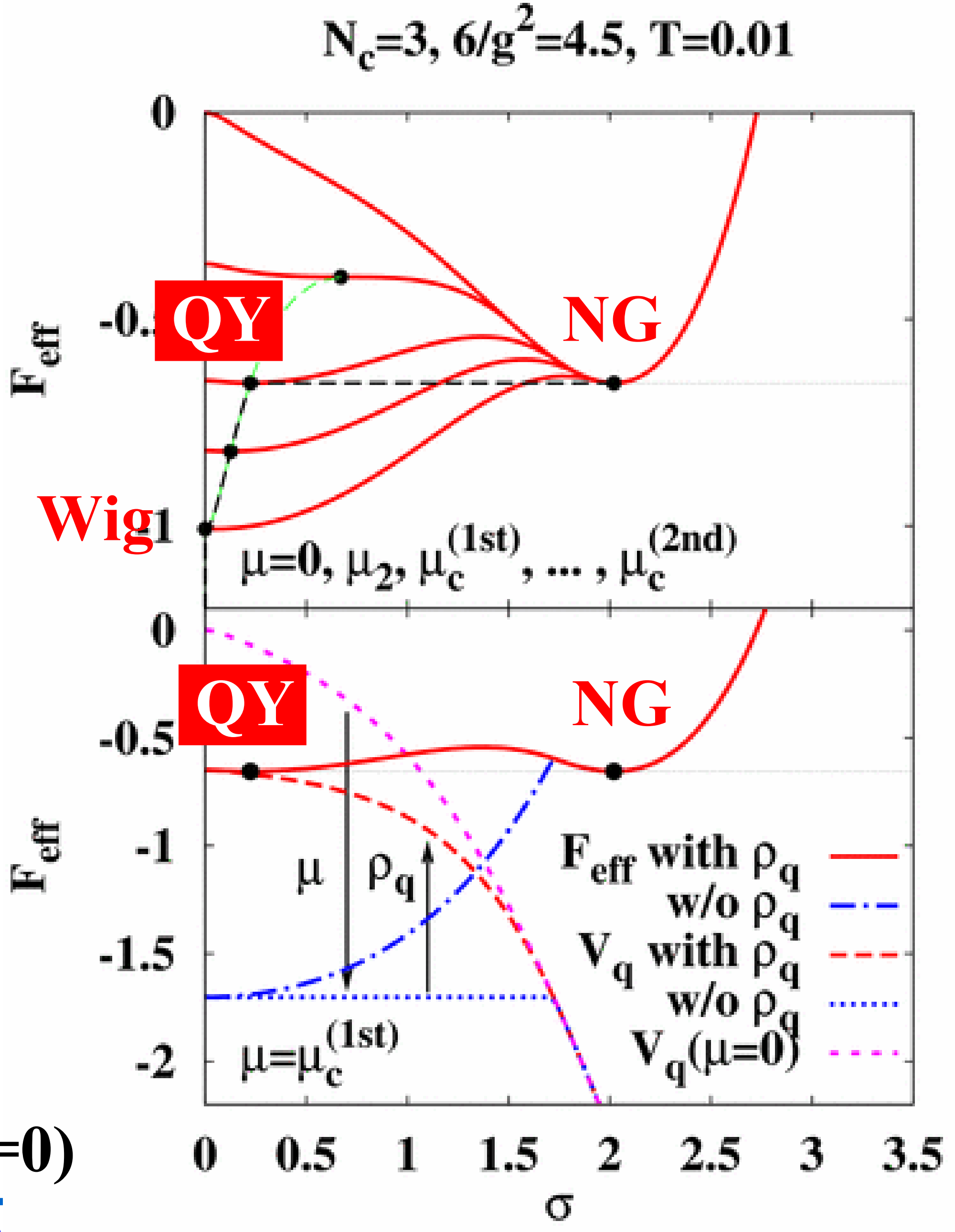
- **Vector field (ϕ_τ) acts more repulsively at smaller σ, and generates a local minimum in the region of $\sigma \ll \sigma_{vac}$**
- **2nd order P.T. condition (C_2 =0) leads to the relation of effective μ and T.**

$$F_{\rm eff} = F(\sigma=0) + C_2\sigma^2 + C_4\sigma^4 + \ldots$$

$$C_2(T, \mu - \beta_\tau \rho_q(\sigma=0)) = 0$$

$$\rightarrow \mu_c^{\rm (2nd)} = f(T) - \beta_\tau \rho_q$$

- **Smaller σ**
  → Smaller const. quark mass
  → Larger ρ_q
  → Smaller $\mu_{\rm eff}$
  → Later P.T. to Wigner phase (σ=0)

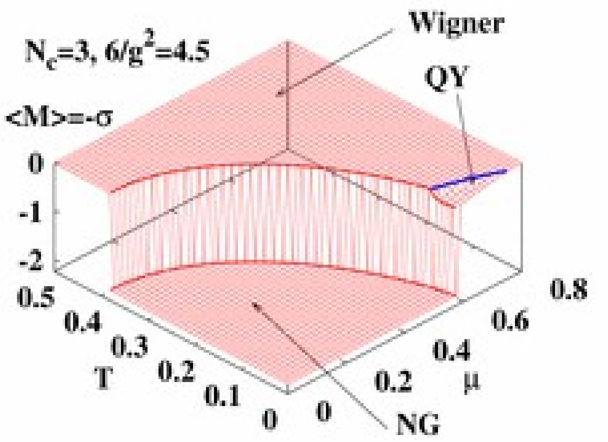


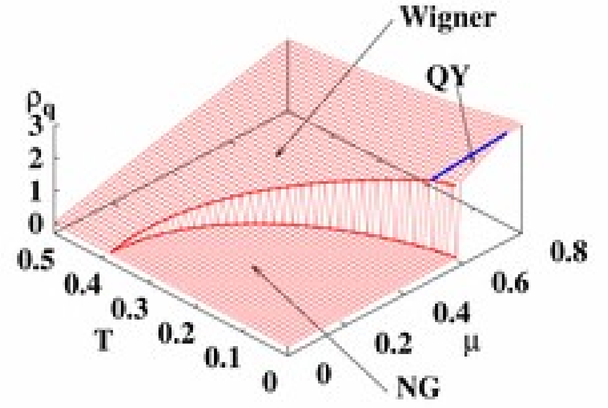
Baryon Density and Polyakov Loop in QY

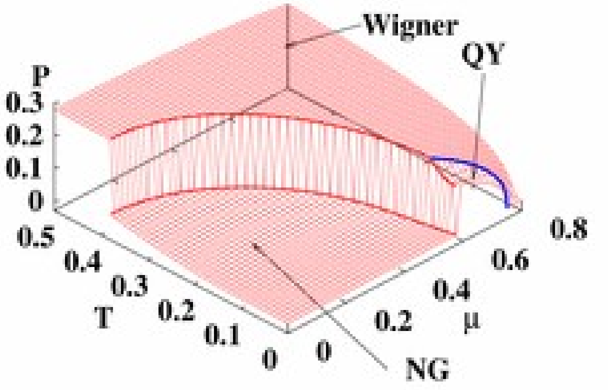
- Example: N_c=3, $6/g^2$=4.5, m_0=0 (χ limit)
- Baryon Density (= $\rho_q/3$)
  - $\rho_q \sim 0$ in Nambu-Goldstone (NG) phase
    $\rho_q > 0$ in Wigner phase
  - ρ_q in QY ~ ρ_q in Wigner phase
- (Quark Driven) Polyakov Loop (=P)
  - Quark driven P ~ O(N_c)
  - P(QY) < P (Wig.)

$$P \equiv \frac{1}{2N_c}\left\langle \mathrm{tr}\left[\prod_\tau U_0 + \prod_\tau U_0^\dagger\right]\right\rangle$$

$$= \frac{X_{N_c-1}\,\cosh\left[\tilde{\mu}/T\right] + X_1\,\cosh\left[(N_c-1)\tilde{\mu}/T\right]}{N_c\left(X_{N_c} + 2\cosh\left[N_c\tilde{\mu}/T\right]\right)}$$

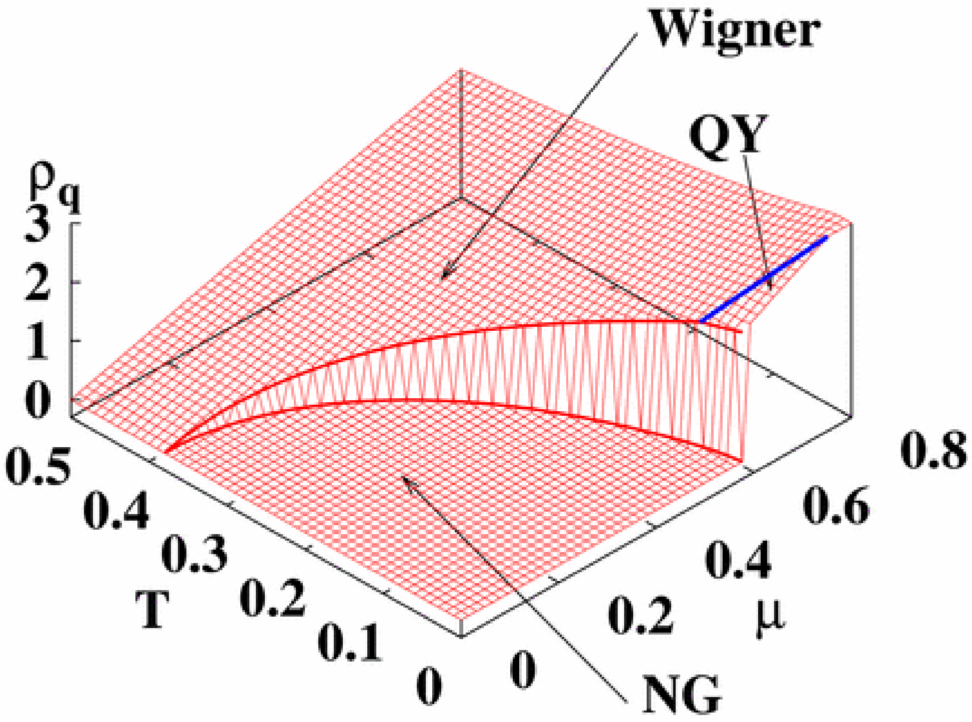






Baryon Density (Quark Number Density)

