伏見汎関数による
非平衡状態でのエントロピー生成の記述

- Introduction
- Entropy production in Quantum Mechanics
- Entropy production in Quantum Field Theory
- Summary

"Towards a Theory of Entropy Production in the Little and Big Bang"

T. Kunihiro, B. Muller, A. Ohnishi, A. Schafer

Eprint:0809.4831

Various Entropies

- エントロピーとは何か？10文字以内で答えよ。
  「乱雑さを表す示量変数」(10文字)
  (学部 1 年生によく出した試験問題)

- von Neumann entropy

$$S_{vN} = -\mathrm{Tr}\left[\hat{\rho}\log\hat{\rho}\right] = -\sum_n w_n \log w_n$$

- Wehrl entropy (Discrete level → phase space)

$$S_{Wehrl} = -\int d\Gamma\, f \log f$$

- Kolmogorov-Sinai entropy = Entropy growth rate in classical non-linear dynamics

$$S_{KS} = \sum_n \lambda_n \theta(\lambda_n)$$

λ_n = Lyapunov exponent

$$\delta X_i = \delta X_i(t=0)\exp(\lambda_i t)$$

Entropy production in Quantum Mechanics

- Pure state → Entropy = 0

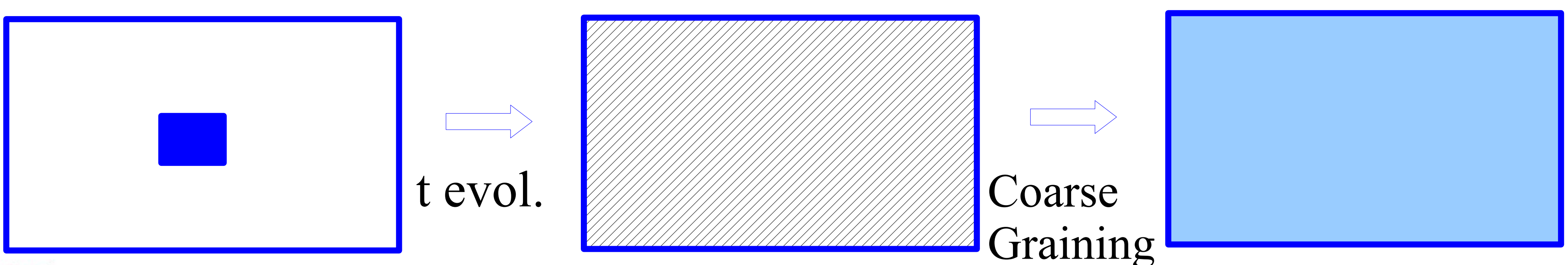
$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \rightarrow \rho_{nm} = 1\,(n=m=\text{occupied}),\,0\,(\text{other})$$

→ We need "something else" than Schrodinger equation !

- Two kind of information loss
  - Interaction with unobserved environment

$$S = -\mathrm{Tr}\left[\rho_S \log \rho_S\right],\quad \rho_S = \mathrm{Tr}_E\,\rho$$

  - Loss of practically obtainable information due to increasing complexity of phase space distribution
    → Some kind of "Coarse Graining" is necessary



Coarse Graining

- N-body → Projection to product wave functions

$$S=\int d\Gamma\left[-f\log f+\sigma(1+\sigma f)\log(1+\sigma f)\right]$$

$$(\sigma=\pm 1 \text{ for bosons/fermions})$$

  - Advanced treatment with spectral function
    T. Kita, JPSJ75('06)114005, Yu. B. Ivanov, J. Knoll, D.N. Voskresensky, NPA672(00)313, A. Nishiyama, arXiv:0810.5003.
    → "single particle energy" is necessary in advanced treatment.

- Smearing of phase space distribution function

  - Wigner function is not always positive.
    → Husimi function

We study the entropy production in quantum mechanics and field theory with Husimi function / functional

