Auxiliary field Monte-Carlo study of the QCD phase diagram at strong coupling

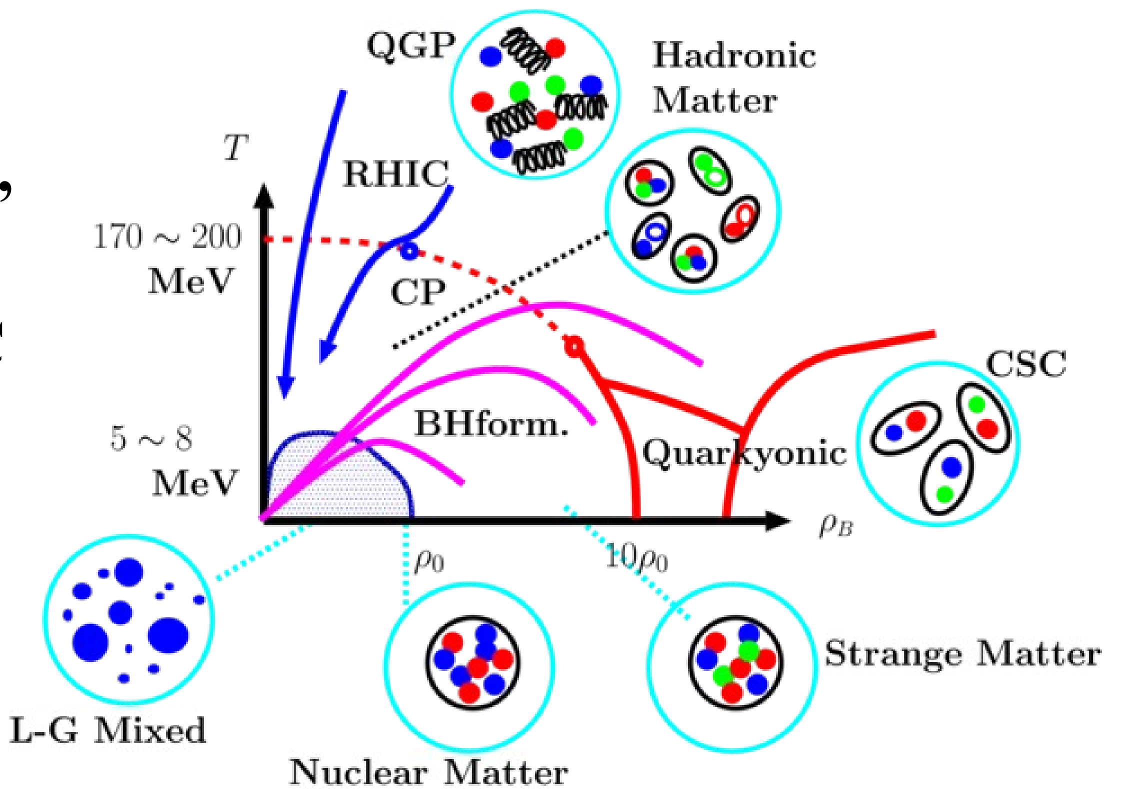
Akira Ohnishi (YITP)
Takashi Z. Nakano (Kyoto U./ YITP)

- **Introduction**
- **Strong coupling lattice QCD (mean field results)**
- **Auxiliary field effective action in the Strong Coupling Limit**
- **Monte-Carlo estimate of the phase boundary**
- **Summary**

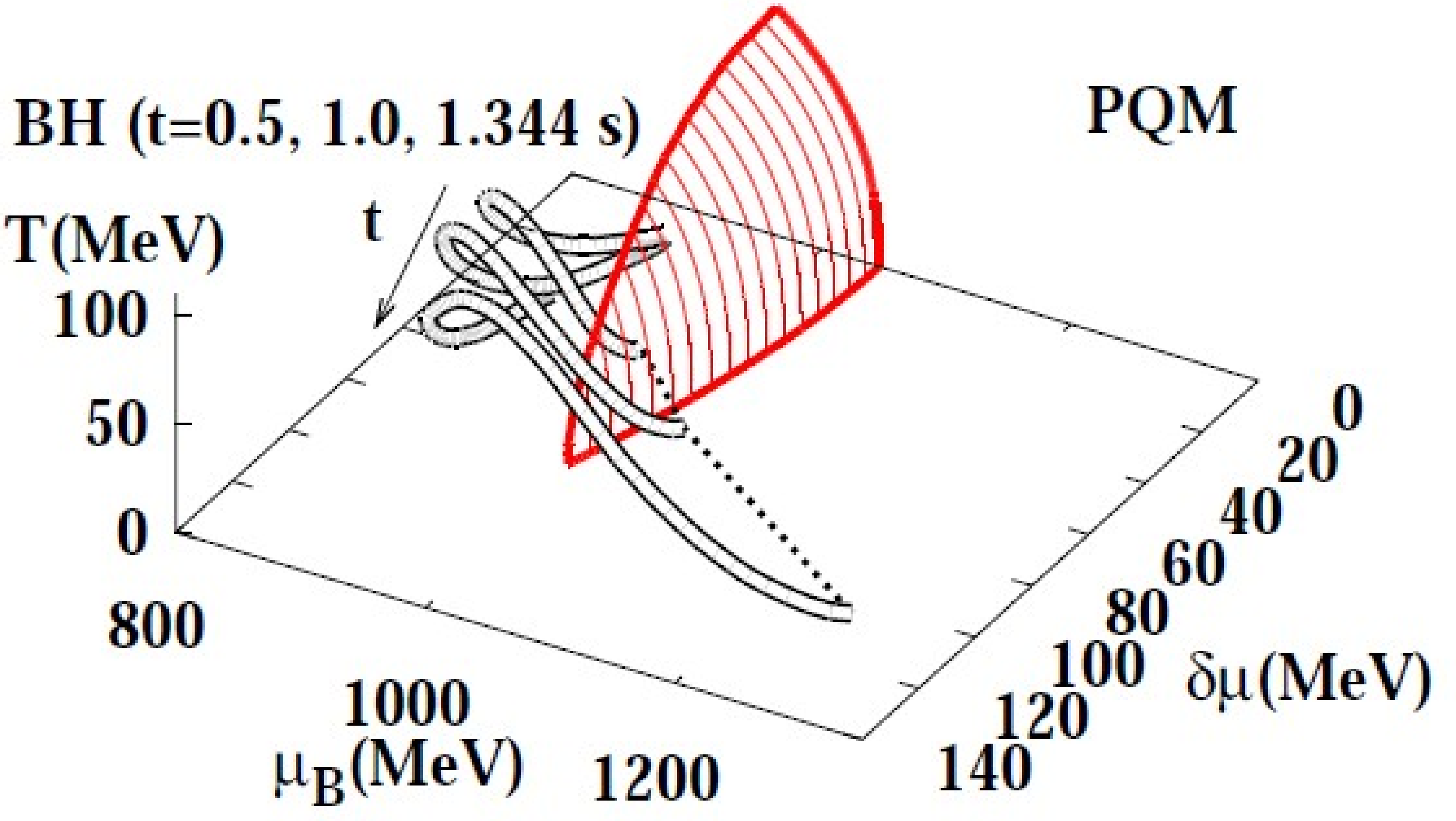
AO, T. Z. Nakano, in prep.