- **Baryon Density (= $\rho_q/3$)**
  - **$\rho_q \sim 0$ in Nambu-Goldstone (NG) phase**
$\rho_q > 0$ in Wigner phase
  - **ρ_q in QY ~ ρ_q in Wigner phase**



N_c=3, 6/g^2=4.5, m_0=0 (χ limit)

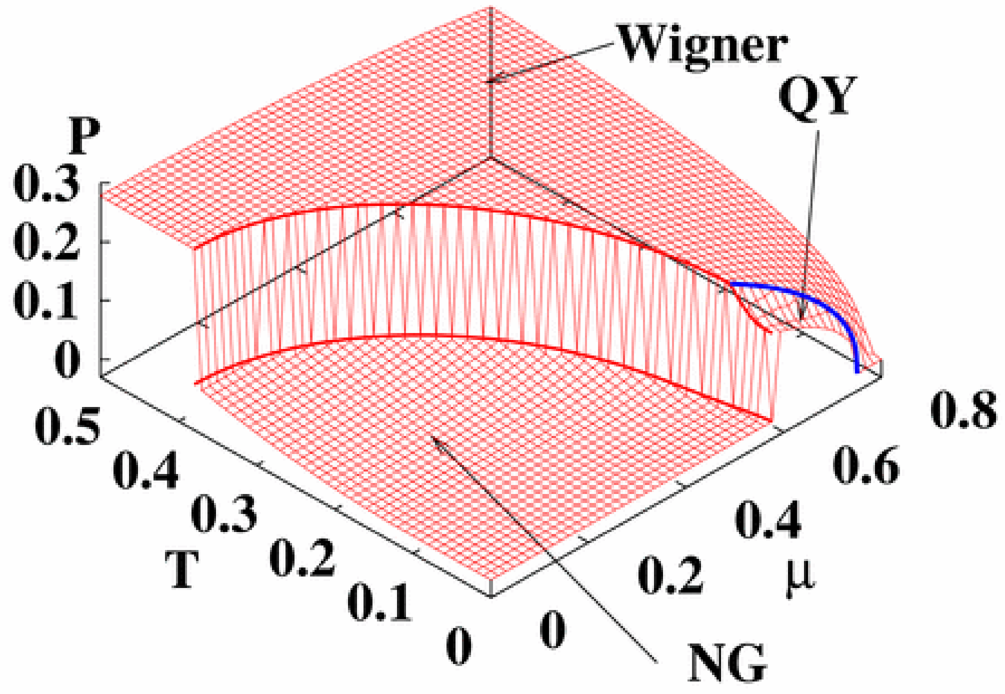
Polyakov Loop

- **(Quark Driven) Polyakov Loop (=P)**

$$P \equiv \frac{1}{2N_c}\left\langle \mathrm{tr}\left[\prod_{\tau} U_0 + \prod_{\tau} U_0^\dagger\right]\right\rangle$$

$$= \frac{X_{N_c-1}\cosh[\tilde{\mu}/T] + X_1 \cosh[(N_c-1)\tilde{\mu}/T]}{N_c\left(X_{N_c} + 2\cosh[N_c\tilde{\mu}/T]\right)}$$

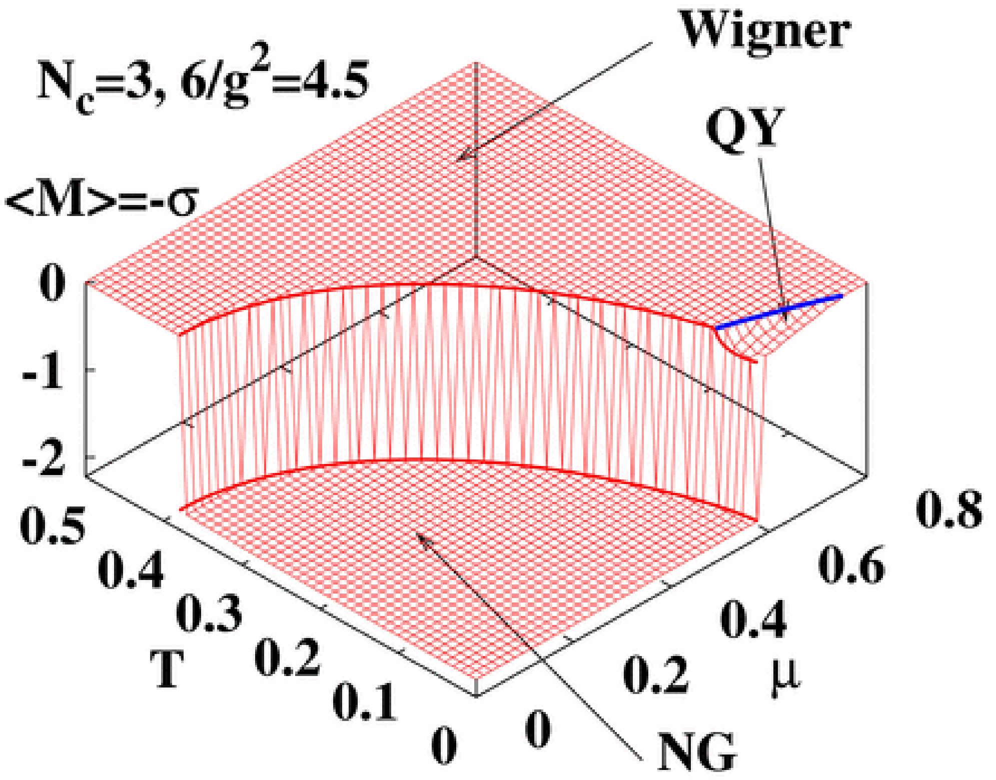
  - **P(QY) is a little smaller than P(Wig.)**
  - **Quark driven P = O(1/N_c)**



N_c=3, $6/g^2$=4.5, m_0=0 (χ limit)

Chiral Condensate

- Chiral Condensate (= σ)
  - $\sigma \sim \sigma_{vac}$ in Nambu-Goldstone (NG) phase
    $\sigma = 0$ in Wigner phase
  - $0 < \sigma \ll \sigma_{vac}$ in QY

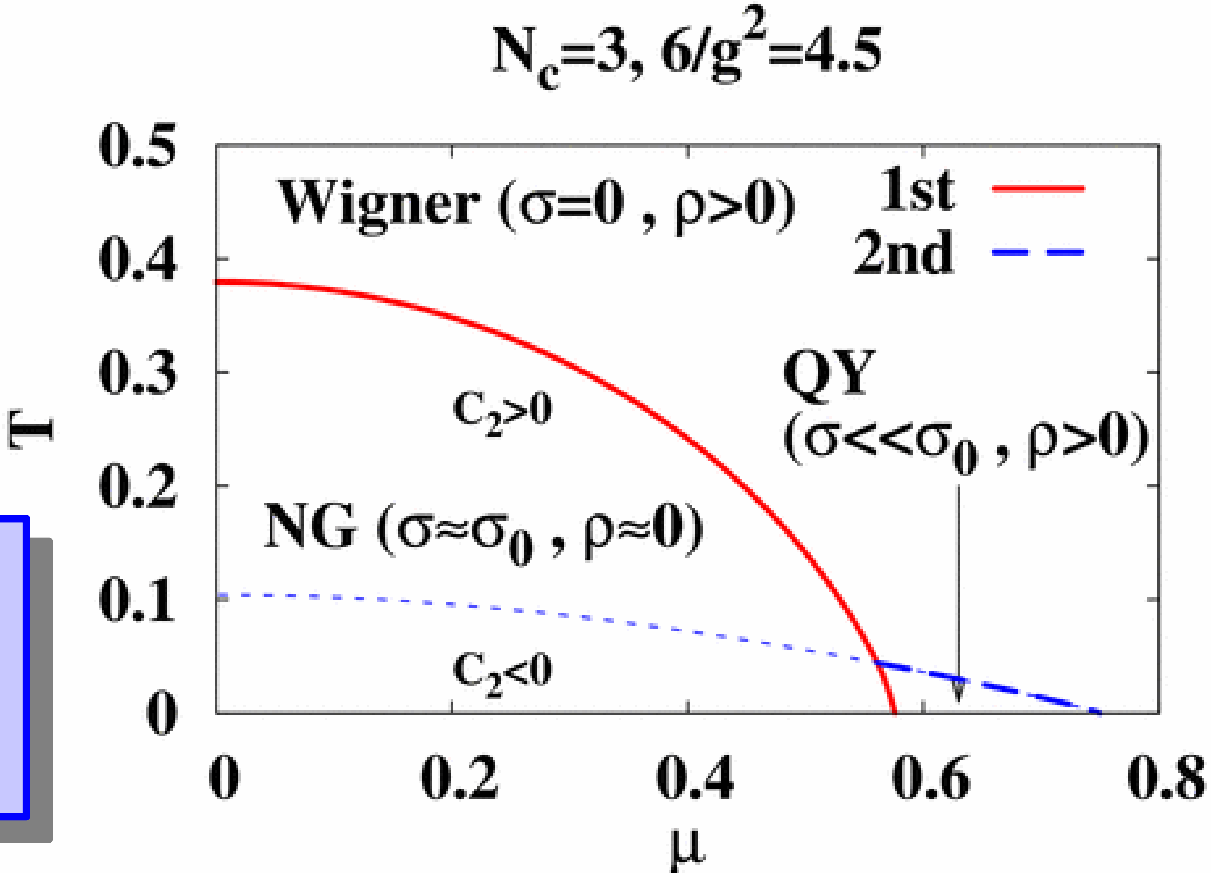


N_c=3, $6/g^2$=4.5, m_0=0 (χ limit)