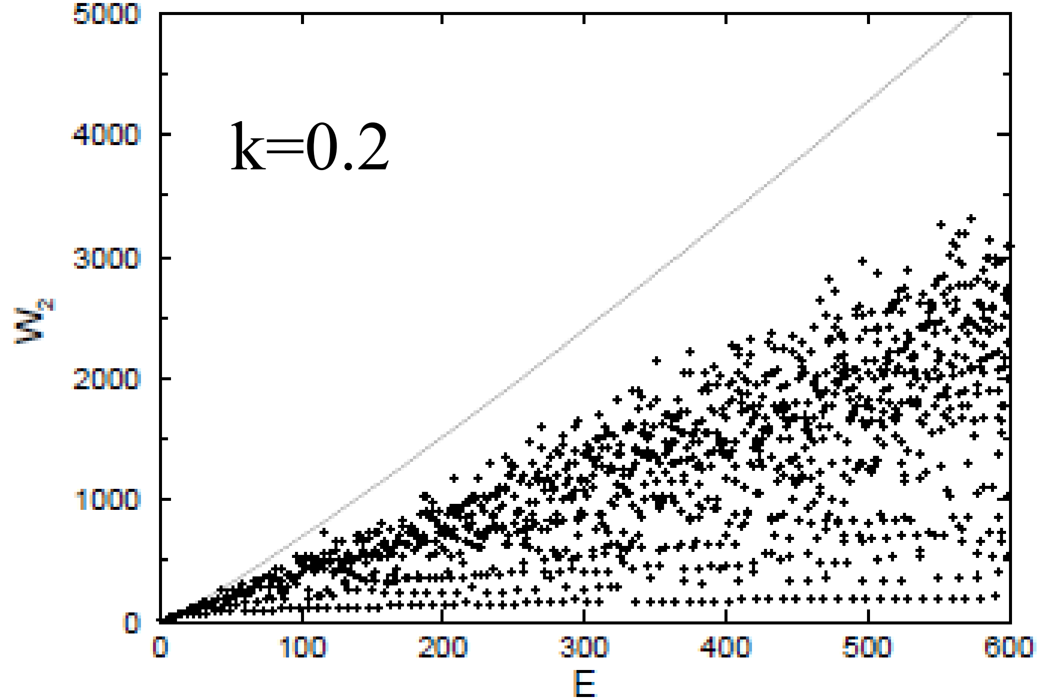
Complexity Degree: Second Moment

- Small average Husimi function would be a measure of chaoticity
  Sugita, Aiba, J. Phys. A 36 (2001), 9081.

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^4 + y^4) - kx^2y^2.$$

$$W_2(\rho_H) = \frac{1}{M_2(\rho_H)},$$

$$M_2(\rho_H) = \int \frac{d\boldsymbol{p}d\boldsymbol{q}}{(2\pi\hbar)^k}\,\rho_H(\boldsymbol{p},\boldsymbol{q})^2.$$



TDHF and Vlasov Equation

- Time-Dependent Mean Field Theory (e.g., TDHF) $i\hbar \frac{\partial \phi_i}{\partial t} = h \phi_i$
- Density Matrix

$$\rho(r, r') = \sum_i^{Occ} \phi_i(r) \phi_i^*(r') \quad \rightarrow \quad \boldsymbol{\rho_W = f} \textbf{ (phase space density)}$$

- TDHF for Density Matrix

$$i\hbar \frac{\partial \rho}{\partial t} = [h, \rho] \quad \rightarrow \quad \frac{\partial f}{\partial t} = \{h_W, f\}_{P.B.} + O(\hbar^2)$$

- Wigner Transformation and Wigner-Kirkwood Expansion

(Ref.: Ring-Schuck)

$$O_W(r, p) \equiv \int d^3 s \exp(-i p \cdot s / \hbar) < r + s/2 | O | r - s/2 >$$

$$(AB)_W = A_W \exp(i\hbar \Lambda) B_W \quad \Lambda \equiv \nabla'_r \cdot \nabla_p - \nabla'_p \cdot \nabla_r \quad (\nabla' \text{ acts on the left})$$

$$[A, B]_W = 2i A_W \sin(\hbar \Lambda / 2) B_W = i\hbar \{A_W, B_W\}_{P.B.} + O(\hbar^3)$$

Test Particle Method

- Vlasov Equation

$$\frac{\partial f}{\partial t} - \{h_W, f\}_{P.B.} = \frac{\partial f}{\partial t} + v \cdot \nabla_r f - \nabla U \cdot \nabla_p f = 0$$

- Classical Hamiltonian

$$h_W(r, p) = \frac{p^2}{2\mathrm{m}} + U(r, p)$$

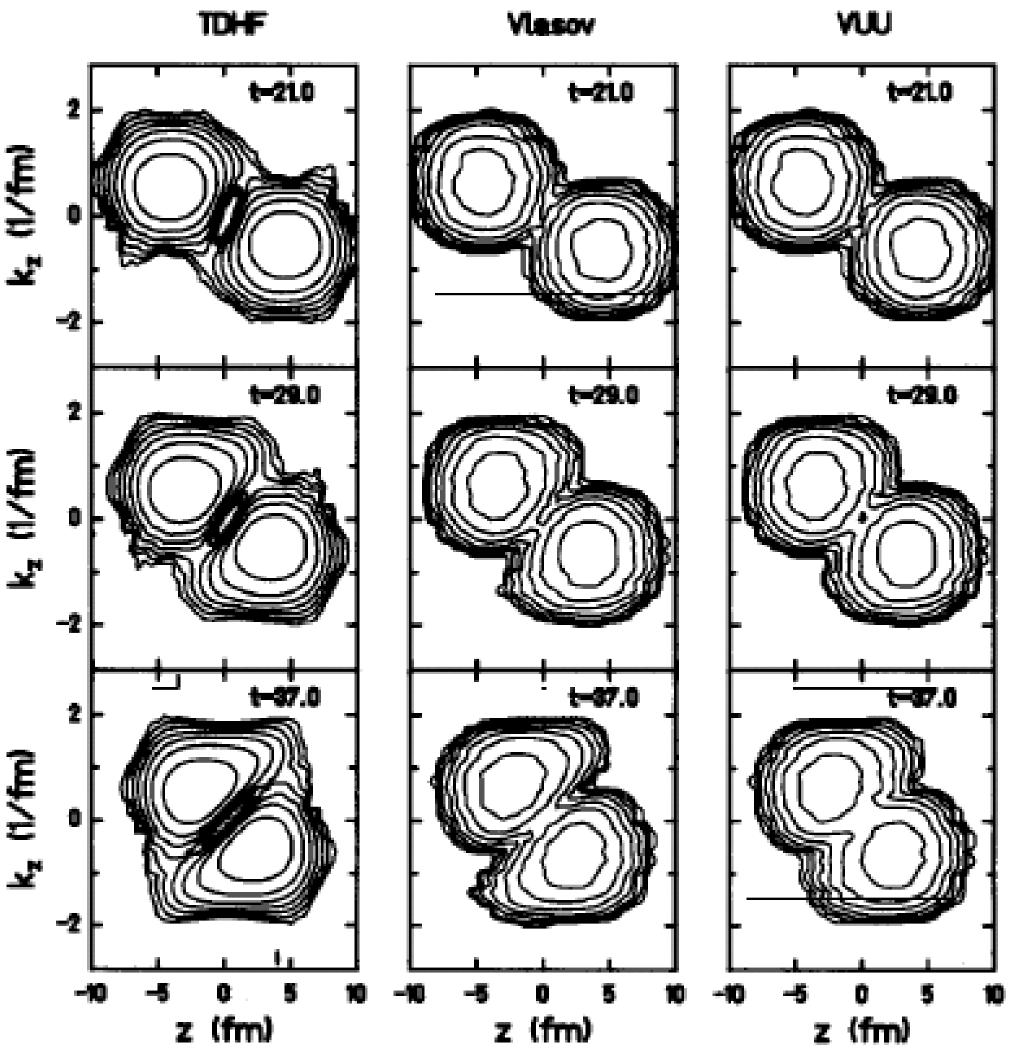
- Test Particle Method (C. Y. Wong, 1982)