QCD Phase diagram

- Phase transition at high T
  - Physics of early universe: Where do we come from ?
  - RHIC, LHC, Lattice MC, pQCD, ….
- High μ transition
  - Physics of neutron stars: Where do we go ?
  - RHIC-BES, FAIR, J-PARC, Astro-H, Grav. Wave, …
  - Sign problem in Lattice MC
    → Model studies, Approximations, Functional RG, ...



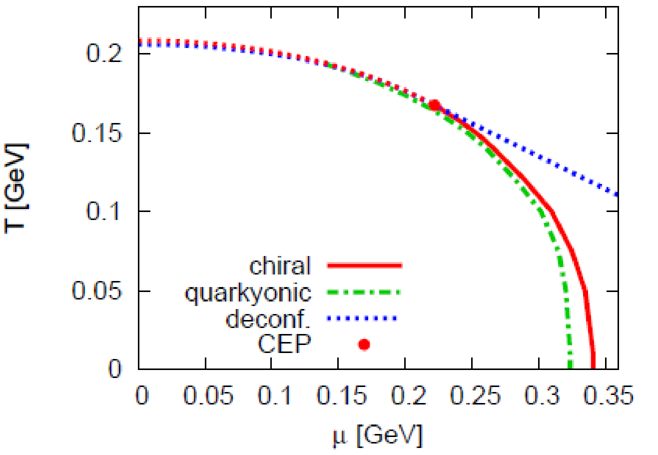
CP sweep during BH formation



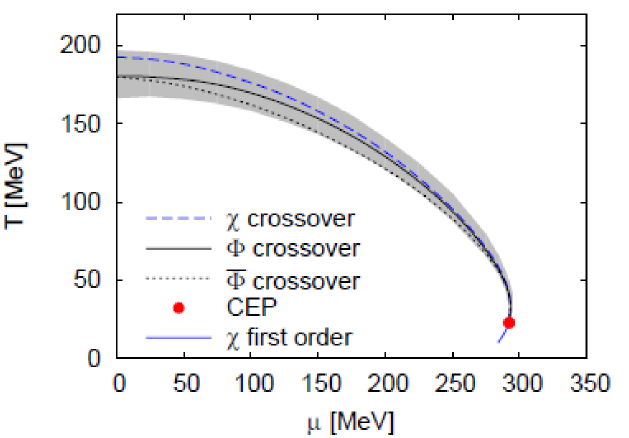
AO, H. Ueda, T. Z. Nakano, M. Ruggieri, K. Sumiyoshi, PLB, to appear [arXiv:1102.3753 [nucl-th]]

QCD based approaches to Cold Dense Matter

- **Effective Models**
  (P)NJL, (P)QM, Random Matrix, …
  E.g.: K.Fukushima, PLB 695('11)387 (PNJL+Stat.).
- **Functional (Exact, Wilsonian) RG**
  E.g.: T. K. Herbst, J. M. Pawlowski, B. J. Schaefer, PLB 696 ('11)58 (PQM-FRG).
- **Expansion / Extrapolation from μ=0**
  - **AC, Taylor expansion, … → μ/T<1**
  - **Cumulant expansion of θ dist. (S. Ejiri, ...)**
- **Strong Coupling Lattice QCD**
  - **Mean field approaches**
  - **Monomer-Dimer-Polymer (MDP) simulation**