Phase Diagram

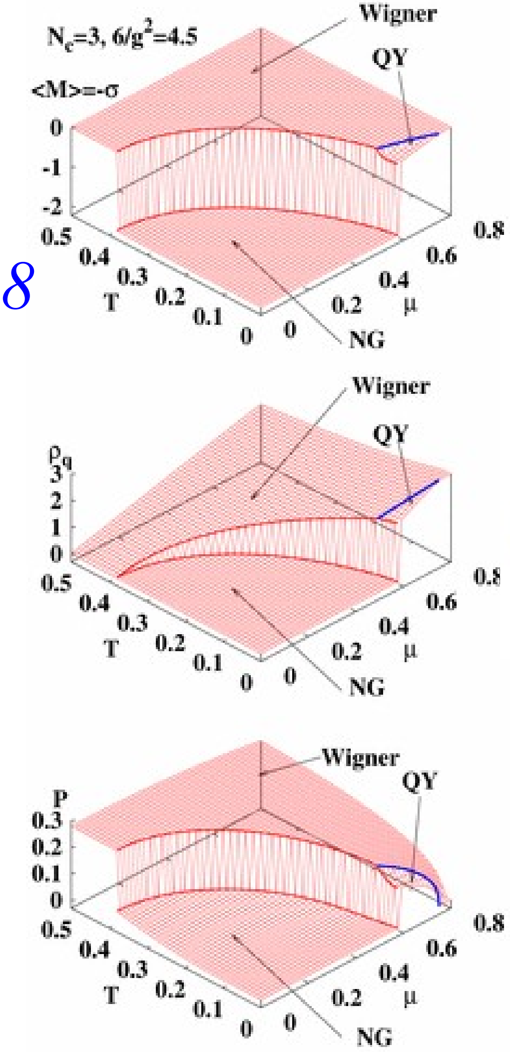
- **Three phases in SC-LQCD with $N_c=3$, $6/g^2 > 3.53$, $m_0=0$ (χ limit)**
  - **Nambu-Goldstone (NG) phase: Large σ, Small ρ_q, Small P**
  - **Winger phase: $\sigma=0$, Large ρ_q, finite P**
  - **Quarkyonic phase:**
    $0 < \sigma \ll \sigma_{vac}$
    $\rho_q(QY) \sim \rho_q(Wig.)$
    $P(QY) < P(Wig.)$
    Quark driven P → 0 at large N_c

QY in SC-LQCD can be regarded as QY at large N_c

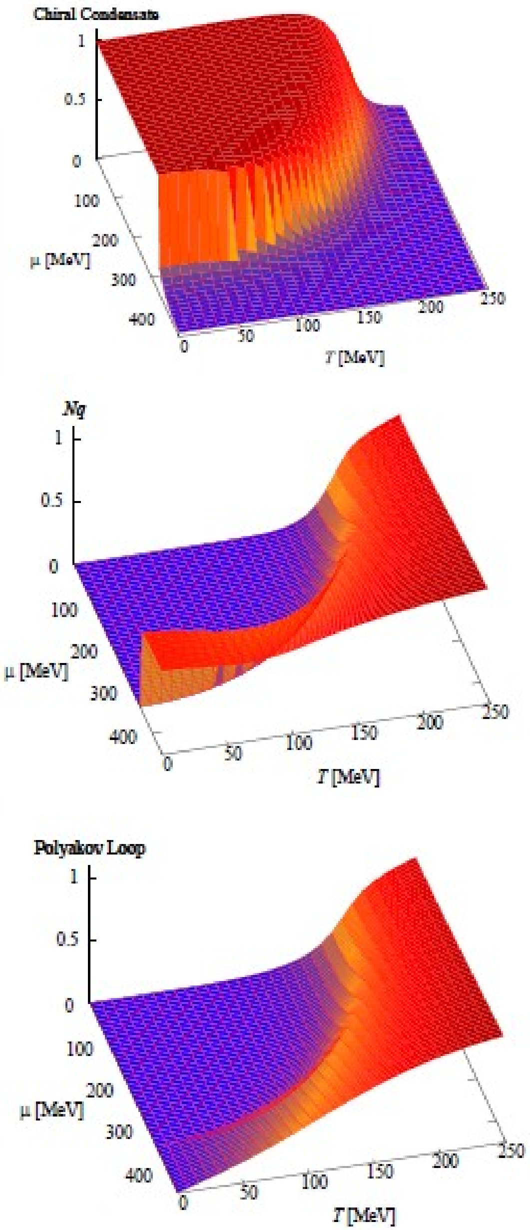


Comparison with Other Models

- **SC-LQCD results are qualitatively similar to 2+1 flavor PNJL Model in Chiral Cond., Baryon Density, and Polyakov Loop**
  Fukushima, PRD77(114028)08



Present

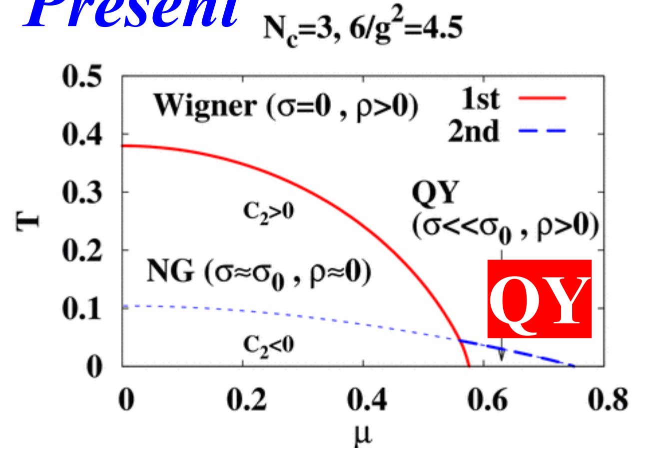


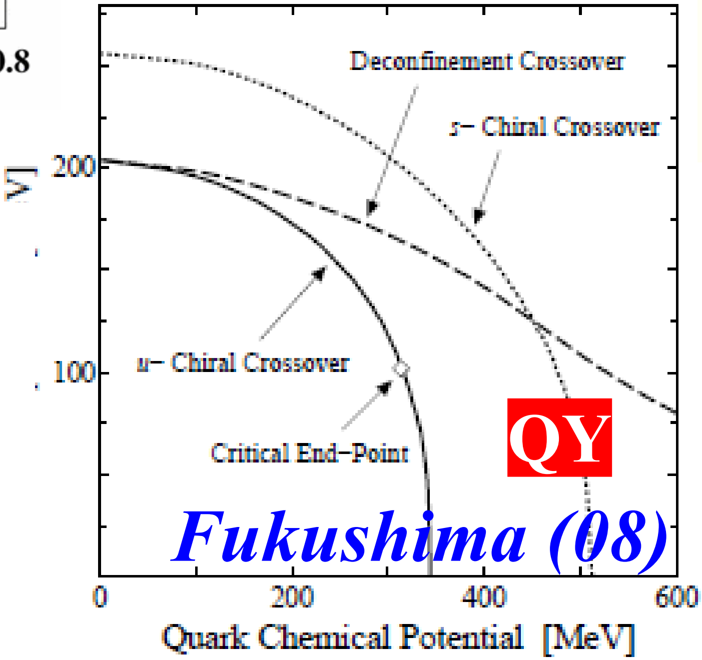
Fukushima, 2008

Comparison with Other Models

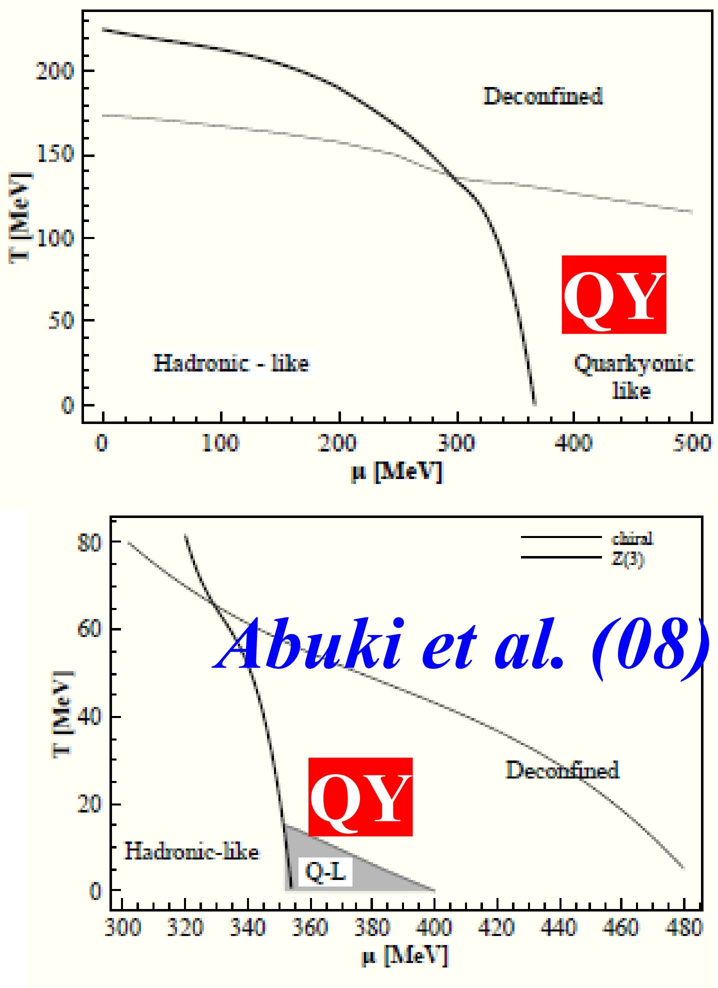
- **Quarkyonic(-like) Area in SC-LQCD is smaller than in PNJL**
  Fukushima (08)
  Abuki, Anglani Gatto, Nardulli, Ruggieril [arXiv:0805.1509]