$$f(r, p) = \frac{1}{N_0} \sum_i^{A N_0} \delta(r - r_i) \delta(p - p_i) \quad \rightarrow \quad \frac{dr_i}{dt} = \nabla_p h_w, \quad \frac{dp_i}{dt} = -\nabla_r h_w,$$

Mean Field Evolution can be simulated
by Classical Test Particles
→ Opened a possibility to Simulate High Energy HIC
including Two-Body Collisions in Cascade

Comarison of TDHF, Vlasov and BUU(VUU)

- Ca+Ca, 40 A MeV
  (Cassing-Metag-Mosel-Niita, Phys. Rep. 188 (1990) 363).



Wigner Function

- Example: Inverted Harmonic Oscillator $\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 - \frac{1}{2}\lambda^2\hat{x}^2$
- Equation of Motion $$\frac{\partial W}{\partial t} = \frac{2}{\hbar} H \sin(\hbar\Lambda/2) W = \{H, W\}_{P.B.}$$

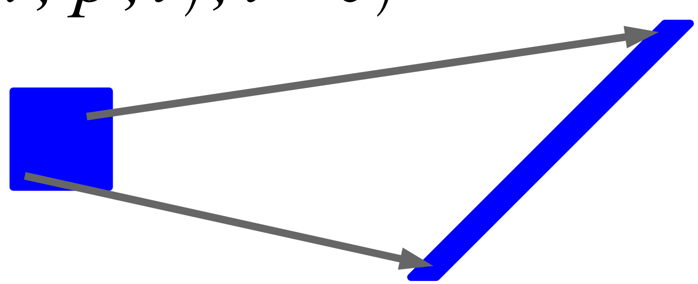
$$(AB)_W = A_W \exp(i\hbar\Lambda) B_W \quad \Lambda \equiv \nabla'_r \cdot \nabla_p - \nabla'_p \cdot \nabla_r \quad (\nabla' \text{ acts on the left})$$

H contains only p^2 and x^2 → No O($\hbar^3$) terms
→ Classical EOM gives exact results

$$\frac{dx}{dt} = p, \quad \frac{dp}{dt} = \lambda^2 x \rightarrow \begin{pmatrix} x \\ p/\lambda \end{pmatrix} = \begin{pmatrix} \cosh\lambda t & \sinh\lambda t \\ \sinh\lambda t & \cosh\lambda t \end{pmatrix} \begin{pmatrix} x_0 \\ p_0/\lambda \end{pmatrix}$$