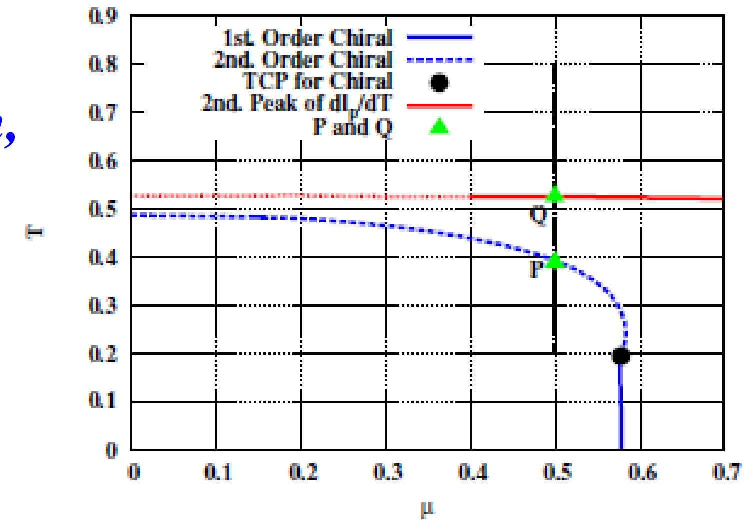
McLerran, Redlich, Sasaki ('09)



Herbst, Pawlowski, Schafer, ('11)

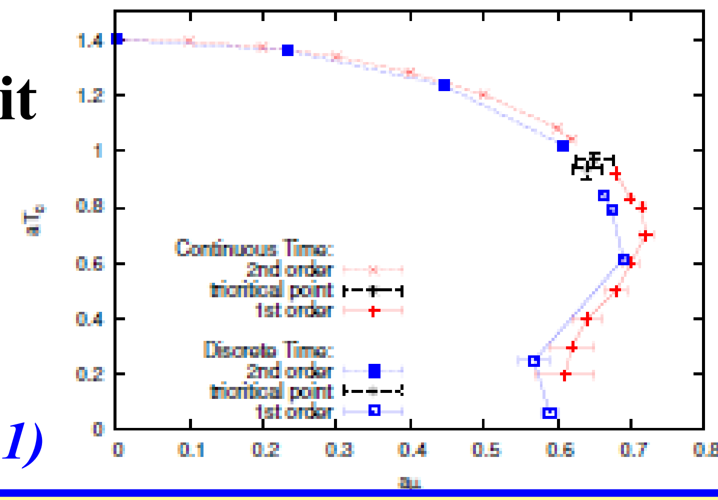
Strong Coupling Lattice QCD for finite μ

- **Mean Field approaches**
  Damagaard, Hochberg, Kawamoto ('85); Bilic, Karsch, Redlich ('92); Fukushima ('03); Nishida ('03); Kawamoto, Miura, AO, Ohnuma ('07).



- **MDP simulation**
  Karsch, Mutter('89); de Forcrand, Fromm ('10); de Forcrand, Unger ('11)
  - **Partition function = sum of config. weights of various loops.**
  - **Applicable only to Strong Coupling Limit ($1/g^2=0$) at present**

Miura, Nakano, AO, Kawamoto, arXiv:1106.1219



de Forcrand, Unger ('11)

Can we include both fluctuation and finite coupling effects ?
→ One of the candidates = Auxiliary field MC