Present

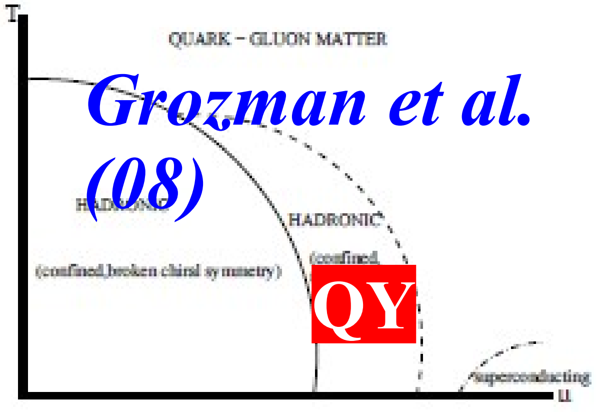




Fukushima (08)



Abuki et al. (08)



Grozman et al. (08)

Conclusions

Miura and AO, arXiv:0806.3357

- **We have investigated the phase diagram in Strong Coupling Lattice QCD with $1/g^2$ corrections.**
  Through the extended Hubbard-Stratonovich transf., we find that vector field for quarks is introduced.
- **Critical Temperature at μ=0 is found to be consistent with MC results by P. de Forcrand, T_c=1/2 (N_τ =2) at $6/g_c^2 \sim 3.6$**
  The ratio $R=N_c\mu_c/T_c$ is also improved.
- **We find that the Quarkyonic (QY) phase at large Nc proposed by McLerran & Pisarski appears also at N_c=3 in SC-LQCD with $6/g^2 > 3.53$, where the chiral symmetry is weakly but spontaneously broken, and the baryon density is high.**

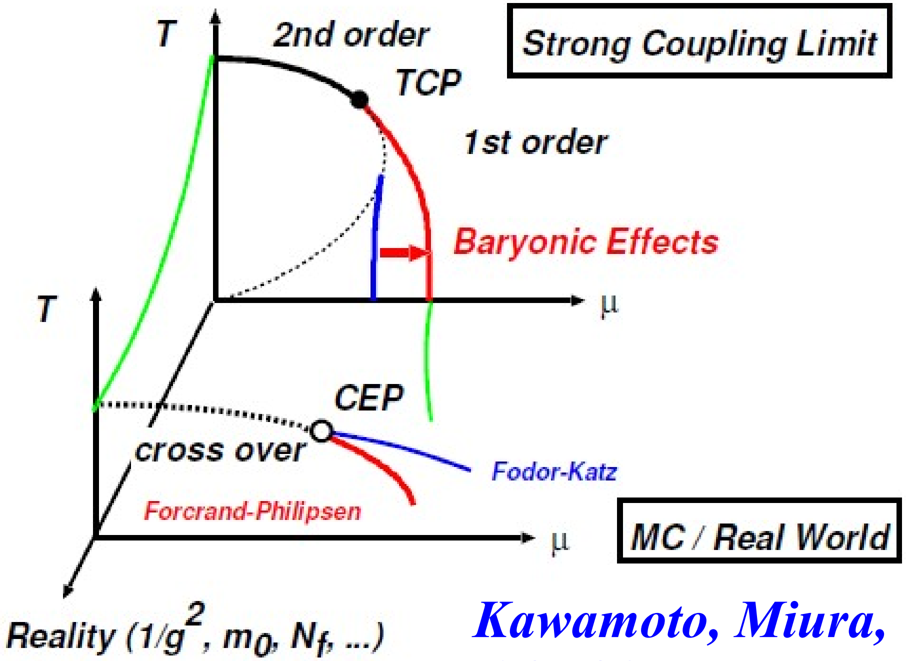
  QY may be the "NEXT" to the hadron phase even at N_c=3.

Backups

Towards the Real Phase Diagram

- **When we increase “Reality” variable,**
  Phase diagram “Shape” may be approximately explained.
    Real World: R=3 μ_c/T_c ~ (6-12)
    SCL-LQCD: R=0.79-1.34
    SC-LQCD with finite β (=6/g^2)~5 → R ~ 4.5

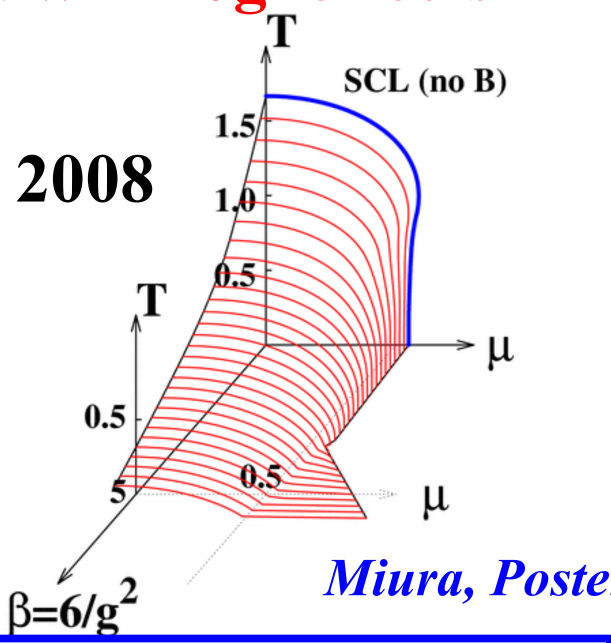
Expectation before Calc.



Kawamoto, Miura, AO, Ohnuma, 2007

Calc. with 1/g^2 effects

2008



Miura, Poster

Gluon Contribution is important at High T

Definitions

$$\begin{aligned}
\mathcal{F}_{\mathrm{eff}} =& \mathcal{F}_{\mathrm{X}}(\sigma, \phi_\tau) + \mathcal{V}_{\mathrm{q}}(m_q(\sigma), \tilde{\mu}(\phi_\tau), T) \ , \\
\mathcal{F}_{\mathrm{X}} =& \frac{1}{2} b_\sigma \sigma^2 + \frac{\beta_\tau}{2} \sigma^2 (m_q^{\mathrm{SCL}})^2 + \frac{3d\beta_s}{2} \sigma^4 - \frac{\beta_\tau}{2} \phi_\tau^2 \ , \\
\mathcal{V}_{\mathrm{q}} =& -T \log \left[X_{N_c}(E_q/T) + 2\cosh(N_c \tilde{\mu}/T) \right] \ , \\
m_q =& m_q^{\mathrm{SCL}}(1 - N_c \beta_\tau) + \beta_\tau \sigma (m_q^{\mathrm{SCL}})^2 + 2d\beta_s \sigma^3 \ , \\
\tilde{\mu} =& \mu - \beta_\tau \phi_\tau \ ,
\end{aligned}$$

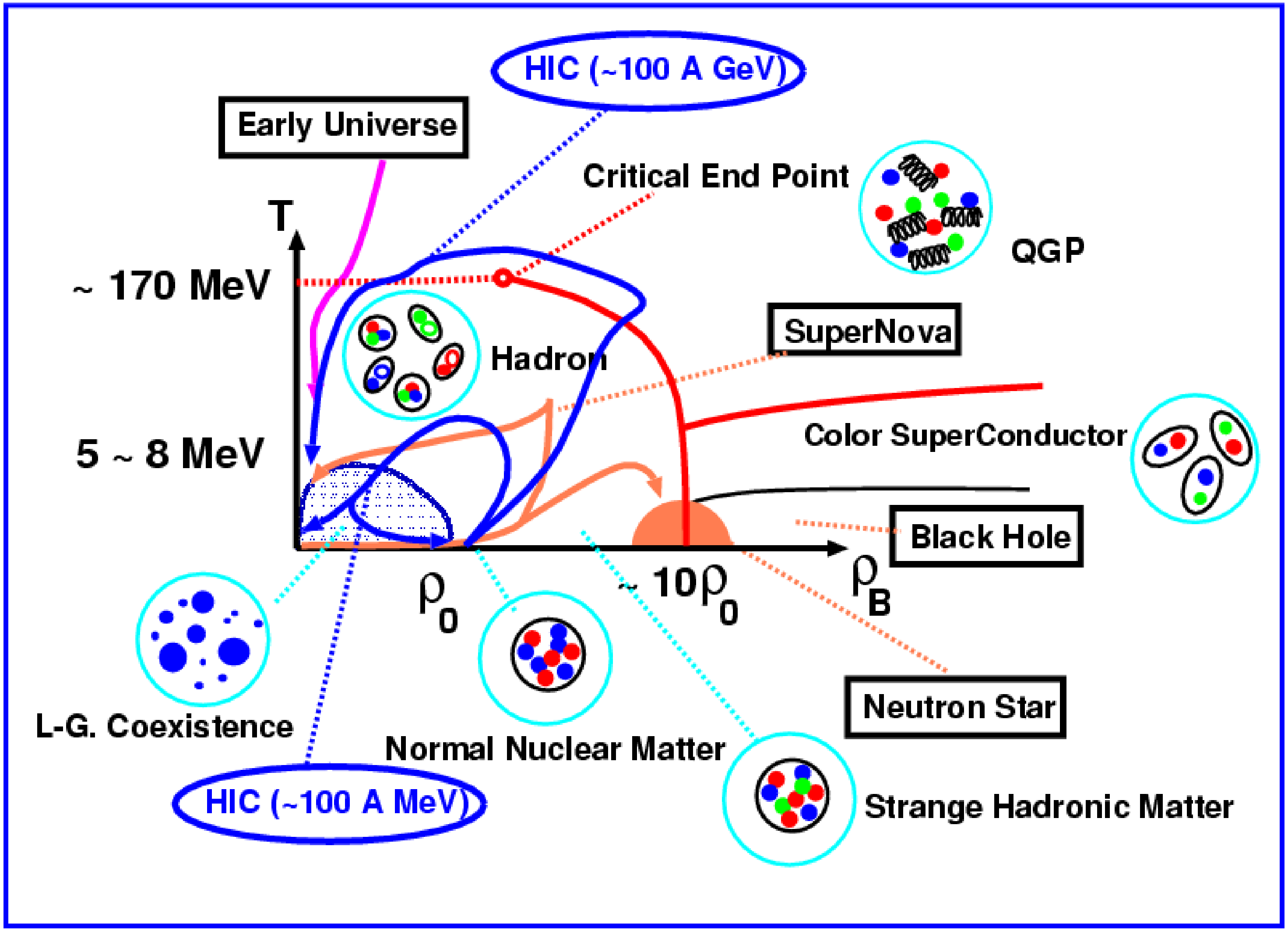
$$\beta_\tau = d/N_c^2 g^2, \ \beta_s = (d-1)/8N_c^4 g^2$$

$$\begin{aligned}
b_\sigma &= d/2N_c, \ m_q^{\mathrm{SCL}} = b_\sigma \sigma + m_0, \\
X_N(x) &= \sinh[(N+1)x]/\sinh x \\
E_q &= \operatorname{arcsinh}(m_q)
\end{aligned}$$

$$\phi_\tau = -\frac{\partial \mathcal{V}_{\mathrm{q}}}{\partial \mu} = \rho_q = \frac{2N_c \sinh(N_c \tilde{\mu}/T)}{X_{N_c} + 2\cosh(N_c \tilde{\mu}/T)}$$

I'm interested in ....