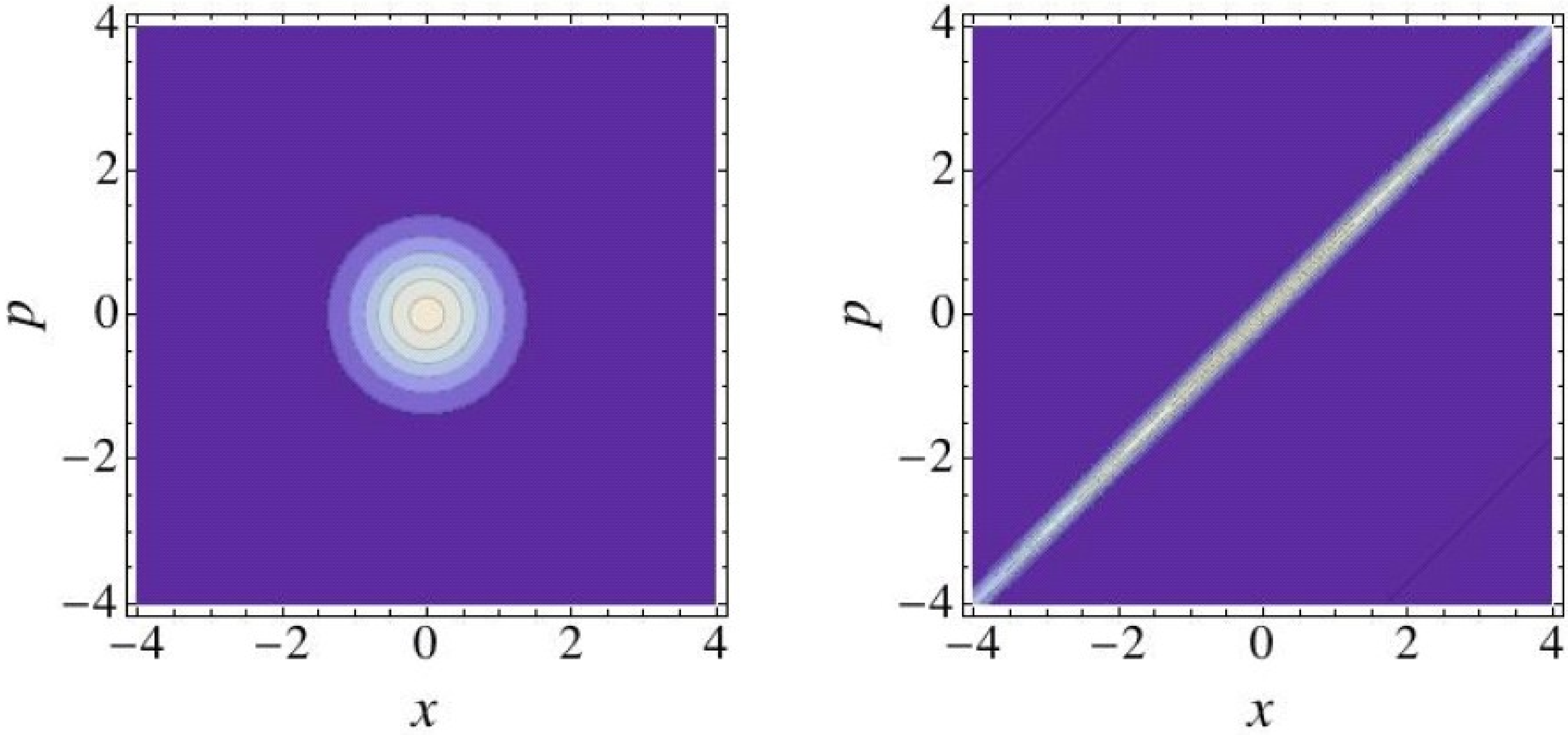
- Solution : Wigner function is constant along the classical path

$$W(x, p; t) = W(x_0(x, p, t), p_0(x, p, t); t=0)$$



Evolution of the Wigner Function

- Liouville theorem → conservation of the phase space volume
  - Exponential growth in (x+p/λ), Exponential narrowing in (x-p/λ)



$$\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 - \frac{1}{2}\lambda^2\hat{x}^2$$ λ=1, λt=0, 2

Wigner-Wehrl Entropy Growth Rate

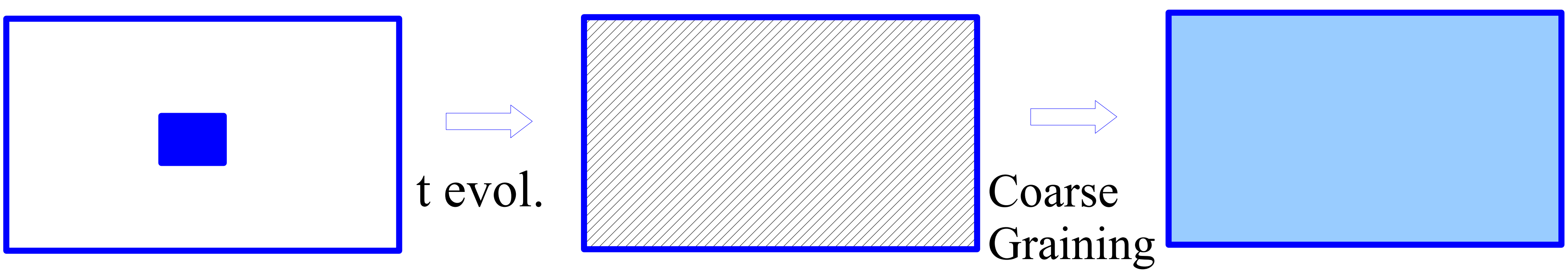
- Wigner-Wehrl entropy

$$S_W(t) = \int d\Gamma\, W(x, p; t) \log W(x, p; t)$$

  - Constant W along the classical path
  - Liouville theorem : J(x(t),p(t)/ x(t=0), p(t=0))=1

$$S_W(t) = \int d\Gamma\, W(x, p; t) \log W(x, p; t)$$
$$= \int d\Gamma_0 W(x_{0,} p_0; t=0) \log W(x_{0,} p_{0,} t=0) = \text{const.}$$

→ # of "touched" phase space cell increases,
but no Entropy Production
→ Coarse Graining is necessary to evaluate the entropy
coming from the complexity in the phase space



Husimi Function

- Husimi Function
  - Coarse grained Wigner function by the Gaussian satisfying uncertainty principle

$$H_\Delta(p,x;t) \equiv \int \frac{dp'\,dx'}{\pi\hbar} \exp\left(-\frac{1}{\hbar\Delta}(p-p')^2 - \frac{\Delta}{\hbar}(x-x')^2\right) W(p',x';t)$$