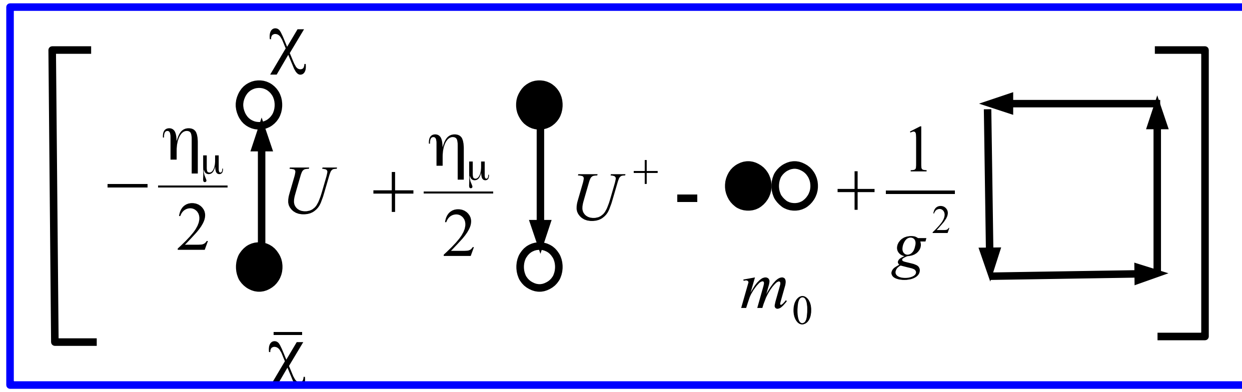
Strong Coupling Lattice QCD

SC-LQCD with Fermions & Polyakov loop: Outline

- **Effective Action & Effective Potential (free energy density)**

$$Z = \int D[\chi, \bar{\chi}, U_0, U_j] \exp$$

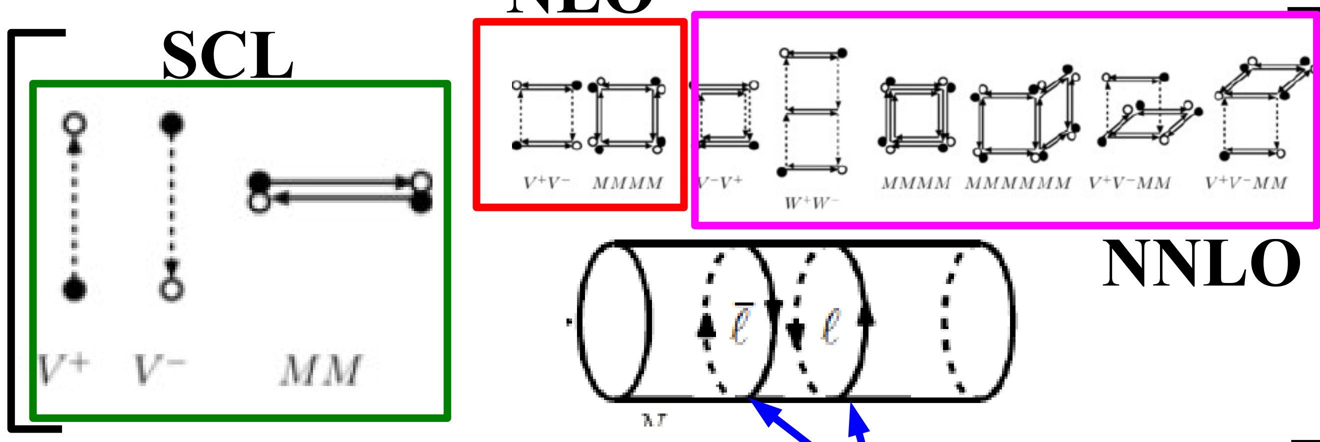
- S_{LQCD}



Spatial link integral

$$\int DU\, U_{ab} U^{+}_{cd} = \delta_{ad}\delta_{bc}/N$$

$$= \int D[\chi, \bar{\chi}, U_0] \exp$$



Bosonization & MF Approx.

$$\approx \int D[\chi, \bar{\chi}, U_0] \exp\left(-S_{\text{eff}}[\chi, \bar{\chi}, U_0, \Phi_{\text{stat.}}]\right)$$

Polyakov loop

Fermion + U_0 integral

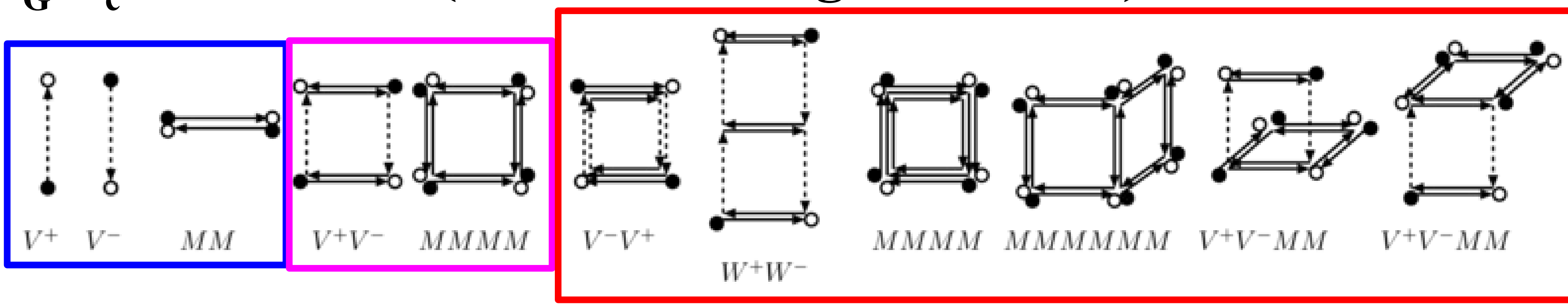
$$\approx \exp\left(-V\, F_{\text{eff}}(\Phi_{\text{stat}}; T, \mu)/T\right)$$

SC-LQCD with Fermions

- **Effective Action with finite coupling corrections**
  Integral of exp(-S_G) over spatial links with exp(-S_F) weight → S_{eff}

$$S_{\rm eff} = S_{\rm SCL} - \log\langle \exp(-S_G)\rangle = S_{\rm SCL} - \sum_{n=1} \frac{(-1)^n}{n!} \langle S_G^n \rangle_c$$

$\langle S_G^n \rangle_c$=Cumulant (connected diagram contr.) *c.f. R.Kubo('62)*



$$S_{\rm eff} = \frac{1}{2}\sum_x (V_x^+ - V_x^-) - \frac{b_\sigma}{2d}\sum_{x,j>0}[MM]_{j,x}$$ *SCL (Kawamoto-Smit, '81)*

$$+\frac{1}{2}\frac{\beta_\tau}{2d}\sum_{x,j>0}[V^+V^- + V^-V^+]_{j,x} - \frac{1}{2}\frac{\beta_s}{d(d-1)}\sum_{x,j>0,k>0,k\neq j}[MMMM]_{jk,x}$$ *NLO (Faldt-Petersson, '86)*

$$-\frac{\beta_{\tau\tau}}{2d}\sum_{x,j>0}[W^+W^- + W^-W^+]_{j,x} - \frac{\beta_{ss}}{4d(d-1)(d-2)}\sum_{\substack{x,j>0,|k|>0,|l|>0\\ |k|\neq j,|l|\neq j,|l|\neq|k|}}[MMMM]_{jk,x}[MM]_{j,x+\hat{l}}$$

$$+\frac{\beta_{\tau s}}{8d(d-1)}\sum_{x,j>0,|k|\neq j}[V^+V^- + V^-V^+]_{j,x}\left([MM]_{j,x+\hat{k}} + [MM]_{j,x+\hat{k}+\hat{0}}\right)$$ *NNLO (Nakano, Miura, AO, '09)*