- **Quark / Hadron / Nuclear Matter EOS and Phase Diagram**

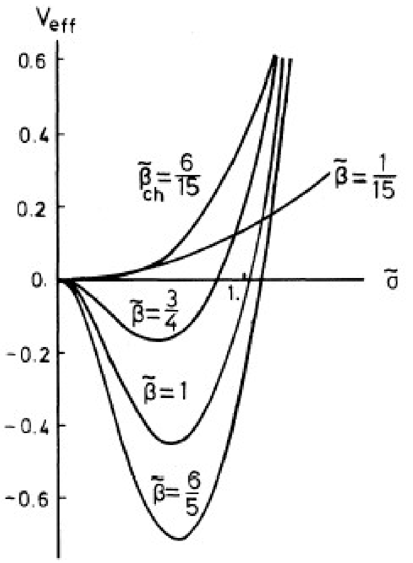


Rich Structure / Astrophysical implications / Accessible in HIC

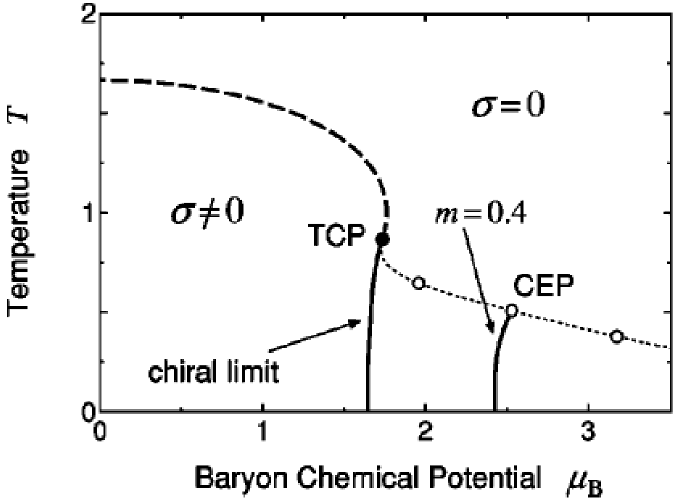
Strong Coupling Lattice QCD

Strong Coupling Limit of Lattice QCD

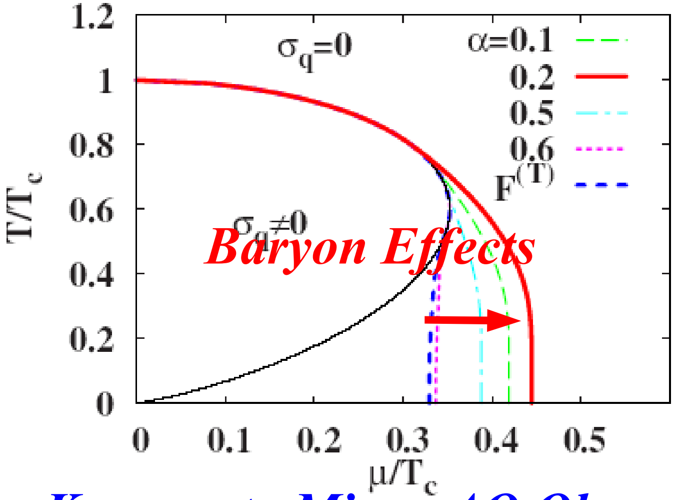
- SCL-LQCD has been a powerful tool in “phase diagram” study !
    - Chiral restoration, Phase diagram, Baryon effects, Hadron masses, Finite coupling effects, ...



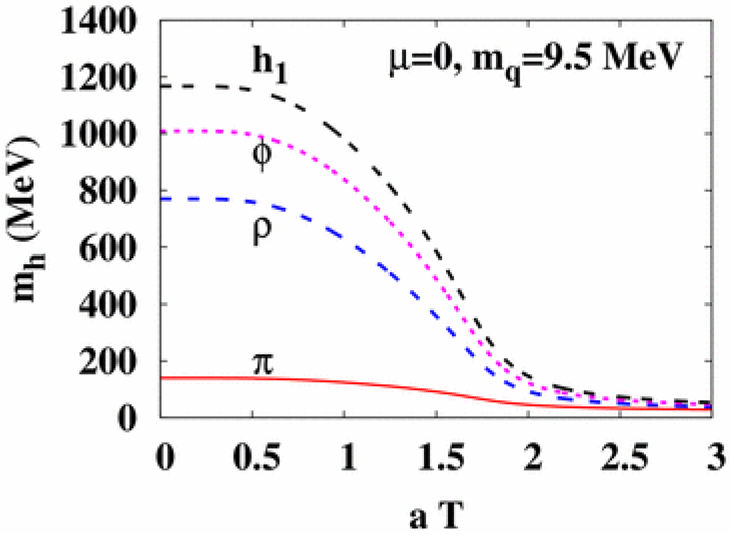
Damgaard,Kawamoto, Shigemoto, PRL53('84),2211



Nishida, PRD69, 094501 (2004



Kawamoto,Miura,AO,Ohnuma, PRD75 (07), 014502.



AO, Kawamoto, Miura, 2008

Lattice QCD (1)

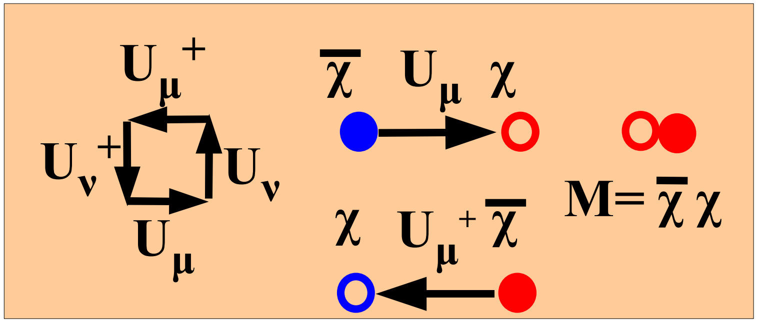
- **QCD Lagrangian**

$$L=\bar{\psi}(i\gamma^{\mu}D_{\mu}-m_0)\psi-\frac{1}{4}tr(F_{\mu\nu}F^{\mu\nu})$$

ψ = Quark, F = Gluon tensor, m_0 = (small) quark mass

- **Lattice Action**

$$S_{\mathrm{QCD}}=S_G+S_F^{(s)}+S_F^{(t)}+m_0\bar{\chi}\chi$$



$$S_G=-\frac{1}{g^2}\sum_{\mathrm{plaq.}}\mathrm{Tr}\,U_{ij}(x)+c.c.$$

$$S_F^{(s)}=\frac{1}{2}\sum_{x,\,j>0}\left(\bar{\chi}_x U_j(x)\chi_{x+\hat{j}}-\bar{\chi}_{x+\hat{j}}U_j^{+}(x)\chi_x\right)$$

$$S_F^{(t)}=\frac{1}{2}\sum_{x}\left(e^{\mu}\bar{\chi}_x U_0(x)\chi_{x+\hat{0}}-e^{-\mu}\bar{\chi}_{x+\hat{0}}U_0^{+}(x)\chi_x\right)$$

χ = starggered fermion (quark)
U= link variable ∈ SU(N_c) (gluon),
μ = quark chemical potential

Lattice QCD (2)

- **Full QCD MC Simulation**
  → MC Integral of Det (Fermion Matrix) over link var. (U)
  - **Big Task !**
    Matrix Size= 4 (spinor) x (Color) x (Space-Time Points)
    Eigen Values are widely distributed
  - **Complex Weight with finite μ**

$$\int d\bar{\chi}\, d\chi\, dU \exp(-S_G+\bar{\chi} A \chi)=\int dU \left| \quad A \quad \right| \updownarrow \mathbf{4\, N_c\, N_\tau\, N_s^3}$$

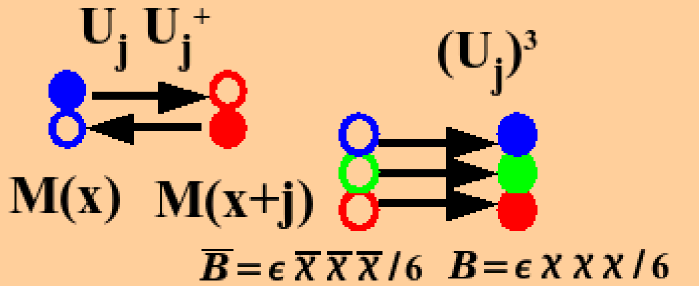
- **Quenched QCD**
  - **Assuming Det = 1 ~ Ignoring Fermion Loops**
  - **Works very well for hadron masses**
- ***Strong Coupling Limit (g → ∞)***
  - ***Pure gluonic action disappears → Analytic evaluation of Fermion Det.***

