Fluctuation Effects on the phase boundary in the strong coupling limit of lattice QCD

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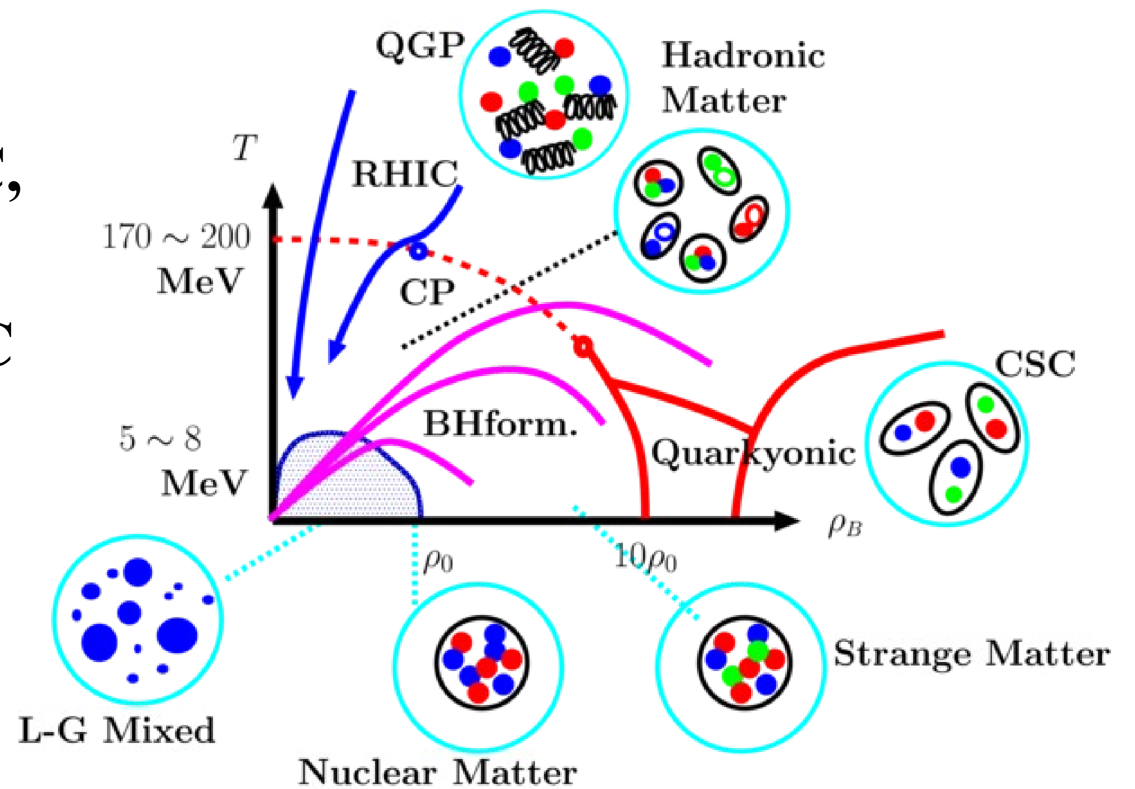
- **Introduction**
- **Mesonic auxiliary field effective action in the strong coupling limit of lattice QCD**
- **Monte-Carlo estimate of the phase boundary**
- **Summary**

QCD Phase diagram

- **Phase transition at high *T***
  - **Physics of early universe: Where do we come from ?**
  - **RHIC, LHC, Lattice MC, pQCD, ….**
- **High μ transition**
  - **Physics of neutron stars: Where do we go ?**
  - **RHIC-BES, FAIR, J-PARC, Astro-H, Grav. Wave, …**
  - **Sign problem in Lattice MC → Model studies and/or Approximations are necessary.**



QCD phase transition in strong coupling limit

- **QCD phase transition at finite μ**
  - **Lattice QCD Monte-Carlo simulation has the sign problem. There are many attempts to avoid the sign problem, but the results at large μ (μ/T > 1 or μ > m_π) are not yet reliable.**
    (Reweighting, AC from Imaginary μ, Taylor expansion, cumulant expansion, …) Fodor, Katz ('02); de Forcrand, Philipsen('02); D'Elia, M. Lombardo ('03); Allton et al. ('04); Ejiri ('08); …
  - **Phase diagram is obtained by using the auxiliary field method of strong coupling lattice QCD, but those works rely on the mean field approximation.**
    Damgaard, Kawamoto, Shigemoto ('84); Damagaard, Hochberg, Kawamoto ('85); Bilic, Karsch, Redlich ('92); Fukushima ('03); Nishida ('03); Kawamoto, Miura, AO, Ohnuma ('07); Miura, Nakano, AO ('09); Miura, Nakano, AO, Kawamoto ('09); Nakano, Miura, AO ('10);
  - **Monomer-Dimer-Polymer (MDP) algorithm was proposed and has been demonstrated to work in the strong coupling limit (SCL), but its extension to finite coupling cases is not easy.**
    Karsch, Mutter ('89), de Forcrand, Fromm ('09), …

Problem in mean field approaches in SC-LQCD

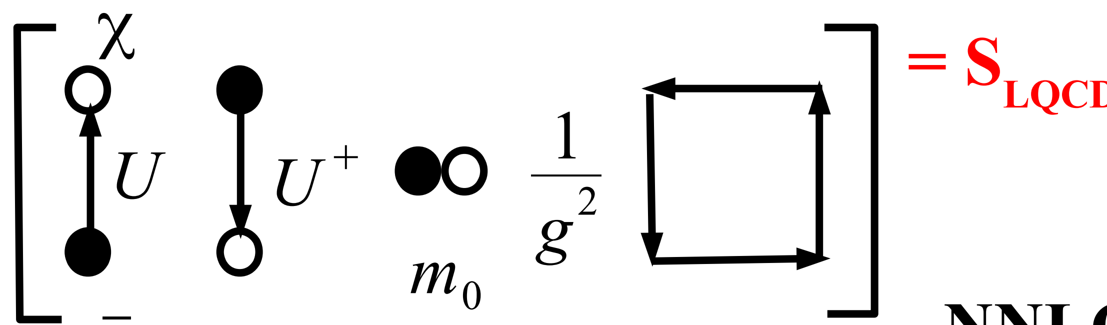
- **Three types of approximations**
  - **Strong coupling expansion ($1/g^2$): Small number of plaquettes**
  - **Large dimensional approximation (1/d): Small number of quarks**
  - **Constant field assumption**
- **Phase diagram in mean field approaches in SCL may not match the phase diagram in MDP simulation result.**
  → 1/d or constant field assumption ?
  - **Higher orders in 1/d expansion: Still difficult to handle (Bosonization breaks chiral and/or gauge symmetry.)**
    Azcoiti et al. ('03); Kawamoto, Miura, AO, Ohnuma ('07); AO, Nakano, Miura ('10).
  - **Fluctuations of aux. fields: Not discussed seriously.**

We try to extend the auxiliary field method of SC-LQCD to include fluctuations.