SC-LQCD Eff. Pot. with Fermions & Polyakov loop

- **Effective potential [free energy density, NLO + LO(Pol. loop)]**

$$\mathcal{F}_{\text{eff}}(\Phi; T, \mu) \equiv -(T \log \mathcal{Z}_{\text{LQCD}})/N_s^d = \mathcal{F}_{\text{eff}}^{\chi} + \mathcal{F}_{\text{eff}}^{\text{Pol}}$$

$$\mathcal{F}_{\text{eff}}^{\chi} \simeq \left(\frac{d}{4N_c} + \beta_s \varphi_s\right)\sigma^2 + \frac{\beta_s \varphi_s^2}{2} + \frac{\beta_\tau}{2}(\varphi_\tau^2 - \omega_\tau^2) - N_c \log Z_\chi$$

$$- N_c E_q - T(\log \mathcal{R}_q(T, \mu) + \log \mathcal{R}_{\bar{q}}(T, \mu))$$

$$\mathcal{R}_q(T, \mu) \equiv 1 + e^{-N_c(E_q - \tilde{\mu})/T} + N_c\left(L_{p,\mathrm{x}} e^{-(E_q - \tilde{\mu})/T} + \bar{L}_{p,\mathrm{x}} e^{-2(E_q - \tilde{\mu})/T}\right)$$

$$\mathcal{F}_{\text{eff}}^{\text{Pol}} \simeq -2T d N_c^2 \left(\frac{1}{g^2 N_c}\right)^{1/T} \bar{\ell}_p \ell_p - T \log \mathcal{M}_{\text{Haar}}(\ell_p, \bar{\ell}_p)$$

aux. fields
w.f. ren.
zero point E.
thermal

quad. coef.
Haar measure

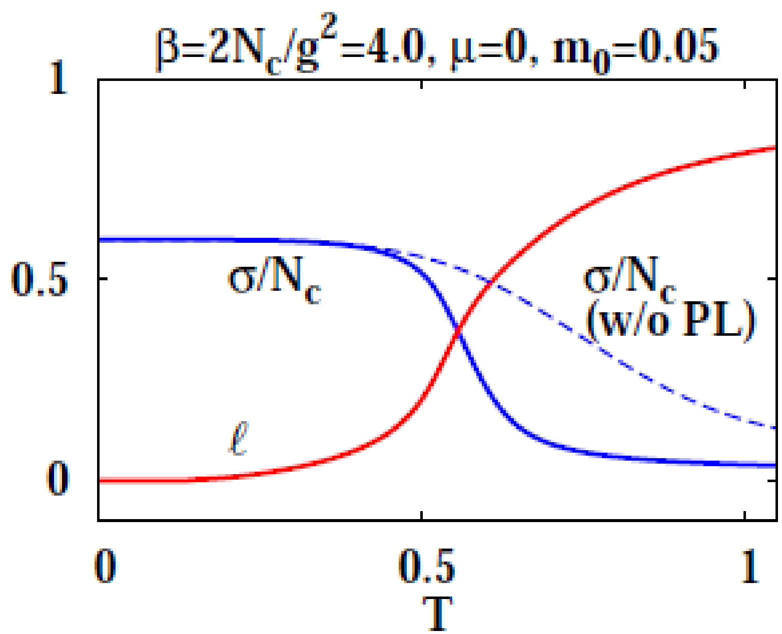
- **Strong coupling lattice QCD with Polyakov loop (P-SC-LQCD)**
  = Polyakov loop extended Nambu-Jona-Lasino (PNJL) model
  (Haar measure method, quadratic term fixed)
  + higher order terms in aux. fields
  - quark momentum integral