  - Expectation value of the density matrix with a coherent state → Semi-Positive definite ($H_\Delta \geq 0$)

$$H_\Delta(p,x;t) = \langle z|\hat{\rho}|z\rangle$$

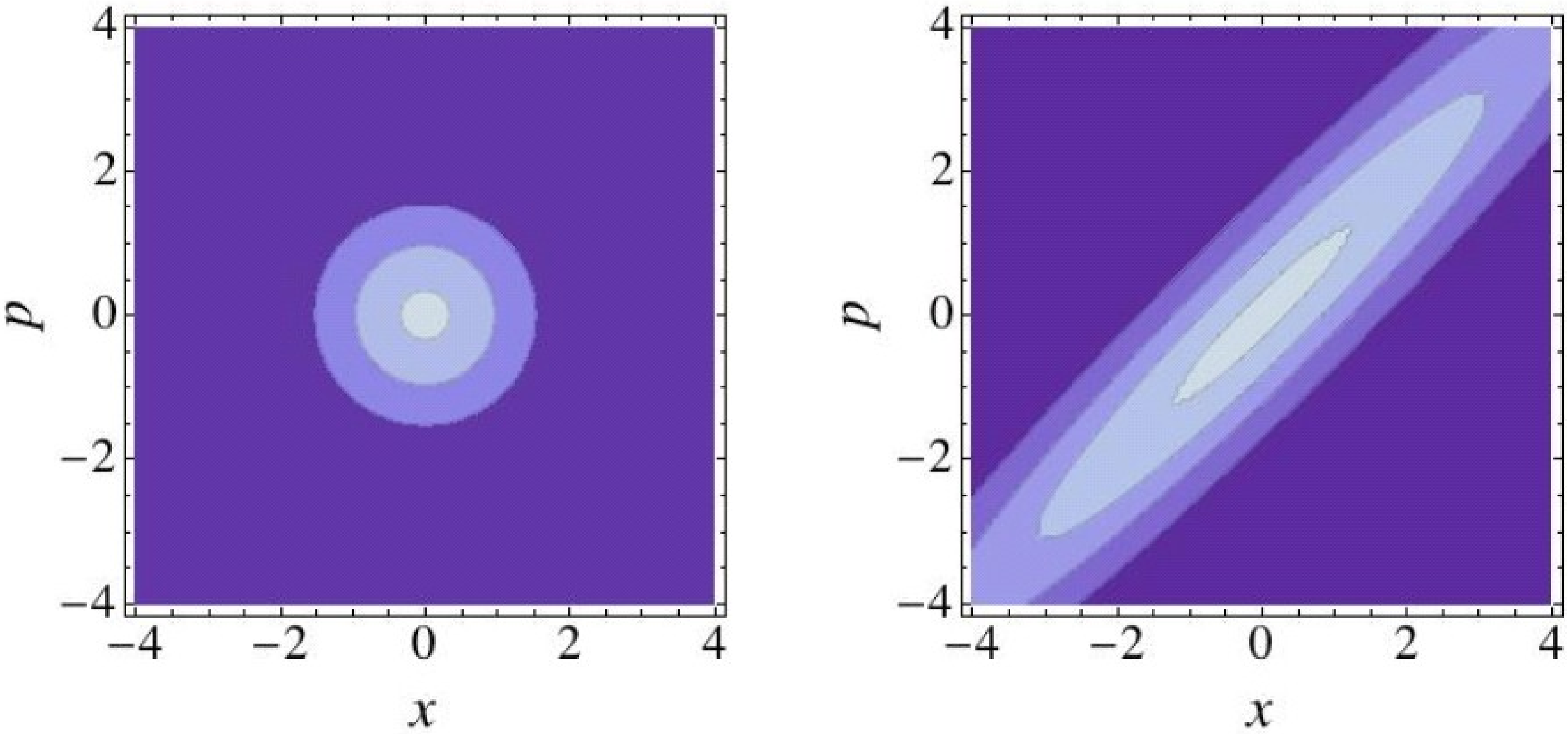
$$|z\rangle = e^{-\bar{z}z/2}\exp(z a^+)|0_\Delta\rangle,\quad z=\sqrt{\nu}\,x+\frac{i}{2\hbar\sqrt{\nu}}p,\quad \nu=\Delta/2\hbar$$

- Husimi-Wehrl Enropyy

$$S_{\mathrm{H},\Delta}(t) \;=\; -\int \frac{dp\,dx}{2\pi\hbar} H_\Delta(p,x;t)\ln H_\Delta(p,x;t)$$

Evolution of the Husimi Function

- Coherent state broadening of phase space
  - Minimum width in (x-p/λ) → phase space dist. func. is smeared !



$$\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 - \frac{1}{2}\lambda^2\hat{x}^2$$ λ=1, λt=0, 2

Husimi-Wehrl Entropy Growth Rate

- Example
  - Initial Cond.= g.s. of HO with freq. ω
  - Hamiltonian = Inverted HO with freq. λ

$$\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 - \frac{1}{2}\lambda^2\hat{x}^2$$

$$S_{\mathrm{H},\Delta}(t) = \ln\frac{\sqrt{A(t)}}{2} + 1 = \frac{1}{2}\ln\frac{A(t)}{4} + 1$$

$$A(t) = 2(\sigma\rho\cosh 2\lambda t + 1 + \delta\delta')$$

$$\frac{dS_{\mathrm{H},\Delta}}{dt} = \int\frac{dp\,dx}{2\pi\hbar}\frac{\partial H_\Delta}{\partial t}\ln H_\Delta + \frac{\partial}{\partial t}\int\frac{dp\,dx}{2\pi\hbar}H_\Delta = \int\frac{dp\,dx}{2\pi\hbar}\frac{\partial H_\Delta}{\partial t}\ln H_\Delta$$
$$= \frac{\lambda\sigma\rho\,\sinh 2\lambda t}{\sigma\rho\,\cosh 2\lambda t + 1 + \delta\delta'} \xrightarrow{t\to\infty} \lambda$$

Kolmogorov-Sinai entropy appears in quantum mechanical problem with Husimi coarse graining !