SCL-LQCD: Tools (1) --- One-Link Integral

- **Group Integral Formulae**

$$\int dU\, U_{ab} U^{+}_{cd} = \frac{1}{N_c}\delta_{ad}\delta_{bc}$$

$$\int dU\, U_{ab} U_{cd} U_{ef} = \frac{1}{6}\epsilon_{ace}\epsilon_{bdf}$$



$$\int dU\, U_{ab} U^{+}_{cd} = \frac{1}{N_c}\delta_{ad}\delta_{bc}$$

$$\int dU\, U_{ab} U_{cd} U_{ef} = \frac{1}{6}\epsilon_{ace}\epsilon_{bdf}$$

$$\begin{aligned}
&\int dU \exp\left(-a\bar{\chi}(x)U\chi(y)+b\bar{\chi}(y)U^{+}\chi(x)\right)\\
&=\int d\,U\left[1-ab\,\bar{\chi}(a)^{a}U_{ab}\chi^{b}(y)\bar{\chi}^{c}(y)U^{+}_{cd}\chi^{d}(x)+\cdots\right]\\
&=1+ab\left(\chi\bar{\chi}\right)(x)\left(\chi\bar{\chi}\right)(y)+\cdots=1+ab\,M(x)M(y)+\cdots\\
&=\exp\left[ab\,M(x)M(y)+\cdots\right]
\end{aligned}$$

Quarks and Gluons → One-Link integral
→ Mesonic and Baryonic Composites

SCL-LQCD: Tools (2) --- 1/d Expansion

- **Keep mesonic action to be indep. from spatial dimension *d***
  → Higher order terms are suppressed at large *d*.

$$\sum_j (\bar{\chi} U_j \chi)(\bar{\chi} U_j^{+} \chi) \to -\frac{1}{N_c} \sum_j M(x) M(x+\hat{j}) = O(1)$$

$$\to M \propto 1/\sqrt{d}\,, \chi \propto d^{-1/4}$$

$$\sum_j (\bar{\chi} U_j \chi)(\bar{\chi} U_j \chi)(\bar{\chi} U_j \chi) \to N_c! \sum_j B(x) B(x+\hat{j}) = O(1/\sqrt{d})$$

$$\sum_j (\bar{\chi} U_j \chi)^2 (\bar{\chi} U_j^{+} \chi)^2 \to \sum_j M^2(x) M^2(x+\hat{j}) = O(1/d)$$

We can stop the expansion in U,
since higher order terms are suppressed !

SCL-LQCD: Tools (3) --- Bosonization

- **We can reduce the power in χ by introducing bosons**

$$\exp\left(\frac{1}{2}M^2\right)=\int d\sigma\exp\left(-\frac{1}{2}\sigma^2-\sigma M\right)$$

Nuclear MFA: $V=-\frac{1}{2}(\bar{\psi}\psi)(\bar{\psi}\psi)\simeq -U(\bar{\psi}\psi)+\frac{1}{2}U^2$

$$\exp\left[-\frac{1}{2}M^2\right]=\int d\varphi\exp\left[-\frac{1}{2}\varphi^2-i\varphi M\right]$$

Reduction of the power of χ
→ Bi-Linear form in χ → Fermion Determinant

SCL-LQCD: Tools (4) --- Grassman Integral

- **Bi-linear Fermion action leads to -log(det A) effective action**

$$\int d\chi\, d\bar{\chi} \exp\left[\bar{\chi} A \chi\right] = det\, A = \exp\left[-(-\log det\, A)\right]$$

$$\int d\chi \cdot 1 = \text{anti-comm. constant} = 0 \quad , \quad \int d\chi \cdot \chi = \text{comm. constant} \equiv 1$$

$$\int d\chi\, d\bar{\chi} \exp\left[\bar{\chi} A \chi\right] = \int d\chi\, d\bar{\chi} \frac{1}{N\,!} (\bar{\chi} A \chi)^N = \cdots = det\, A$$

Constant σ → - log σ interaction (Chiral RMF)

- **Temporal Link Integral, Matsubara product, Staggered Fermion,**

→ I will explain next time ....

Effective Potential in SCL-LQCD (Zero T)

- **QCD Lattice Action (Zero T treatment)**

Kawamoto, Smit, 1981

$$S = \cancel{S_G} + S_F + m_0 \bar{\chi}\chi$$

Strong Coupling Limit

$$\to -\frac{1}{2}(\bar{\chi}\chi) V_M (\bar{\chi}\chi) + m_0 \bar{\chi}\chi$$

One-link integral (1/d expansion*)

$$\to \frac{1}{2}\sigma V_M^{-1}\sigma + \bar{\chi}(\sigma + m_0)\chi$$

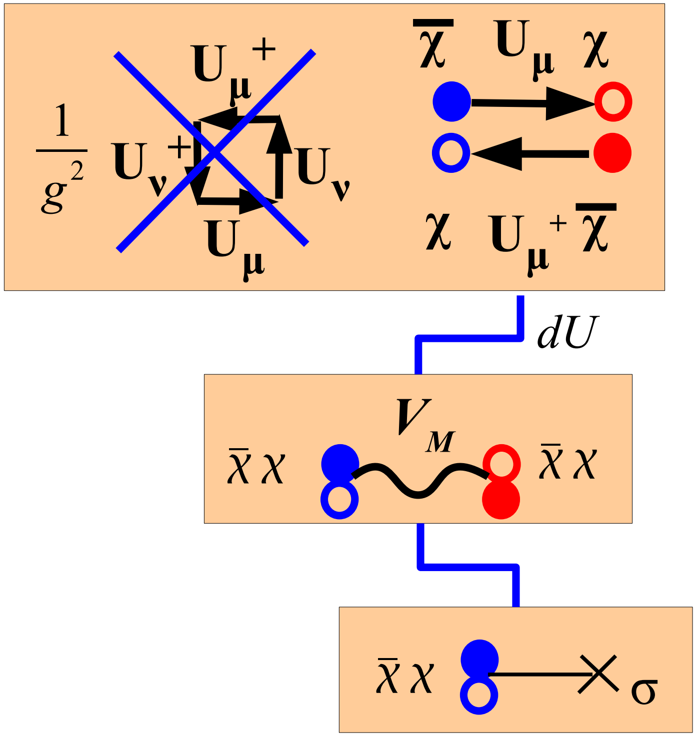
Bosonization

$$\to \frac{1}{2}\sigma V_M^{-1}\sigma - N_c \sum_x \log(\sigma(x) + m_0)$$

Fermion Integral

$$= L^d N_\tau \left[\frac{N_c}{d+1}\bar{\sigma}^2 - N_c \log(\bar{\sigma} + m_0) \right]$$

Effective Potential



*** d = Spatial dim.**

Fermion Matrix = Just a number
→ Simple Logarithmic Effective Potential for σ

$$V_\sigma = \frac{1}{2} a_\sigma \sigma^2 - b_\sigma \log \sigma$$

Effective Potential in SCL-LQCD (Zero T)

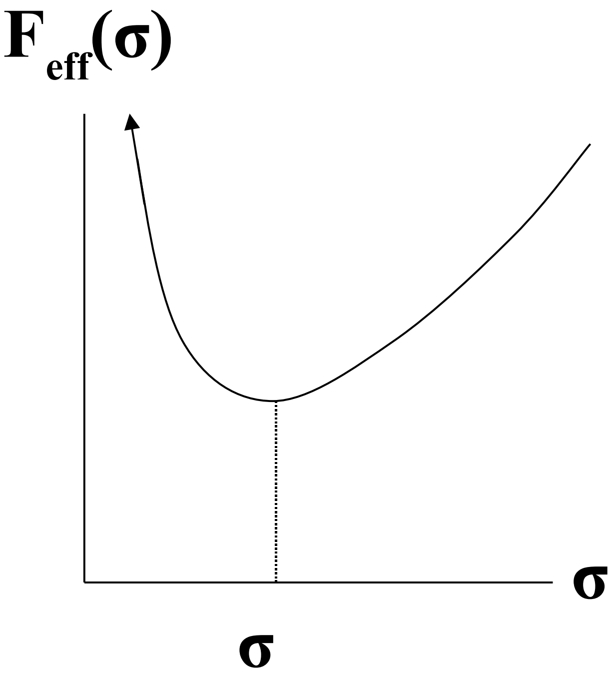
- **Effective Pot. at Zero T**

  Kawamoto, Smit, 1981
  Kluberg-Stern, Morel, Napoly, Petersson, 1981

$$F_{eff}(\sigma)=\frac{1}{2}a_\sigma\sigma^2-b_\sigma\log\sigma$$

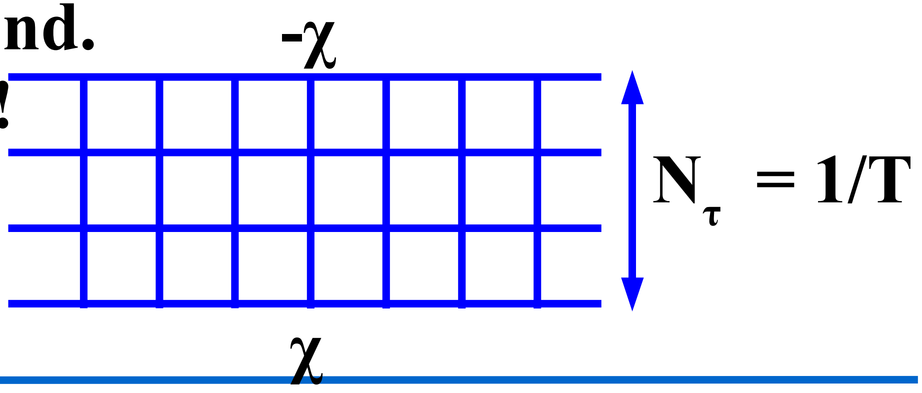
Spontaneous Chiral Symmetry breaking at T=0 is naturally explained !

No Phase Transition ?



- **Grassman integral at each space-time point in Zero T treatment**
  → "Temporal" Correlation
  and Anti-periodic Boundary Cond.
  would be important at Finite T !