Auxiliary field effective action in SCL-LQCD

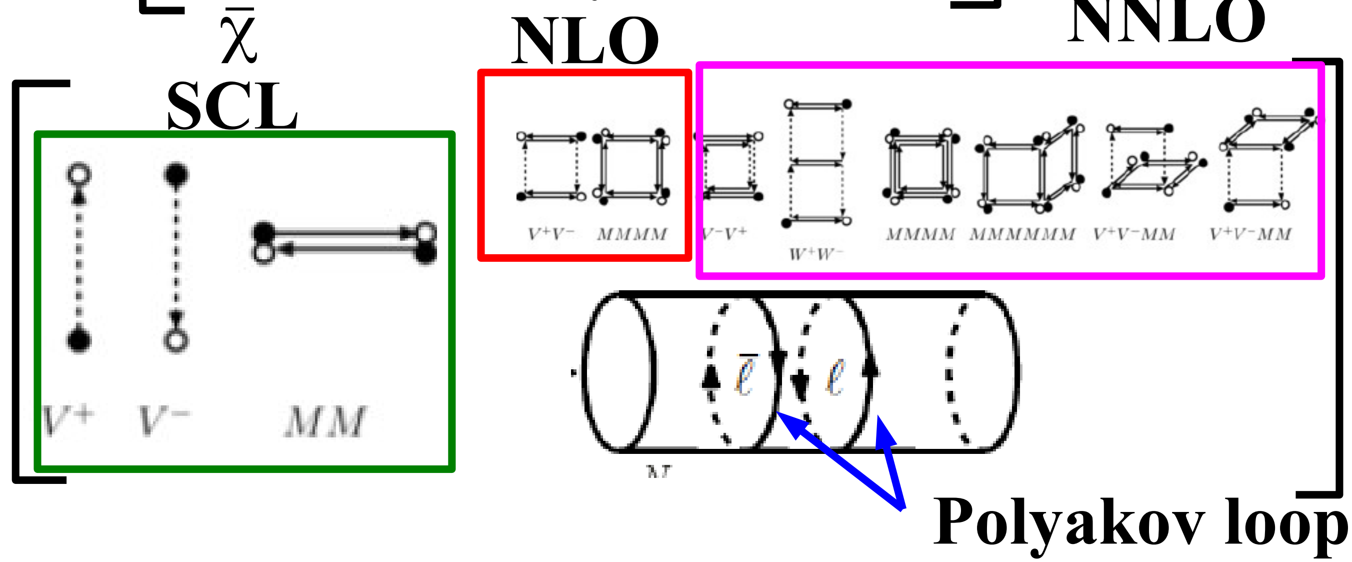
SC-LQCD Procedures

$$Z = \int D[\chi, \bar{\chi}, U_{0,} U_j] \exp$$



Spatial link integral
(1/g² and 1/d exp.)

$$= \int D[\chi, \bar{\chi}, U_0] \exp$$



Bosonization

$$\approx \int D[\chi, \bar{\chi}, U_{0,} \Phi] \exp\left(-S_{\rm eff}[\chi, \bar{\chi}, U_{0,} \Phi]\right)$$

Fermion det.
& U_0 integral

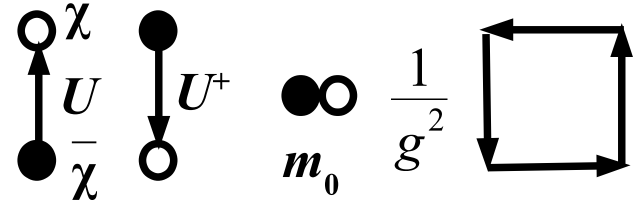
Mean (const.) field approx.

$$\approx \exp\left(-V\, F_{\rm eff}(\Phi; T, \mu)/T\right)$$

Lattice QCD action → SCL quark & U_0 action

- **Lattice QCD action with (unrooted) staggered Fermion**

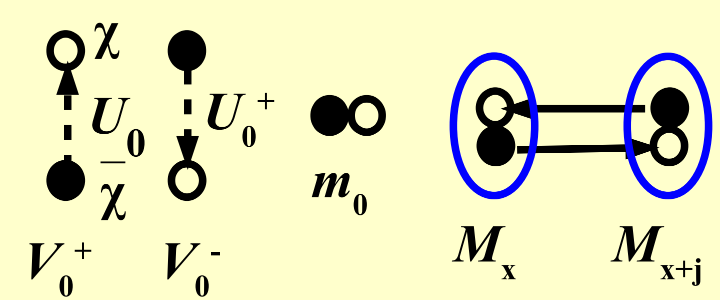
$$S_{LQCD}=S_F+S_G \quad S_G=-\frac{1}{g^2}\sum_{plaq.}\mathrm{tr}\left[U_P+U_P^+\right]f_P$$



$$S_F=\frac{1}{2}\sum_x\left[V^+(x)-V^-(x)\right]+\frac{1}{2\gamma}\sum_{x,j}\eta_\mu(x)\left[\bar\chi_x U_{x,j}\chi_{x+\hat j}-\bar\chi_{x+\hat j}U^+_{x,j}\chi_x\right]+\sum_x\frac{m_0}{\gamma}M_x$$

$$V^+(x)=e^\mu\bar\chi_x U_{x,0}\chi_{x+\hat 0},\quad V^-(x)=e^{-\mu}\bar\chi_{x+\hat 0}U^+_{x,0}\chi_x,\quad M_x=\bar\chi_x\chi_x,\quad a_\tau=a/\gamma,\quad f_P=1\ \text{or}\ 1/\gamma$$

- **Strong coupling expansion (Strong coupling limit)**
  - **Ignore plaquette action ($1/g^0$)**
  - **Integrate out spatial link variables of min. quark number diagrams (1/d expansion)**



$$\int dU\,U_{ab}U^+_{cd}=\delta_{ad}\delta_{bc}/N_c$$

$$S_{\text{eff}}=\frac{1}{2}\sum_x\left[V^+(x)-V^-(x)\right]-\frac{1}{4N_c\gamma^2}\sum_{x,j}M_xM_{x+\hat j}+\frac{m_0}{\gamma}\sum_x M_x$$

Introduction of Auxiliary Fields

- *MM* term = Four Fermi (two-body) interacting term
  → Bosonization

Non-Local NJL type ?