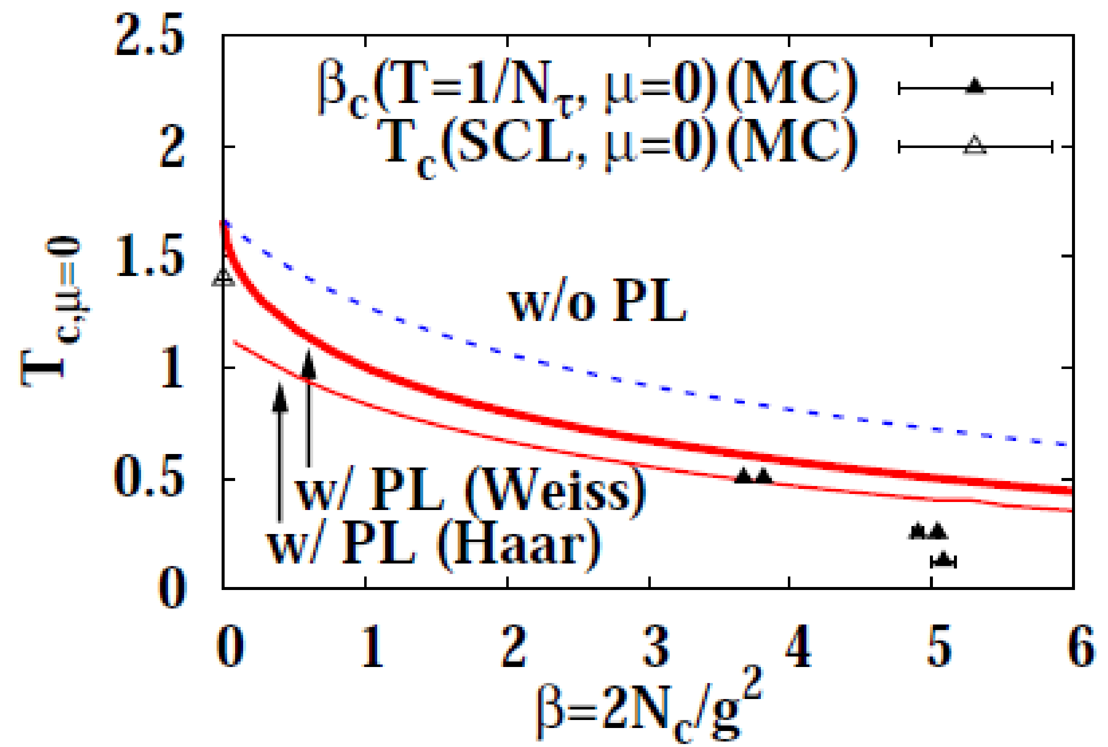
P-SC-LQCD at μ=0

T. Z. Nakano, K. Miura, AO, PRD 83 (2011), 016014 [arXiv:1009.1518 [hep-lat]]

- **P-SC-LQCD reproduces $T_c(\mu=0)$ in the strong coupling region ($\beta = 2N_c/g^2 \leq 4$)**

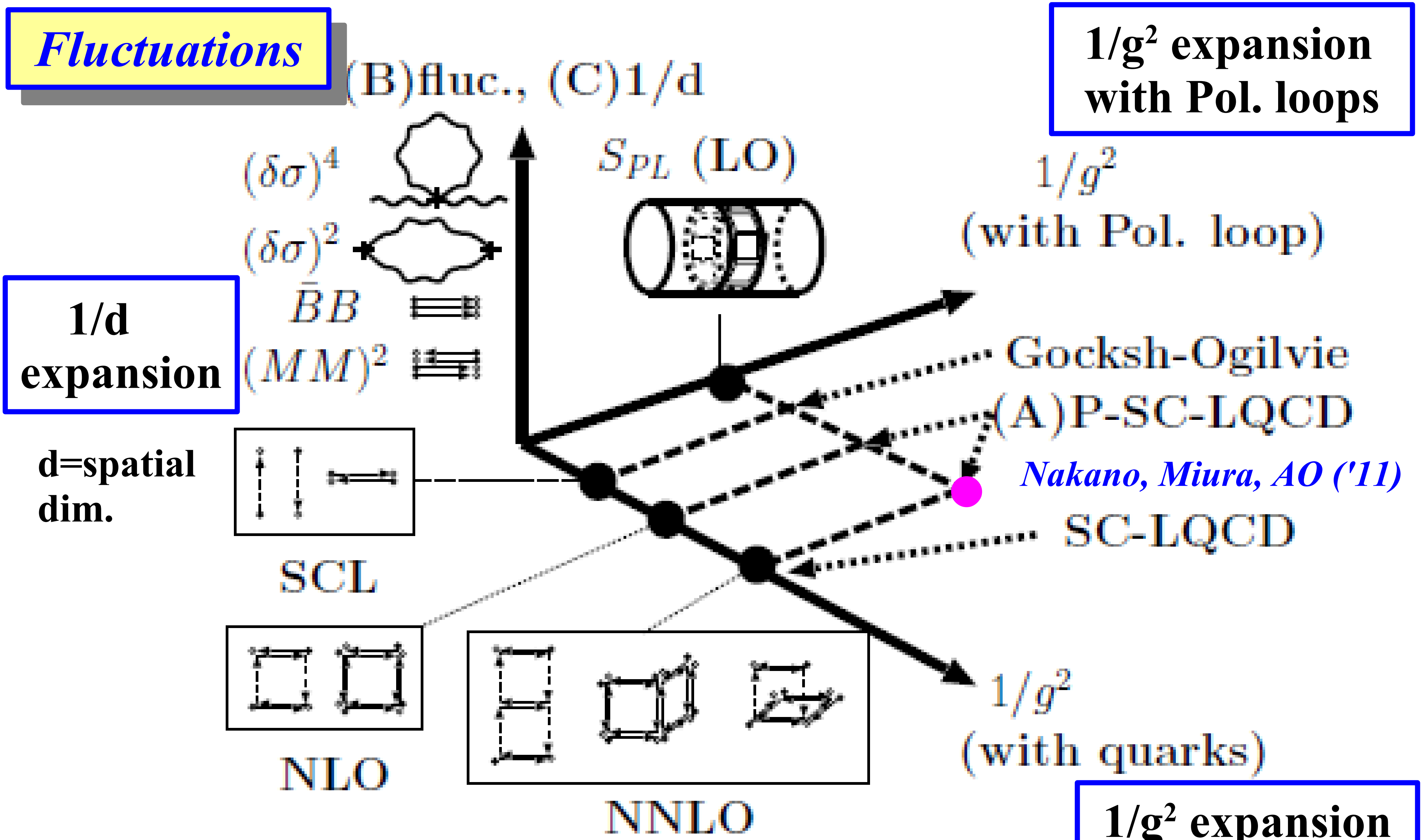
  MC data: SCL (Karsch et al. (MDP), de Forcrand, Fromm (MDP)), N_τ =2 (de Forcrand, private), N_τ=4 (Gottlieb et al.('87), Fodor-Katz ('02)), N_τ =8 (Gavai et al.('90))





Lattice Unit

Approximations in Pol. loop extended SC-LQCD



Approximations in Pol. loop extended SC-LQCD

- **Strong coupling expansion**
  - **Fermion terms: LO($1/g^0$, SCL), NLO($1/g^2$), NNLO ($1/g^4$)**
  - **Plaquette action: LO ($1/g^{2N\tau}$)**
- **Large dimensional approximation**
  - **1/d expansion (d=spatial dim.)**
    → Smaller quark # configs. are preferred.
    $\Sigma_j M_x M_{x+j} = O(1/d^0) \to M \propto d^{-1/2} \to \chi \propto d^{-1/4}$
  - **Only LO ($1/d^0$) terms are mainly evaluated.**
- **(Unrooted) staggered Fermion**
  - **Nf=4 in the continuum limit.**
- **Mean field approximation**
  - **Auxiliary fields are assumed to be constant.**