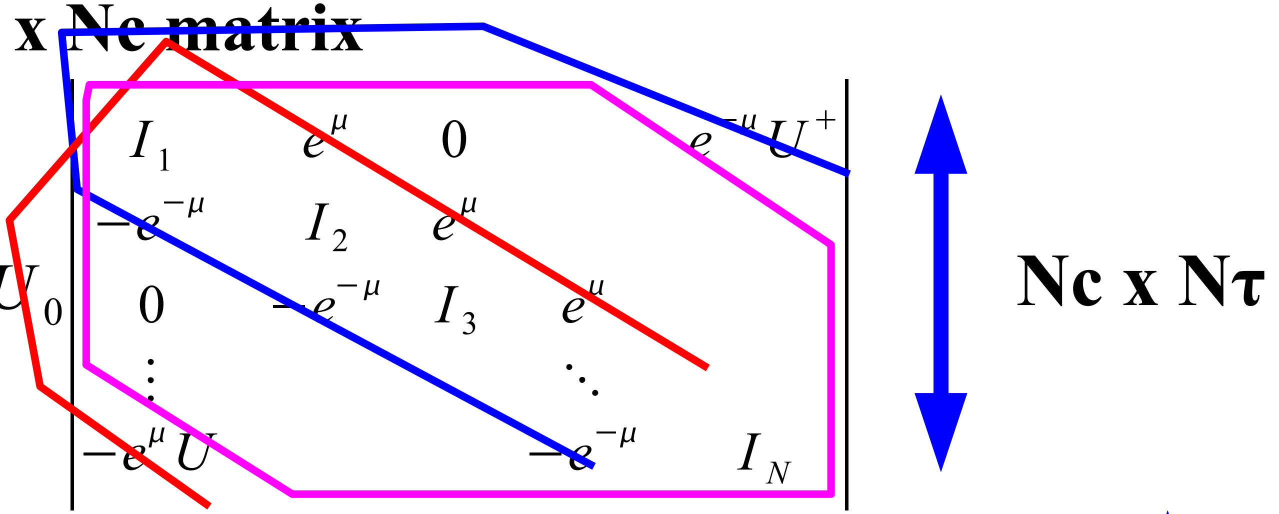
Let's go to Finite T



Fermion Determinant

Faldt, Petersson, 1986

- **Fermion action is separated to each spatial point and bi-linear → Determinant of Nτ x Nc matrix**



$$\exp(-V_{\rm eff}/T)=\int dU_0 \begin{vmatrix} I_1 & e^{\mu} & 0 & & e^{-\mu}U^{+} \\ -e^{-\mu} & I_2 & e^{\mu} & & \\ 0 & -e^{-\mu} & I_3 & e^{\mu} & \\ \vdots & & & \ddots & \\ -e^{\mu}U & & & -e^{-\mu} & I_N \end{vmatrix}$$

Nc x Nτ

$$=\int dU_0 \, det\left[\underline{X_N[\sigma]\otimes\mathbf{1}_c}+\underline{e^{-\mu/T}U^{+}}+\underline{(-1)^{N_\tau}e^{\mu/T}U}\right]$$

Nc

$$I_k=2(\sigma(k)+m_0)$$

$$X_N=\begin{vmatrix} I_1 & e^{\mu} & 0 & \cdots & & e^{-\mu} \\ -e^{-\mu} & I_2 & e^{\mu} & & & 0 \\ 0 & -e^{-\mu} & I_3 & & & 0 \\ \vdots & & & \ddots & & \\ & & & & I_{N-1} & e^{\mu} \\ -e^{\mu} & 0 & 0 & \cdots & -e^{-\mu} & I_N \end{vmatrix}-\left[e^{-\mu/T}+(-1)^N e^{\mu/T}\right]$$

Effective Potential in SCL-LQCD (Time dependence...)

- **Zero T, no Baryon** *Kawamoto, Smit, 1981*

$$\mathcal{F}_{\rm eff}^{(0)} = \frac{1}{2} b_\sigma^{(0)} \sigma^2 - N_c \log(b_\sigma^{(0)} \sigma + m_0)$$

- **Zero T, with Baryon**
  Damgaad, Hochberg, Kawamoto, 1985

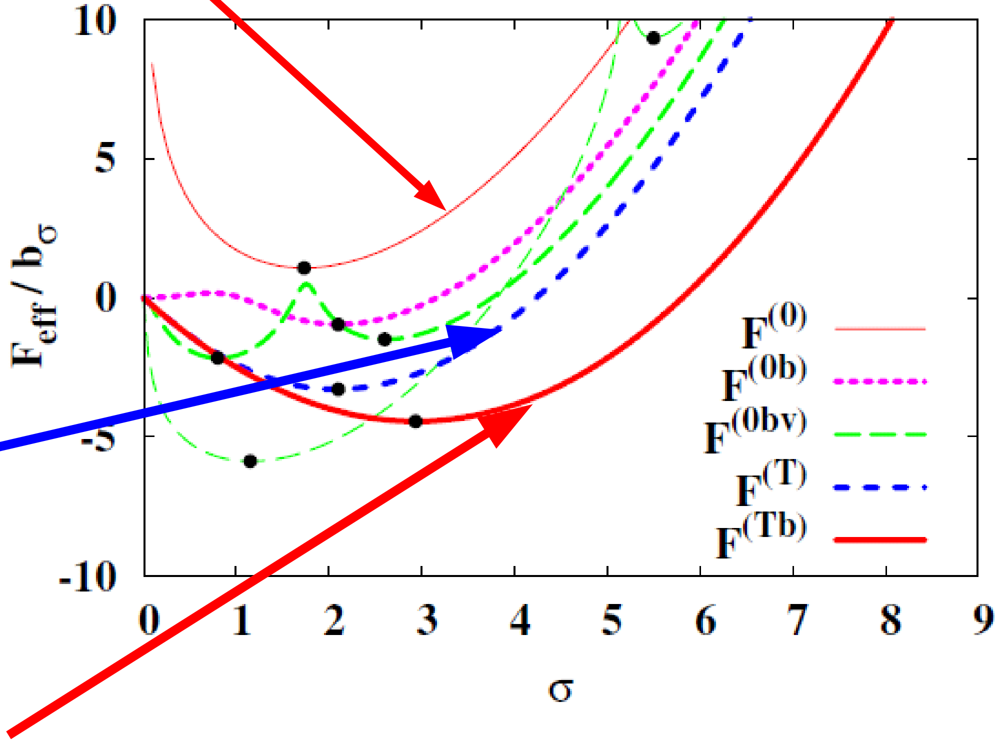
$$\mathcal{F}_{\rm eff}^{(0{\rm b})} = \frac{1}{2} b_\sigma^{(0)} \sigma^2 + F_{\rm eff}^{(b\mu)}(4{m_q}^3; T, \mu)$$

- **Finite T, no Baryon**
  Fukushima, 2004; Nishida, 2004

$$\mathcal{F}_{\rm eff}^{(T)} = \frac{1}{2} b_\sigma^{(T)} \sigma^2 + F_{\rm eff}^{(q)}(m_q)$$

- **Finite T, with Baryon**

$$\mathcal{F}_{\rm eff} = \frac{b_\sigma}{2} \sigma^2 + F_{\rm eff}^{(q)}(m_q) + \Delta F_{\rm eff}^{(b)}(g_\sigma \sigma)$$

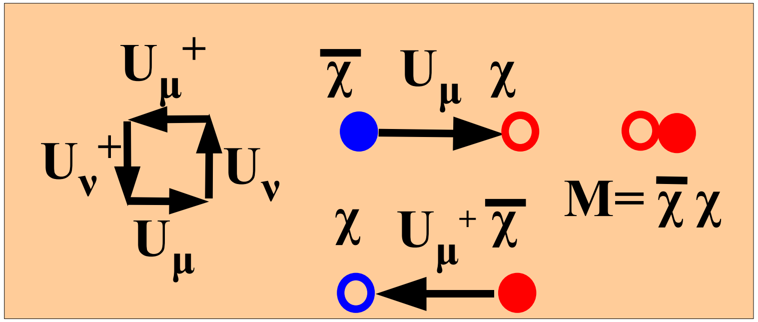


$$F_{\rm eff}^{(q)}(m_q) = -T \log\left(\frac{\sinh((N_c+1)E(m_q)/T)}{\sinh(E(m_q)/T)} + 2\cosh(N_c \mu / T) \right)$$

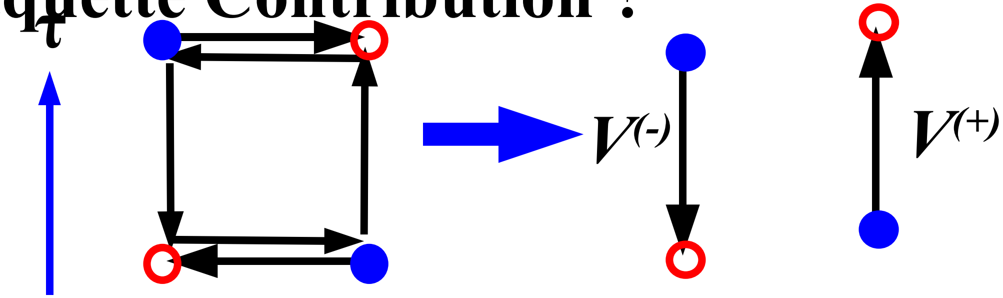
How do we get vector field ?