$$S^{(s)} = -\frac{1}{4N_c\gamma^2}\sum_{x,j} M_x M_{x+\hat{j}} = -\frac{L^3}{4N_c\gamma^2}\sum_{\tau,\mathbf{k}} f_M(\mathbf{k})\,\tilde{M}_{\mathbf{k}}(\tau)\,\tilde{M}_{-\mathbf{k}}(\tau)$$

$$= \frac{L^3}{4N_c\gamma^2}\sum_{\tau,\mathbf{k},f_M(\mathbf{k})>0} f_M(\mathbf{k})\Big[\varphi_{\mathbf{k}}(\tau)^2 + \phi_{\mathbf{k}}(\tau)^2 + \varphi_{\mathbf{k}}(\tilde{M}_{\mathbf{k}} + \tilde{M}_{-\mathbf{k}}) - i\phi_{\mathbf{k}}(\tilde{M}_{\mathbf{k}} - \tilde{M}_{-\mathbf{k}})$$

$$+ \varphi_{\bar{\mathbf{k}}}(\tau)^2 + \phi_{\bar{\mathbf{k}}}(\tau)^2 + i\varphi_{\bar{\mathbf{k}}}(\tilde{M}_{\bar{\mathbf{k}}} + \tilde{M}_{-\bar{\mathbf{k}}}) + \phi_{\bar{\mathbf{k}}}(\tilde{M}_{\bar{\mathbf{k}}} - \tilde{M}_{-\bar{\mathbf{k}}})\Big]$$

$$= \frac{\Omega}{2N_c\gamma^2}\sum_{k,f_M(\mathbf{k})>0} f_M(\mathbf{k})\,[\sigma_k^*\sigma_k + \pi_k^*\pi_k] + \frac{1}{2N_c\gamma^2}\sum_x M_x\,[\sigma(x) + i\varepsilon(x)\pi(x)]$$

$$\sigma(x) = \sum_{k,f_M(\mathbf{k})>0} f_M(\mathbf{k})e^{ikx}\sigma_k\ ,\quad \pi(x) = \sum_{k,f_M(\mathbf{k})>0} f_M(\mathbf{k})e^{ikx}\pi_k$$

$$\sigma_k = \varphi_k + i\phi_k\ ,\ \pi_k = \varphi_{\bar{k}} + i\phi_{\bar{k}}$$

$$V_{x,y} = \frac{1}{2}\sum_j\left(\delta_{x+\hat{j},y} + \delta_{x-\hat{j},y}\right)\ ,\quad f_M(\mathbf{k}) = \sum_j \cos k_j\ ,\quad \bar{\mathbf{k}} = \mathbf{k} + (\pi,\pi,\pi)$$

Fermion Determinant

Faldt, Petersson, 1986

- **Fermion action is separated to each spatial point and bi-linear → Determinant of Nτ x Nc matrix**

$$\exp(-V_{\mathrm{eff}}/T)=\int dU_0 \begin{vmatrix} I_1 & e^{\mu} & 0 & & e^{-\mu}U^{+} \\ -e^{-\mu} & I_2 & e^{\mu} & & \\ 0 & -e^{-\mu} & I_3 & e^{\mu} & \\ \vdots & & & \ddots & \\ -e^{\mu}U & & & -e^{-\mu} & I_N \end{vmatrix}$$

Nc x Nτ

$$=\int dU_0 \det\left[X_N[\sigma]\otimes \mathbf{1}_c + e^{-\mu/T}U^{+} + (-1)^{N_\tau} e^{\mu/T} U\right]$$

Nc

$$= X_N^3 - 2X_N + 2\cosh(3N_\tau \mu)$$

$$I_\tau/2 = [\sigma(x)+i\varepsilon(x)\pi(x)]/2N_c\gamma^2 + m_0/\gamma$$

$$X_N = B_N + B_{N-2}(2;N-1)$$

$$B_N = I_N B_{N-1} + B_{N-2}$$

$$B_N = \begin{vmatrix} I_1 & e^{\mu} & 0 & & \\ -e^{-\mu} & I_2 & e^{\mu} & & \\ 0 & -e^{-\mu} & I_3 & e^{\mu} & \\ \vdots & & & \ddots & \\ & & & -e^{-\mu} & I_N \end{vmatrix}$$

Auxiliary Field Monte-Carlo Integral

- **Effective action of Auxiliary Field**

$$S_{\text{eff}}=\frac{\Omega}{4N_c\gamma^2}\sum_{k,\,f_M(\boldsymbol{k})>0} f_M(\boldsymbol{k})\left[\sigma_k^*\sigma_k+\pi_k^*\pi_k\right] -\sum_{\boldsymbol{x}}\log\left[X_N(\boldsymbol{x})^3-2X_N(\boldsymbol{x})+2\cosh(3N_\tau\mu)\right]$$

  - **μ dependence appears only in the log.**
  - **σ_k, π_k have to be generated in momentum space, while X_N requires σ(x) and π(x) → Fourier transf. in each step.**

$$\Sigma(x)=\frac{1}{2N_c\gamma^2}\left[\sigma(x)+i\varepsilon(x)\pi(x)\right]+\frac{m}{\gamma}$$

$$\sigma(x)=\sum_{k,\,f_M(\boldsymbol{k})>0} f_M(\boldsymbol{k})e^{ikx}\sigma_k,\quad \pi(x)=\sum_{k,\,f_M(\boldsymbol{k})>0} f_M(\boldsymbol{k})e^{ikx}\pi_k$$

  - **X_N is complex, and this action has the sign problem. But the sign problem is less severe at larger μ.**

Auxiliary Field Monte-Carlo (σMC) estimate of the phase boundary

Numerical Calculation

- 4^4 asymmetric lattice + Metropolis sampling of σ_k and π_k.
- Metropolis sampling
  = One of the typical (popular) method of importance sampling

$$P_{B\to A} = 1$$

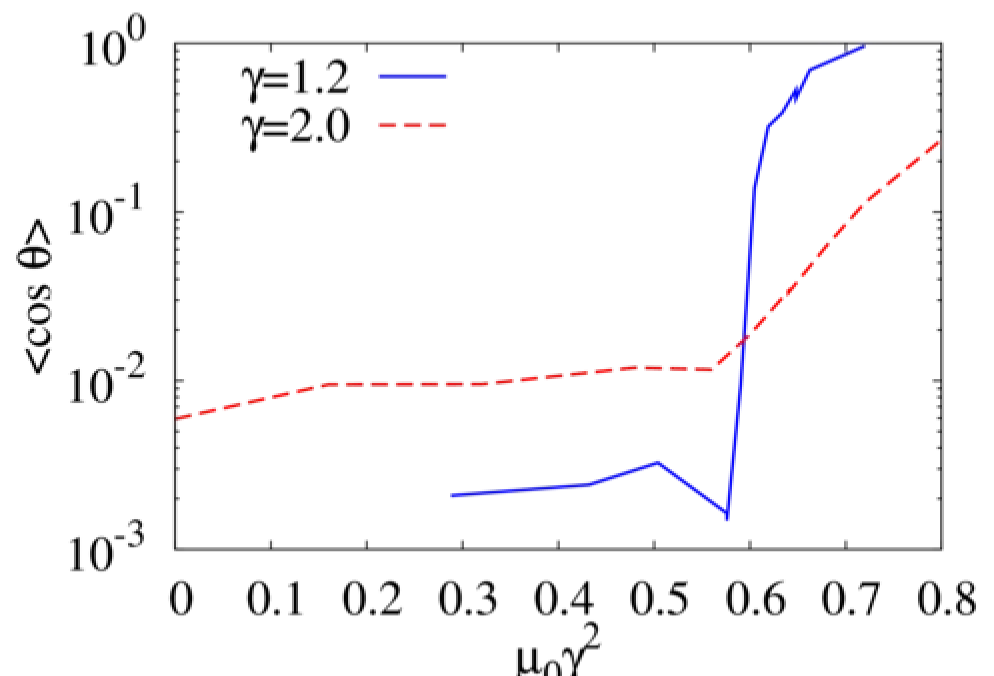
Config. A, $S_{eff}(A)$ ⇄ Config. B, $S_{eff}(B)$ $\qquad S_{eff}(A) < S_{eff}(B)$