Wigner Functional

- Canonical variables
  (x, p) : Classical and Quantum Mechanics
  (Φ,Π): Field Theory
  → Wigner ***Functional***

  - Coordinate representation

$$W[\Pi(x),\Phi(x);t] = \int \mathcal{D}\varphi(x)\; e^{-i\int dx\,\Pi(x)\varphi(x)} \times \langle \Phi(x)+\frac{1}{2}\varphi(x)|\,\hat{\rho}(t)\,|\Phi(x)-\frac{1}{2}\varphi(x)\rangle$$

  - Momentum representation

$$W[\Phi(p),\Pi(p);t] = \int \mathcal{D}\varphi(p)\,\exp\left[-i\int_0^\infty dp\,\left(\Pi^*(p)\varphi(p)+\Pi(p)\varphi^*(p)\right)\right]$$

$$\text{X}\,\langle \Phi(p)+\frac{1}{2}\varphi(p)|\,\hat{\rho}(t)\,|\Phi(p)-\frac{1}{2}\varphi(p)\rangle$$

Equation of Motion of Wigner Functional

- Hamiltonian in the momentum representation

$$\hat{H}_0 = \int_0^\infty \frac{dp}{2\pi} \left(\hat{\Pi}^\dagger(p)\hat{\Pi}(p) + (p^2 + m^2)\hat{\Phi}^\dagger(p)\hat{\Phi}(p) \right)$$

$$\hat{H} = \hat{H}_0 + V[\Phi]$$

$$\frac{\partial W[\Phi, \Pi; t]}{\partial t} = \{H, W\}_{PB} + O(\hbar^2)$$

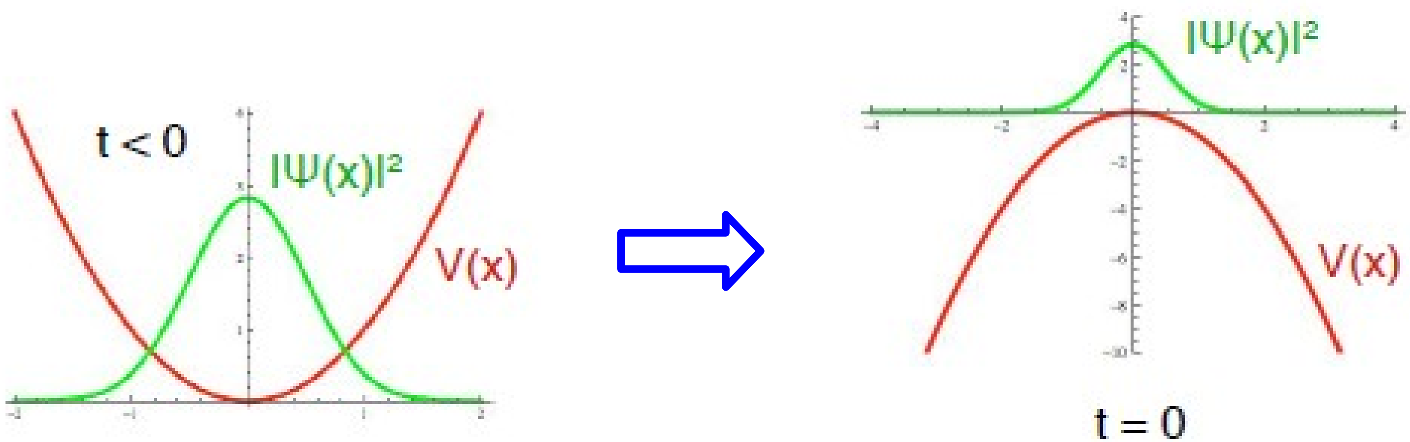
  - As far as the power of Φ and Π is less than or equal to 2, classical EOM gives correct time evolution.
    → Similar treatment to the quantum mechanics

Roll-Over Transition

- Spontaneous symmetry breaking of the vacuum
  → Simple example: roll-over transition

$$\hat{H}(t) = \frac{p^2}{2} + \frac{m(t)^2}{2} x^2$$

with $$m^2(t) = m^2\,\theta(-t) - \mu^2\,\theta(t)$$



Wigner Functional during Roll-Over

- Wigner functional is constant along classical path

$$W[\Pi,\Phi;t] = C\,\exp\left[-\int \frac{dp}{2\pi}\left(\frac{|\Pi_p^0|^2}{E_p} + E_p|\Phi_p^0|^2\right)\right]$$

  - Unstable modes

$$\Phi_p^0 = \Phi_p(t)\cosh\lambda_p t - \frac{\Pi_p(t)}{\lambda_p}\sinh\lambda_p t$$
$$\Pi_p^0 = \Pi_p(t)\cosh\lambda_p t - \lambda_p\,\Phi_p(t)\sinh\lambda_p t$$