- **Relativistic Mean Field**
  = Baryon + Scalar Field + Vector Field
  → Saturation of Nuclear Matter, Binding Energy of Nuclei, ...
- **Vector Meson is indispensable in RMF !**
  - **Couples to baryon number density**
  - **Modifies the baryon Energy → Shifts μ effectively**

$$E=\sqrt{m^{¿2}+p^2}+g_\omega\omega \rightarrow E-\mu=\sqrt{m^{¿2}+p^2}-(\mu-g_\omega\omega)$$



- **Plaquette Contribution ?**

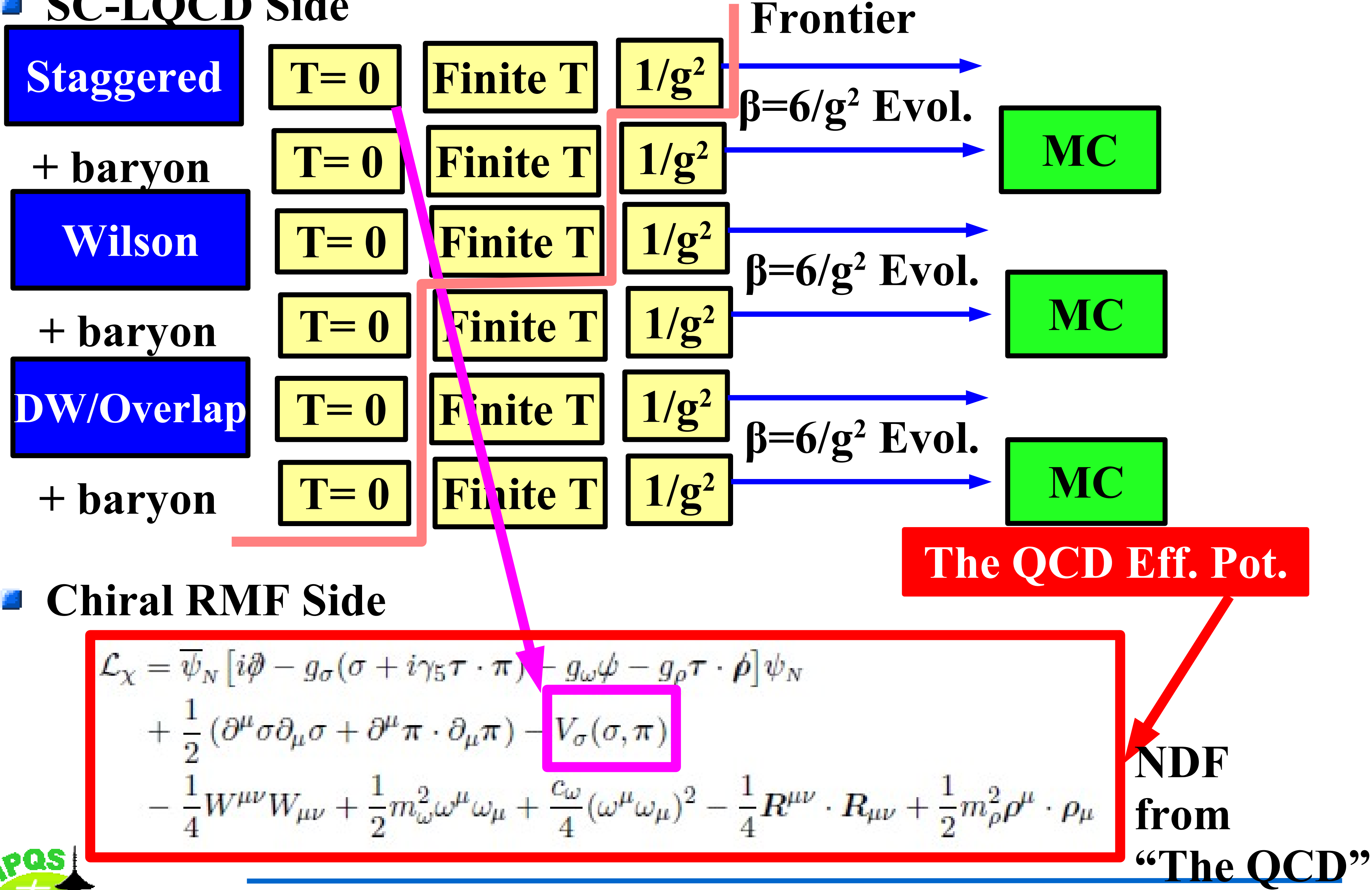


Plaquette

Spatial Link Integral

$$(V_x^{(+)}=e^{\mu}\bar{\chi}_x U_0(x)\chi_{x+\hat{0}}, \quad V_x^{(-)}=e^{-\mu}\bar{\chi}_{x+\hat{0}}U_0^{+}(x)\chi_x)$$

A Long Way to the "Ultimate Goal"



- **SC-LQCD Side**

Frontier

Staggered	T= 0	Finite T	$1/g^2$	
	T= 0	Finite T	$1/g^2$	**MC**
+ baryon				
Wilson	T= 0	Finite T	$1/g^2$	
	T= 0	Finite T	$1/g^2$	**MC**
+ baryon				
DW/Overlap	T= 0	Finite T	$1/g^2$	
	T= 0	Finite T	$1/g^2$	**MC**
+ baryon				

$\beta=6/g^2$ Evol.

$\beta=6/g^2$ Evol.

$\beta=6/g^2$ Evol.

The QCD Eff. Pot.

- **Chiral RMF Side**

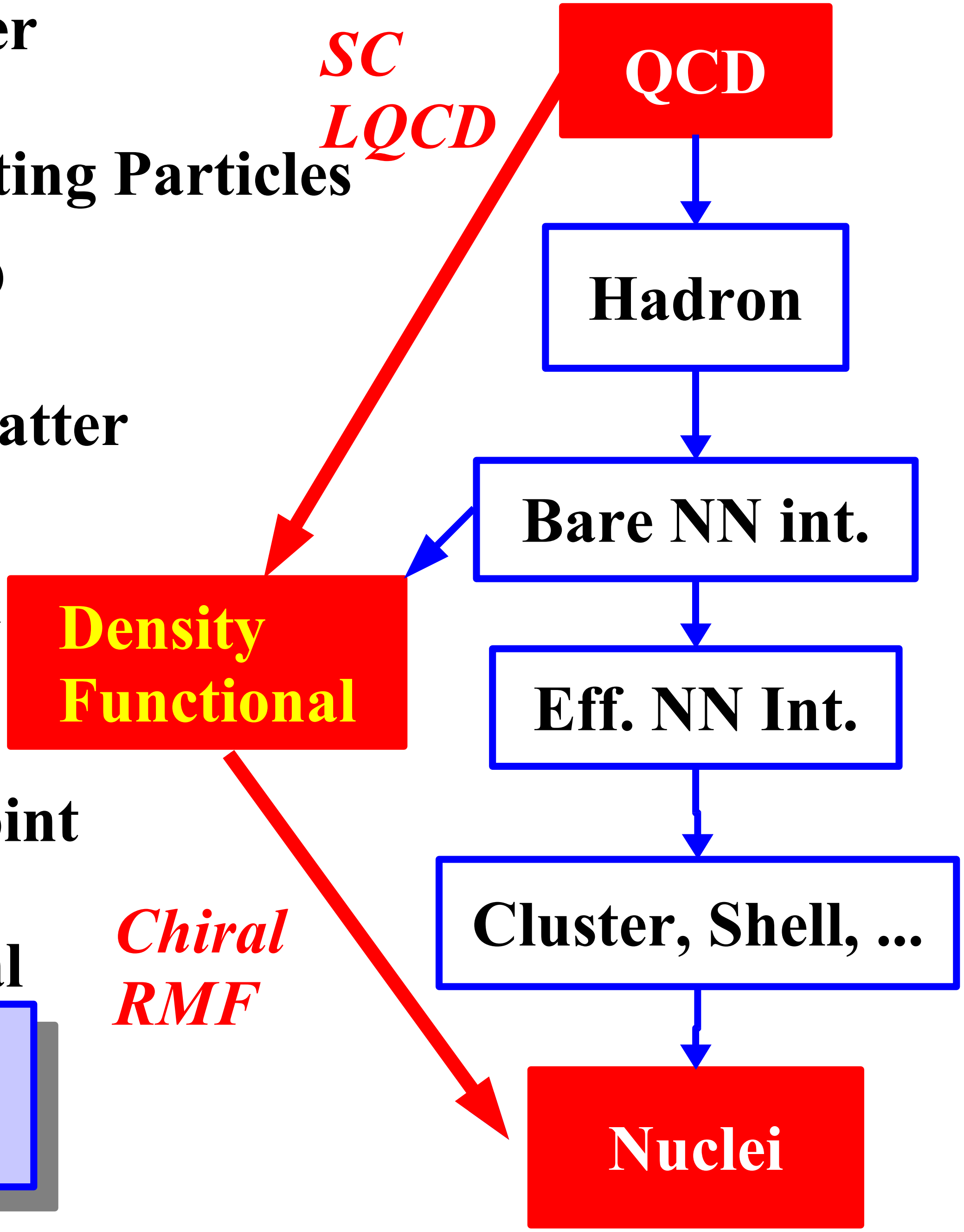
$$\mathcal{L}_\chi = \overline{\psi}_N\left[i\partial\!\!\!/ - g_\sigma(\sigma + i\gamma_5\tau\cdot\pi) - g_\omega\omega\!\!\!/ - g_\rho\tau\cdot\rho\!\!\!/\right]\psi_N + \frac{1}{2}\left(\partial^\mu\sigma\partial_\mu\sigma + \partial^\mu\pi\cdot\partial_\mu\pi\right) - \boxed{V_\sigma(\sigma,\pi)} - \frac{1}{4}W^{\mu\nu}W_{\mu\nu} + \frac{1}{2}m_\omega^2\omega^\mu\omega_\mu + \frac{c_\omega}{4}(\omega^\mu\omega_\mu)^2 - \frac{1}{4}R^{\mu\nu}\cdot R_{\mu\nu} + \frac{1}{2}m_\rho^2\rho^\mu\cdot\rho_\mu$$

NDF from "The QCD"

Summary

- **Finite T and/or Finite μ Matter**
  serves as a benchmark test
  of Theory of Strongly Interacting Particles
- **Strong Coupling Lattice QCD**
  is a good tool to attack
  Quark/Hadron/Nuclear Matter
  at Finite T & μ
- **Relativistic Mean Field model**
  with the effective potential
  derived from SCL-LQCD
  seems to be a good starting point
  to give
  Nuclear Density Functional

It is one of
the New Frontiers in QCD



SC
LQCD

QCD

Hadron

Bare NN int.

Density
Functional

Eff. NN Int.

Cluster, Shell, ...

Chiral
RMF

Nuclei