$$P_{A\to B} = \exp[S_{eff}(A) - S_{eff}(B)]$$

  - Trial prob.: $P^{try}_{A\to B} = P^{try}_{B\to A}$ (detailed balance)
  - Pickup prob.: According to S_{eff}.
  - In equilibrium, $P(A)\, P_{A\to B} = P(B)\, P_{B\to A} \to P(A) \propto \exp[-S_{eff}(A)]$
- Typical sampling size: Thermalization=5×10^4, Sample=2×10^6

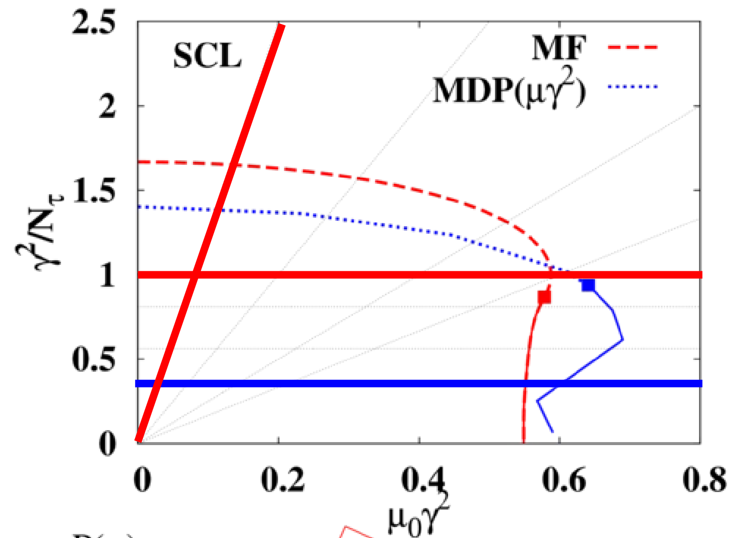
Numerical Calculation (cont.)

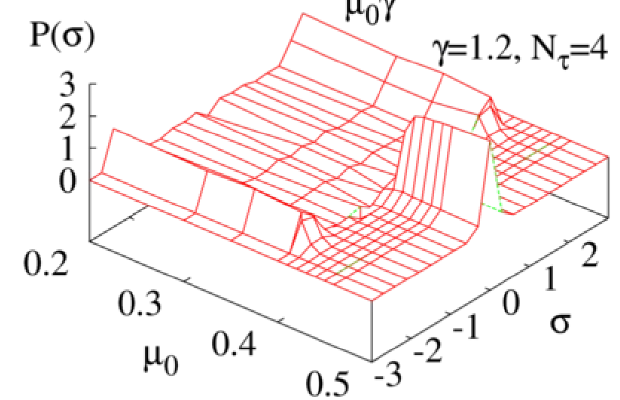
- **Jump size is chosen to be new sampling prob. ~ 0.5**
  - **Always full Update of σ_k and π_k (This may not be very efficient.)**
- **Initial cond. = const. σ (σ=-2.5, -2.0, ..., 2.5)**
  - **Chiral limit (m=0) simulation → Symmetry in σ ↔ - σ**
  - **Deep Seff min. at $\sigma \sim \sigma_{vac}$ at low *T***
- **Sign problem is not severe in 4^4 lattice.**
  - **<cos θ> ~ a few x 10^{-3} or more.**
- **Computer: My PC (Core i7)**

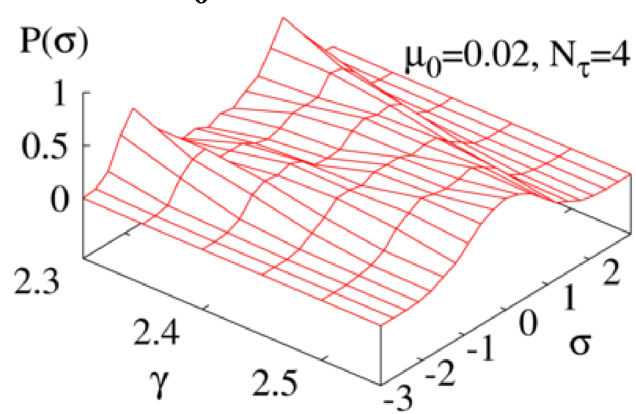


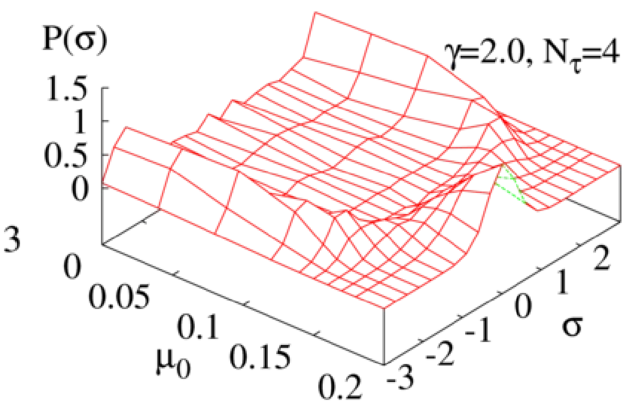
Results (1) --- σ distribution

- **Low T simulation [γ=1.2, N_τ=4 (T=0.36)]**
  - **Two peaks ($\sigma \sim \sigma_{vac}$) → One peak ($\sigma = 0$)**
    → First order phase transition
  - **Transition takes place at $\mu_0 \sim 0.4$**
    ($\mu = \mu_0\gamma^2 \sim 0.58$)
- **Medium T simulation [γ=2, N_τ=4 (T=1)]**
  - **Two peaks merges to be one**
    → Second order phase transition
  - **Transition takes place at $\mu_0 \sim 0.16$**
    ($\mu = \mu_0\gamma^2 \sim 0.64$)