  - Stable modes

$$\Phi_p^0 = \Phi_p(t)\cos\omega_p t - \frac{\Pi_p(t)}{\omega_p}\sin\omega_p t$$
$$\Pi_p^0 = \Pi_p(t)\cos\omega_p t + \omega_p\,\Phi_p(t)\sin\omega_p t$$

Husimi-Wehrl Entropy

- Husimi-Wehrl entropy

$$S_{\mathrm{H},\Delta}(t) = \int \frac{D\Pi\, D\Phi}{2\pi} H_\Delta \ln H_\Delta$$

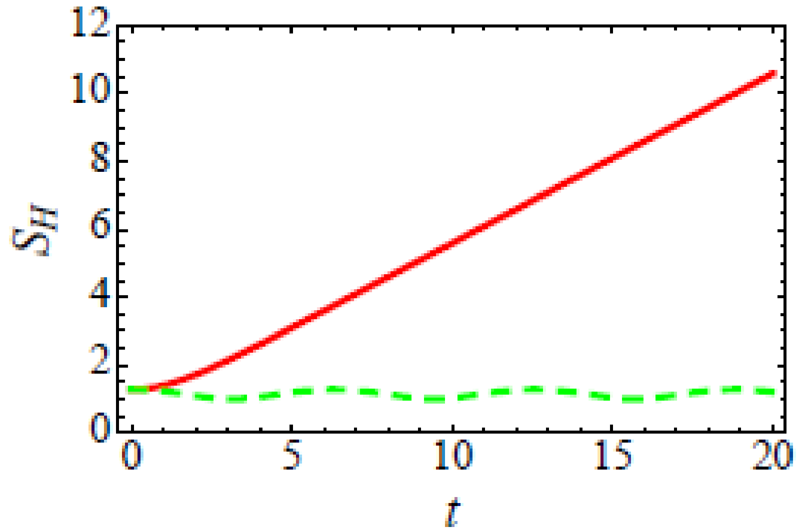
$$= V \int_{|\mathbf{p}|<\mu} \frac{dp}{2\pi} \left[\frac{1}{2} \ln \frac{A_\mathbf{p}(t)}{4} + 1\right] + V \int_{|\mathbf{p}|>\mu} \frac{dp}{2\pi} \left[\frac{1}{2} \ln \frac{\bar{A}_\mathbf{p}(t)}{4} + 1\right]$$

$$\frac{dS_{\mathrm{H},\Delta}}{dt} = V \int_{|\mathbf{p}|<\mu} \frac{dp}{2\pi} \frac{\sigma_\mathbf{p}(\Delta^2 + \lambda_\mathbf{p}^2)\sinh 2\lambda_\mathbf{p} t}{A_\mathbf{p}(t)\Delta} + V \int_{|\mathbf{p}|>\mu} \frac{dp}{2\pi} \frac{\bar{\delta}_\mathbf{p}(\omega_\mathbf{p}^2 - \Delta^2)\sin 2\omega_\mathbf{p} t}{\bar{A}_\mathbf{p}(t)\Delta}$$

$$\xrightarrow{t\to\infty} V \int_{-\mu}^{\mu} \frac{dp}{2\pi} \lambda_\mathbf{p} = \frac{V\mu^2}{8}$$

$$A_\mathbf{p}(t) = \frac{\Delta^2 + \lambda_\mathbf{p}^2}{\lambda_\mathbf{p}\Delta} \cosh 2\lambda_\mathbf{p} t + 2 + \delta_\mathbf{p} \frac{\Delta^2 - \lambda_\mathbf{p}^2}{\lambda_\mathbf{p}\Delta},$$

$$\bar{A}_\mathbf{p}(t) = \frac{\Delta^2 + \omega_\mathbf{p}^2}{\omega_\mathbf{p}\Delta} + 2 + \delta_\mathbf{p} \frac{\Delta^2 - \omega_\mathbf{p}^2}{\omega_\mathbf{p}\Delta} \cos 2\omega_\mathbf{p} t$$



Summary

- Early stage entropy production and/or thermalization is one of the largest remaining problem in RHIC physics.
- Entropy production of isolated quantum system requires some kind of coarse graining.
- Here we have discussed the entropy production in quantum mechanical and field theoretical problems by using the Wigner function/functional, and its coarse graining, Husimi function/functional.
- With Husimi function(al), the entropy growth rate is found to be described by the Kolmogorov-Sinai (KS) entropy, which is the sum of the positive Lyapunov exponent, in the case of inverted HO potential and Roll-over transitions.