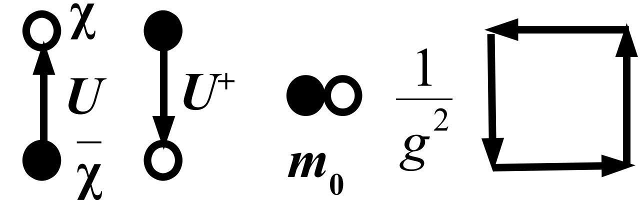
⇒ ***This work: Auxiliary Field MC in SCL***

Auxiliary field effective action in the Strong Coupling Limit

Strong Coupling Expansion

- **Lattice QCD action (aniso. lattice, unrooted staggered Fermion)**

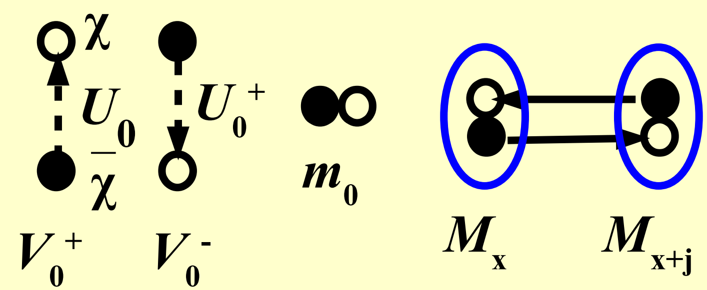
$$S_{LQCD}=S_F+S_G \quad S_G=-\frac{1}{g^2}\sum_{plaq.}\mathrm{tr}\left[U_P+U_P^{+}\right]f_P$$



$$S_F=\frac{1}{2}\sum_x\left[V^{+}(x)-V^{-}(x)\right]+\frac{1}{2\gamma}\sum_{x,j}\eta_j(x)\left[\bar{\chi}_x U_{x,j}\chi_{x+\hat{j}}-\bar{\chi}_{x+\hat{j}}U_{x,j}^{+}\chi_x\right]+\sum_x\frac{m_0}{\gamma}M_x$$

$$V^{+}(x)=e^{\mu/\gamma^2}\bar{\chi}_x U_{x,0}\chi_{x+\hat{0}},\quad V^{-}(x)=e^{-\mu/\gamma^2}\bar{\chi}_{x+\hat{0}}U_{x,0}^{+}\chi_x,\quad M_x=\bar{\chi}_x\chi_x,\quad a_\tau=a/\gamma,\quad f_P=1\ \text{or}\ 1/\gamma$$

- **Strong coupling expansion (Strong coupling limit)**
  - **Ignore plaquette action ($1/g^2$)**
  - **Integrate out spatial link variables of min. quark number diagrams (1/d expansion)**



$$\int dU\, U_{ab}U_{cd}^{+}=\delta_{ad}\delta_{bc}/N_c$$

$$S_{\text{eff}}=\frac{1}{2}\sum_x\left[V^{+}(x)-V^{-}(x)\right]-\frac{1}{4N_c\gamma^2}\sum_{x,j}M_x M_{x+\hat{j}}+\frac{m_0}{\gamma}\sum_x M_x$$

Introduction of Auxiliary Fields

- **Bosonization of *MM* term (Four Fermi (two-body) interaction)**

$$S_F^{(s)} = -\alpha \sum_{j,x} M_x M_{x+\hat{j}} = -\alpha \sum_{x,y} M_x V_{x,y} M_y \quad \left[V_{x,y} = \frac{1}{2} \sum_j \left(\delta_{x+\hat{j}\,y} + \delta_{x-\hat{j},y} \right) \right]$$

- **Meson matrix (*V*) has positive and negative eigen values**

$$f_M(\boldsymbol{k}) = \sum_j \cos k_j \quad , \quad f_M(\bar{\boldsymbol{k}}) = -f_M(\boldsymbol{k}) \quad \left[\bar{\boldsymbol{k}} = \boldsymbol{k} + (\pi, \pi, \pi) \right]$$

- **Negative mode = "High" momentum mode**
  → Involves a factor exp(i π (x_1+x_2+x_3)) = (-1)(x_1+x_2+x_3)**
  in coordinate representation

- **Bosonization of Negative mode: Extended HS transf.**
  → Introducing " *i* " gives rise to the sign problem.
  Miura, Nakano, AO (09), Miura, Nakano, AO, Kawamoto (09)

$$\exp(\alpha A B) = \int d\varphi\, d\phi \exp\left[-\alpha \left(\varphi^2 - (A+B)\varphi + \phi^2 - i(A-B)\phi \right) \right]$$
$$\approx \exp\left[-\alpha \left(\bar{\psi}\psi - A\psi - \bar{\psi} B \right) \right]_{\text{stationary}}$$