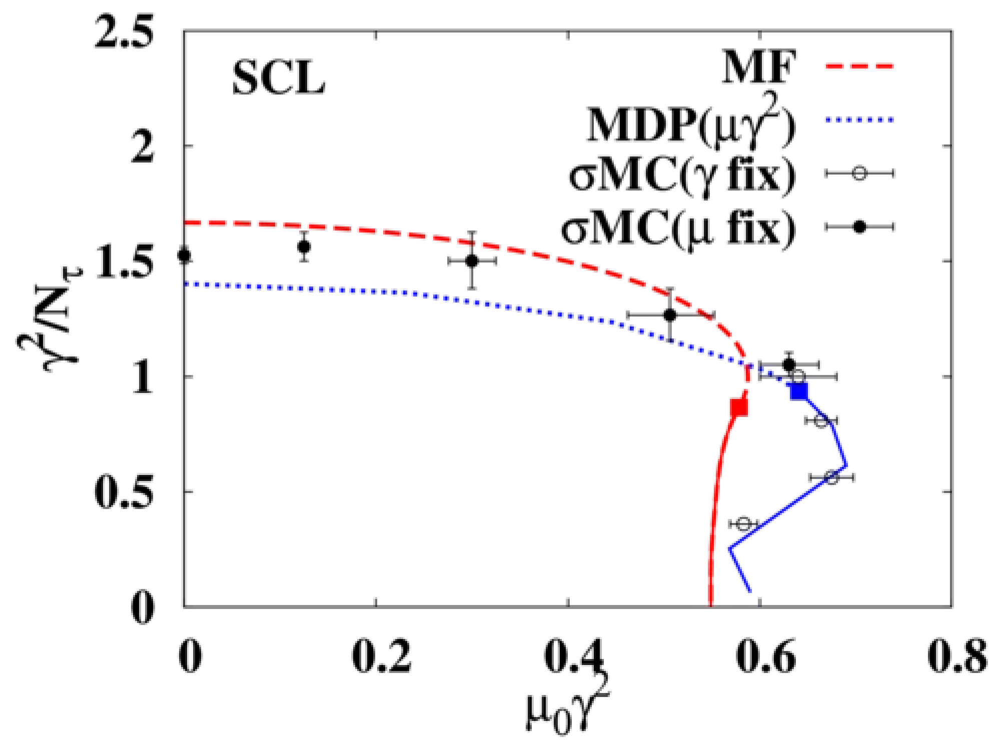


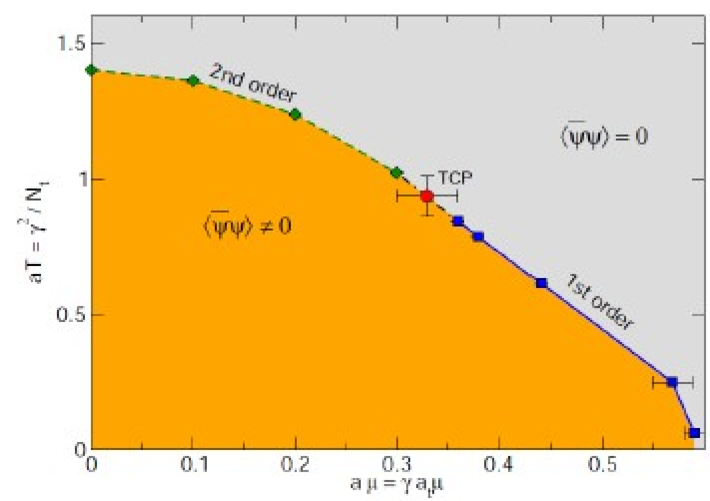




Results (2) --- phase boundary

- T, μ are assumed to be given by γ^2/N_τ, $\mu_0\gamma^2$
- Fluctuation of aux. field modifies the phase boundary.
  - Lower transition T
  - Larger transition μ
- σMC results are close to MDP results.
  - σMC overestimate T_c in μ~0 region by ~ 7 %.
- When $\mu\gamma^2$ scaling is assumed, σMC and MDP results are reasonably match MF results.

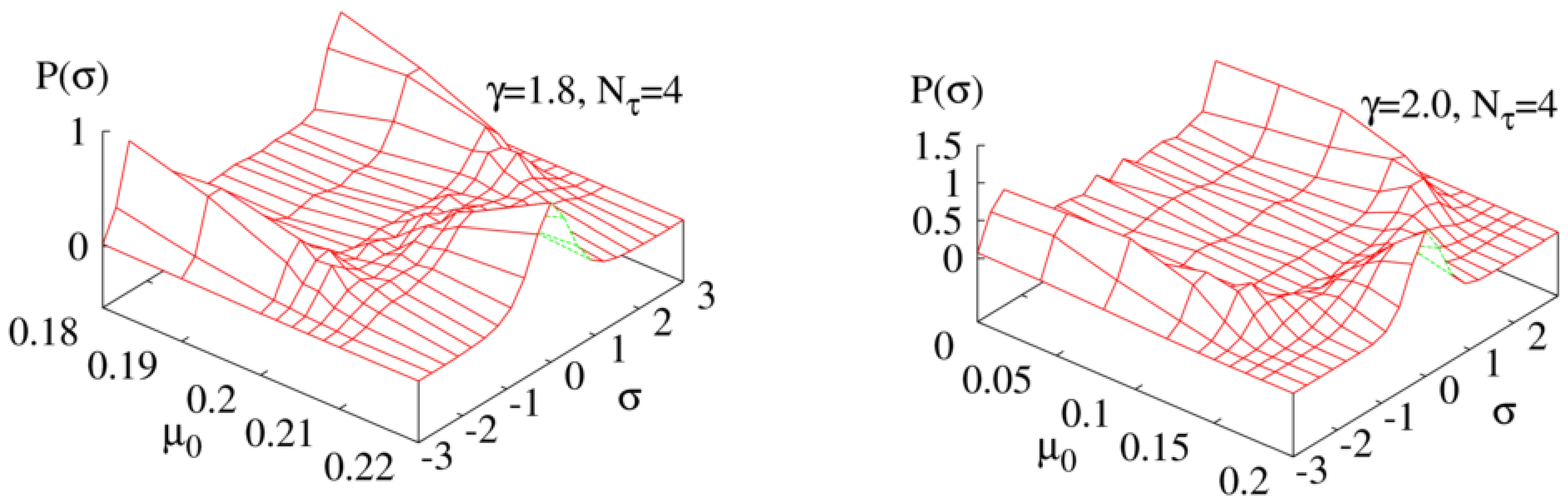




de Forcrand, Fromm ('09)