Discussion

- Is the Husimi-Wehrl entropy consistent with the von Neumann entropy in thermal equilibrium ?
  → Thermal equilibrium with one dim HO potential

- von Neumann entropy

$$S_{vN} \equiv -\sum_{n=0}^{\infty} w_n \ln w_n = \frac{\beta\hbar\omega}{e^{\beta\hbar\omega}-1} - \ln(1-e^{-\beta\hbar\omega})$$
$$= -\bar{n}\ln\bar{n} + (\bar{n}+1)\ln(\bar{n}+1)\ .$$

$$\bar{n} = \frac{1}{Z_\beta}\sum_{n=0}^{\infty} n w_n = \frac{1}{e^{\beta\hbar\omega}-1}$$

- Wigner-Wehrl entropy

$$W(z) = B_\beta \exp(-B_\beta\,\bar{z}z) \quad B_\beta = 2\tanh(\beta\hbar\omega/2) = 1/(\bar{n}+1/2)$$

$$S_W = 1 + \ln\left(\bar{n} + \frac{1}{2}\right)$$

Discussion

- Husimi-Wehrl entropy
  Coherent state

$$|z\rangle = e^{-\bar{z}z/2}\exp(z\hat{a}^\dagger)\,|0\rangle = e^{-\bar{z}z/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\,|n\rangle$$

Husimi function

$$H(z) = \langle z|\,\hat{\rho}_{\rm th}\,|z\rangle = \frac{e^{-\bar{z}z}}{Z_\beta}\sum_{n=0}^{\infty}\frac{(\bar{z}z)^n}{n!}e^{-n\beta\hbar\omega}$$

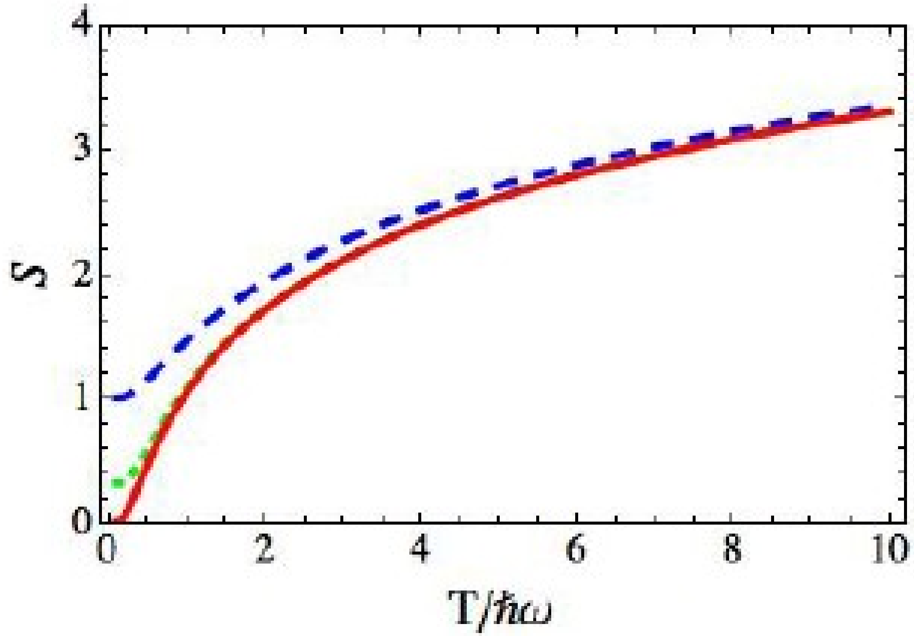
$$= \frac{1}{Z_\beta}\exp\left[-\bar{z}z\left(1-e^{-\beta\hbar\omega}\right)\right] = A_\beta\exp\left(-A_\beta\bar{z}z\right)$$

$$A_\beta = 1-e^{-\beta\hbar\omega} = 1/(\bar{n}+1)$$

Husimi-Wehrl entropy

$$S_H = 1-\ln A_\beta = 1+\ln(\bar{n}+1)$$

With Husimi-Wehrl entropy, coarse graining effects also appears in equilibrium.



Entropy expression in Wave Packet Statistics

- Wave packet statistical mechanics (Ohnishi-Randrup method)
  - Partition Function

$$\mathcal{Z}_\beta = \int d\Gamma \langle z| e^{-\beta\mathcal{H}} |z\rangle = \int d\Gamma \exp\left[-\int_0^\beta d\beta' \mathcal{H}_{\beta'}(z)\right] ,$$

$$\mathcal{H}_\beta(z) = -\frac{\partial}{\partial\beta} \ln \langle z| e^{-\beta\mathcal{H}} |z\rangle = \langle z| \mathcal{H} e^{-\beta\mathcal{H}} |z\rangle / \langle z| e^{-\beta\mathcal{H}} |z\rangle$$
$$= \langle z| \mathcal{H} |z\rangle - \beta\sigma^2_{\mathcal{H}}(z) + \mathcal{O}(\beta^2) ,$$

  - Entropy

$$S = -\frac{\partial F}{\partial T} = \frac{\partial}{\partial T}(T \ln \mathcal{Z}_\beta) = -\beta\frac{\partial}{\partial\beta} \ln \mathcal{Z}_\beta + \ln \mathcal{Z}_\beta$$
$$= \int d\Gamma \frac{\langle z| e^{-\beta\mathcal{H}} |z\rangle}{\mathcal{Z}_\beta} \left[\beta\mathcal{H}_\beta(z) + \ln \mathcal{Z}_\beta\right] .$$

$$S_H - S = \int d\Gamma\, H(z) \left[\int_0^\beta d\beta' \mathcal{H}_{\beta'}(z) - \beta\mathcal{H}_\beta(z)\right]$$
$$\simeq \int d\Gamma\, H(z) \left[\frac{1}{2}\beta^2\sigma^2_{\mathcal{H}}(z) + \mathcal{O}(\beta^3)\right] .$$

We have systematic manner how to calculate entropy in equilibrium. Is there any similar method in Non-Eq. cases where single particle description may not be good enough ?