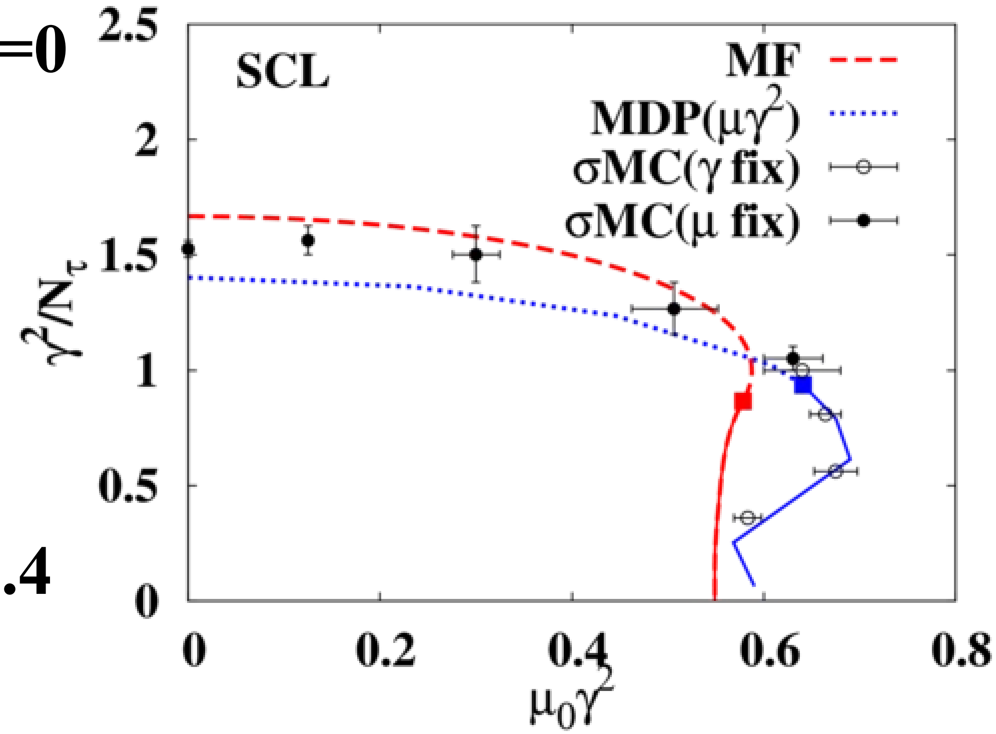
Discussion (1): First or Second order ?

- **First order in small lattice**
  → The peak in the Winger phase grows and overcome NG peak(s) on phase boundary
- **Second order transition**
  → Two peaks merges to be one.
- **γ=1.8 → First order, γ=2.0 → Second order**

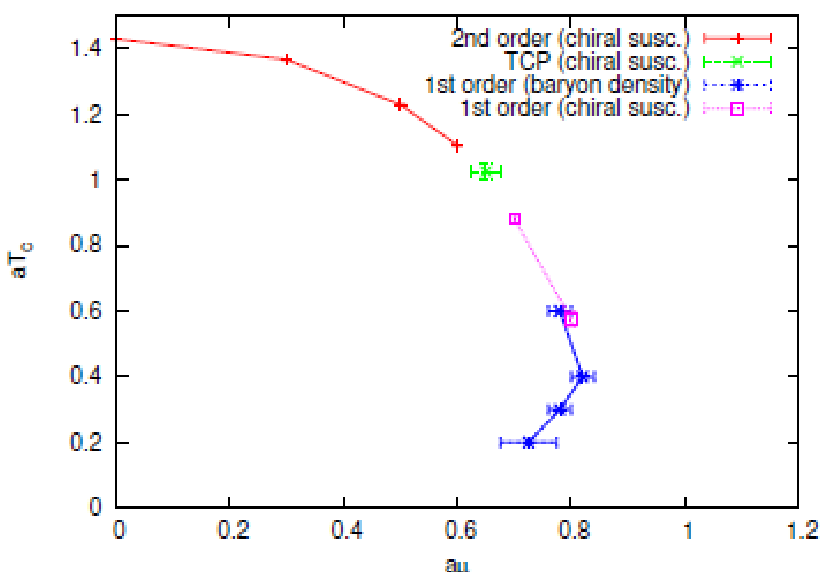


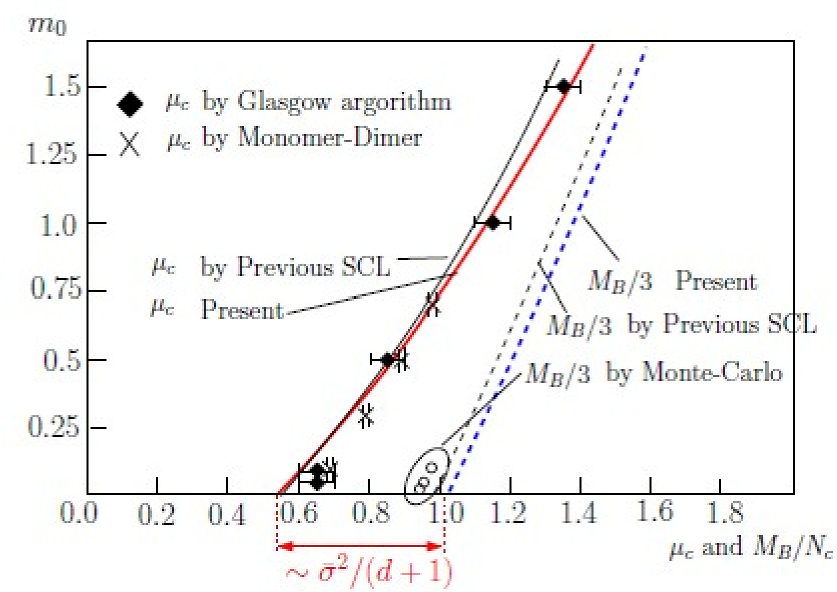
Discussion (2): Comparison with other calc.

- **Transition chem. potential at T=0 ~ 0.6 (Glasgow, MDP)**
  → σMC may underestimate $\mu_c(T=0)$...



- **Continuum time MDP**
  Unger, de Forcrand, QM 2011
  → max. μ_c appears around T=0.4





Summary

- **We have proposed an auxiliary field MC method, abbreviated as σMC method, to simulate the SCL-LQCD quark-U_0 action (LO in strong coupling ($1/g^0$) and 1/d ($1/d^0$) expansion; DKS action) without further approximation.**
  c.f. Determinantal MC by Abe, Seki
- **Sign problem is not easy, <cos θ> ~ a few x 10^{-3} at low *T* on 4^4 lattice, but it is less severe at finite μ.**
- **Phase boundary is moderately modified from MF results by auxiliary field fluctuations, if $\mu = \gamma^2\mu_0$ scaling is adopted.**
- **σMC results are compatible with MDP results, while the shift of T_c at μ=0 is around half.**
- **By-product: X_N deviates from σ=0 MF value in the Wigner phase.
  → "Meson" mass in the Wigner phase ?**

Thank you

Strong Coupling Limit of Lattice QCD

- **Effective Potential** *Fukushima ('04), Nishida ('04)*

$$F_{\rm eff} = \frac{N_c}{d}\bar{\sigma}^2 + V_{\rm eff}(\bar{\sigma}, T, \mu)$$

$$V_{\rm eff} = -T \log\left[\frac{\sinh((N_c+1)E_q/T)}{\sinh(E_q/T)} + 2\cosh(N_c\mu/T)\right] \quad (E_q(m) = \mathrm{arcsinh}\, m)$$

- **Meson propagator**
  - **Meson self-energy comes from the quark determinant, whose derivative (minor det.) is obtained from recursion relation.** *Faldt, Petersson ('86)*

$$G^{-1}(\boldsymbol{k}, \omega) = V_M^{-1}(\boldsymbol{k}) + \mathrm{F.T.}\frac{\partial^2 V_{\rm eff}}{\partial m(\tau)\partial m(\tau')}$$

$$\exp(-V_{\rm eff}/T) = \int dU_0 \begin{vmatrix} I_1 & e^{\mu} & 0 & & & e^{-\mu}U^+ \\ -e^{-\mu} & I_2 & e^{\mu} & & & \\ 0 & -e^{-\mu} & I_3 & e^{\mu} & & \\ \vdots & & & \ddots & & \\ -e^{\mu}U & & & & -e^{-\mu} & I_N \end{vmatrix} \quad (I_k = 2(\sigma_k + m_0))$$

Prescriptions related to lattice staggered fermions

- **Mass = Pole energy of G at "zero" momentum**
  - **"Zero" momentum: $\underline{k}$ = -$\underline{k}$ (vector) → $\underline{k}$ = (0,0,0), (0,0,π), (0, π, 0)**

$$\kappa(\boldsymbol{k}) = \sum_{j=1}^{d} \cos k_j = -3, -1, 1, 3 \quad \text{for zero momentum} \, (\boldsymbol{k} = -\boldsymbol{k})$$

Four different types of meson appear !
(Bound state with doubler)

  - **"Zero" Euclidean energy: ω= -ω → ω = 0 or π**

→ Search for the pole with ($\underline{k}$, ω)= (δ_π, δ_π, δ_π, iM +δ_π) (δ_π =0 or π)

$$G^{-1}(\boldsymbol{k} = '\boldsymbol{0}', \omega = iM + \delta\pi) = \frac{2N_c}{\kappa} + \frac{4N_c}{d} \frac{\bar{\sigma}(\bar{\sigma} + m_0)}{\pm\cosh M + \cosh 2E_q} = 0$$

Hadron Mass in SCL-LQCD (Finite T)

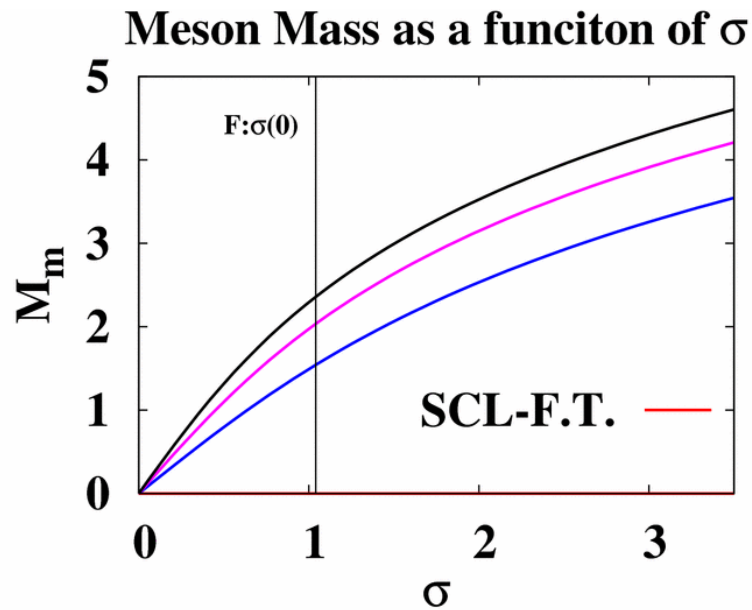
AO, N. Kawamoto, K. Miura, Mod. Phys. Lett. A 23 (2008)2459.

- **Meson Mass**

$$G^{-1}(\boldsymbol{k},\omega)=\frac{2N_c}{\kappa(\boldsymbol{k})}+\frac{4N_c\bar{\sigma}}{d}\frac{\bar{\sigma}+m_0}{\cos\omega+\cosh 2E_q}$$

$$\kappa(\boldsymbol{k})=\sum_{i=1}^{d}\cos k_i \quad\rightarrow\quad \kappa=-d,-d+2,\ldots d$$

$$M=2\,\mathrm{arcsinh}\sqrt{(\bar{\sigma}+m_0)\left(\frac{d+\kappa}{d}\bar{\sigma}+m_0\right)}$$



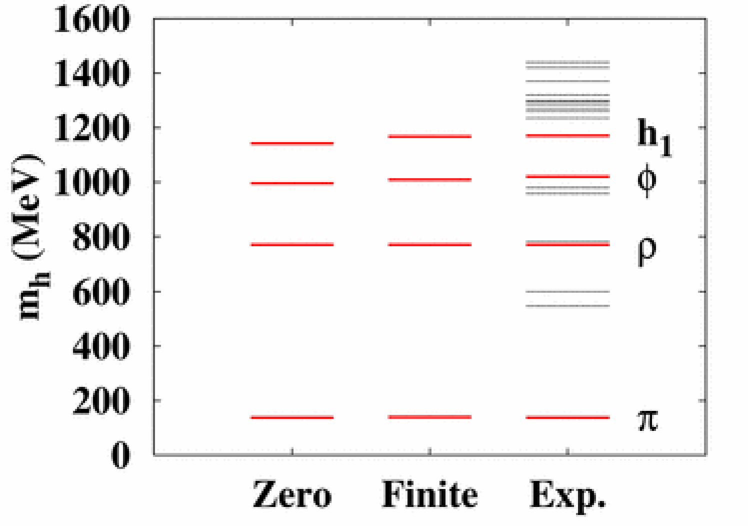
- **Equilibrium condition: $\partial V_{\mathrm{eff}}/\partial\sigma = -2N_c\sigma/d$**
  → Meson masses are determined by the chiral condensate, σ.
- **Chiral condensate is a function of (T, μ).**

→ ***Approximate Brown-Rho scaling emerges in SCL-LQCD***

- **Many eservations: SCL-LQCD, LO in 1/d expansion, staggered fermion, mean field app. (no feed back of fluc.), ...**

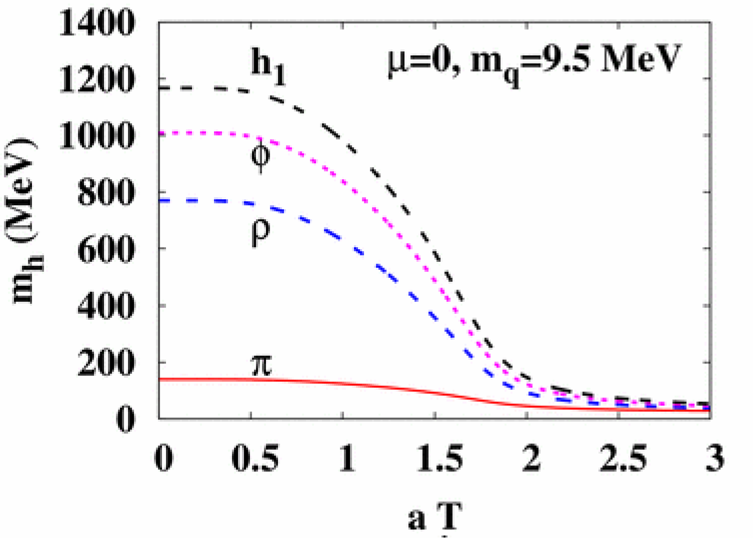
Medium Modification of Meson Masses

- **Scale fixing**
  - **Search for σ_{vac} to minimize free E.**
  - **Assign κ=-3, -1 as π and ρ**
  - **Determine m_0 and a^{-1} (lattice unit) to fit m_π/m_ρ (a=497 MeV)**



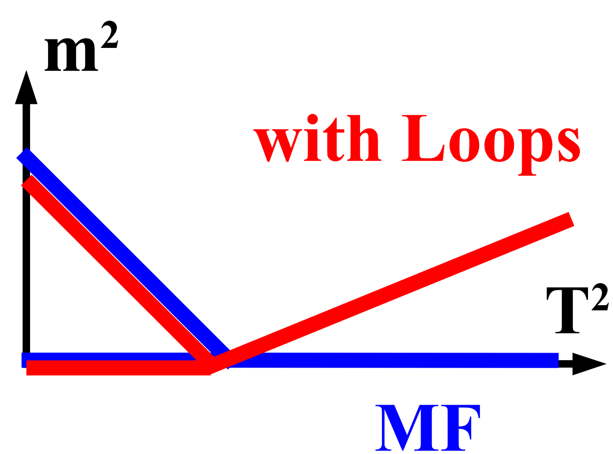
- **Medium modification**
  - **Search for σ(T, μ) → Meson mass**
  - **Vacuum mass ~ Zero T results**

Kluberg-Stern, Morel, Petersson, 1982;
Kawamoto, Shigemoto, 1982



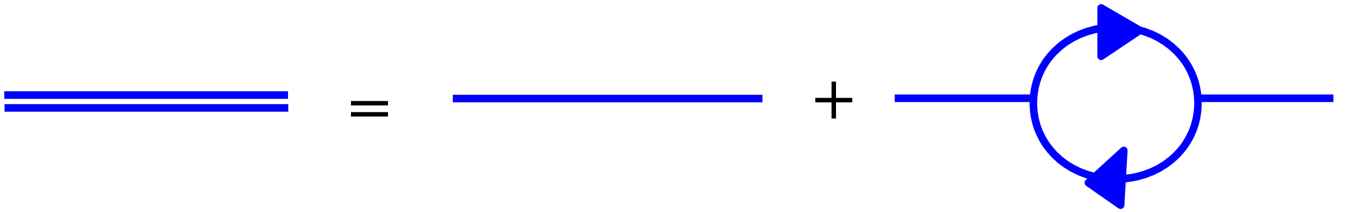
Summary

- Chiral condensates and Polyakov loop at finite T and μ are investigated with SC-LQCD.
  - Partial restoration of χ sym. is expected at finite T and/or μ in SC-LQCD and P-SC-LQCD.
  - Qualitative behavior is similar to NJL and PNJL results.
  - Quantitative differences to be further discussed
    → T_c and μ_c, Density gap at finite μ,Critical point, ...
- Meson masses at finite T and μ are studied in SCL-LQCD.
  - Results with mean field approx. shows Brown-Rho scaling behavior.
  - Loop effects of mesons are expected to enhance meson masses after χ restoration
    Hatsuda, Kunihiro / Kapusta text book
  - Finite coupling effects and self-consistent treatment (SD type) would be interesting.

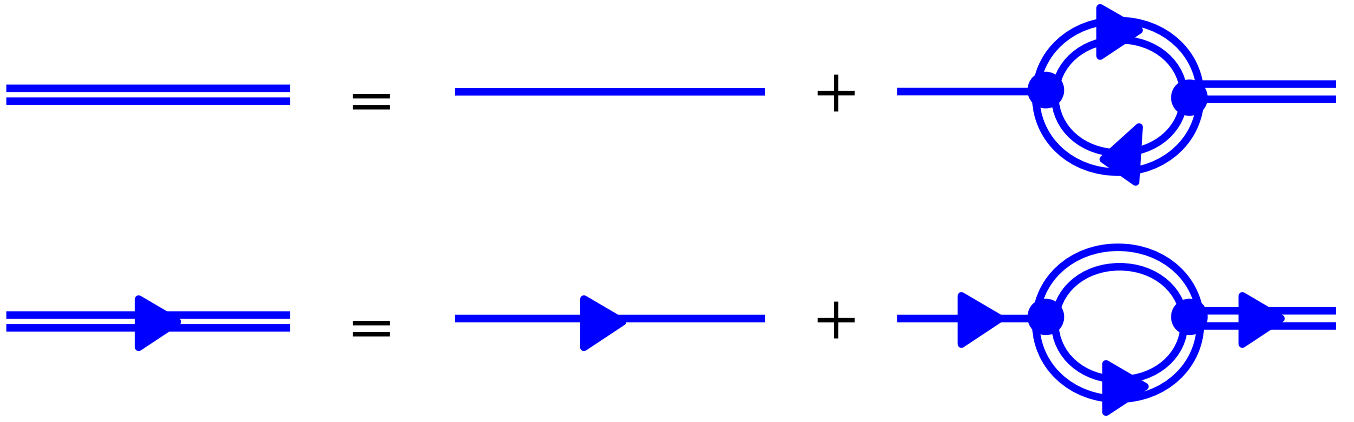


Homework: Can we do it ?

- **Present treatment**



- **Self-consistent treatment**



Is it possible to carry out the self-consistent calculation of meson and quark propagator in SC-LQCD hopefully with NLO/NNLO/PL effects (in two weeks) ?