Negative

Introduction of Auxiliary Fields

$$S^{(s)} = -\frac{1}{4N_c\gamma^2}\sum_{x,j} M_x M_{x+\hat{j}} = -\frac{L^3}{4N_c\gamma^2}\sum_{\tau,\mathbf{k}} f_M(\mathbf{k})\,\tilde{M}_{\mathbf{k}}(\tau)\,\tilde{M}_{-\mathbf{k}}(\tau)$$

$$= \frac{L^3}{4N_c\gamma^2}\sum_{\tau,\mathbf{k},f_M(\mathbf{k})>0} f_M(\mathbf{k})\Big[\varphi_{\mathbf{k}}(\tau)^2 + \phi_{\mathbf{k}}(\tau)^2 + \varphi_{\mathbf{k}}(\tilde{M}_{\mathbf{k}} + \tilde{M}_{-\mathbf{k}}) - i\phi_{\mathbf{k}}(\tilde{M}_{\mathbf{k}} - \tilde{M}_{-\mathbf{k}})$$

$$+ \varphi_{\bar{\mathbf{k}}}(\tau)^2 + \phi_{\bar{\mathbf{k}}}(\tau)^2 + i\varphi_{\bar{\mathbf{k}}}(\tilde{M}_{\bar{\mathbf{k}}} + \tilde{M}_{-\bar{\mathbf{k}}}) + \phi_{\bar{\mathbf{k}}}(\tilde{M}_{\bar{\mathbf{k}}} - \tilde{M}_{-\bar{\mathbf{k}}})\Big]$$

$$= \frac{\Omega}{2N_c\gamma^2}\sum_{k,f_M(\mathbf{k})>0} f_M(\mathbf{k})\left[\sigma_k^*\sigma_k + \pi_k^*\pi_k\right] + \frac{1}{2N_c\gamma^2}\sum_x M_x\left[\sigma(x) + i\varepsilon(x)\pi(x)\right]$$

$$\Omega = L^3 N_\tau$$

$$\sigma(x) = \sum_{k,f_M(\mathbf{k})>0} f_M(\mathbf{k})e^{ikx}\sigma_k\ ,\quad \pi(x) = \sum_{k,f_M(\mathbf{k})>0} f_M(\mathbf{k})e^{ikx}\pi_k$$

$$\sigma_k = \varphi_k + i\phi_k\ ,\ \pi_k = \varphi_{\bar{k}} + i\phi_{\bar{k}}$$

$$V_{x,y} = \frac{1}{2}\sum_j\left(\delta_{x+\hat{j},y} + \delta_{x-\hat{j},y}\right)\ ,\quad f_M(\mathbf{k}) = \sum_j \cos k_j\ ,\quad \bar{\mathbf{k}} = \mathbf{k} + (\pi,\pi,\pi)$$

Phase cancellation mechanism in σMC

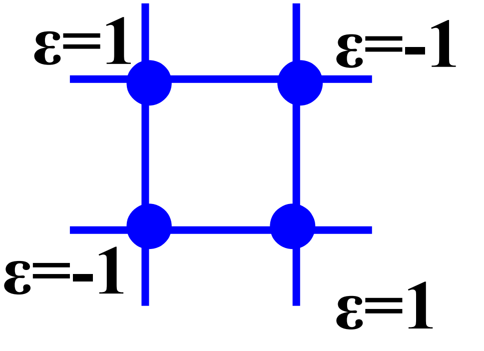
- **Bosonized effective action**

$$S_{\rm eff}=\frac{1}{2}\sum_{x,y}\bar{\chi}_x D_{x,y}\chi_y+\frac{\Omega}{4N_c\gamma^2}\sum_{k,f_M(\boldsymbol{k})>0} f_M(\boldsymbol{k})\left[\sigma_k^*\sigma_k+\pi_k^*\pi_k\right]$$

$$D_{x,y}=\delta_{x+\hat{0},y}\delta_{\boldsymbol{x},\boldsymbol{y}}e^{\mu/\gamma^2}U_{x,0}-\delta_{x,y+\hat{0}}\delta_{\boldsymbol{x},\boldsymbol{y}}e^{-\mu/\gamma^2}U^{+}_{y,0}+2\left[\Sigma_x+\frac{m_0}{\gamma}\right]\delta_{x,y}\quad,\quad \Sigma_x=\frac{\sigma_x+i\varepsilon_x\pi_x}{2N_c\gamma^2}$$

$$\sigma(x)=\sum_{k,f_M(\boldsymbol{k})>0} f_M(\boldsymbol{k})e^{ikx}\sigma_k,\quad \pi(x)=\sum_{k,f_M(\boldsymbol{k})>0} f_M(\boldsymbol{k})e^{ikx}\pi_k$$

- **Fermion matrix is spatially separated**
  → Fermion det at each point
- **Imaginary part (π) involves**
  $\varepsilon=(-1)^{x0+x1+x2+x3}=\exp(i\,\pi(x_0+x_1+x_2+x_3))$
  → Phase cancellation of nearest neighbor spatial site det for π field having low k



Fermion Determinant

Faldt, Petersson, 1986

- **Fermion action is separated to each spatial point and bi-linear → Determinant of Nτ x Nc matrix**

$$\exp(-V_{\mathrm{eff}}/T)=\int dU_0 \begin{vmatrix} I_1 & e^{\mu} & 0 & & e^{-\mu}U^{+} \\ -e^{-\mu} & I_2 & e^{\mu} & & \\ 0 & -e^{-\mu} & I_3 & e^{\mu} & \\ \vdots & & & \ddots & \\ -e^{\mu}U & & & -e^{-\mu} & I_N \end{vmatrix}$$

Nc x Nτ

$$=\int dU_0 \det\left[X_N[\sigma]\otimes \mathbf{1}_c + e^{-\mu/T}U^{+} + (-1)^{N_\tau} e^{\mu/T} U\right]$$

Nc

$$=X_N^3-2X_N+2\cosh(3N_\tau\mu)$$

$$I_\tau/2=[\sigma(x)+i\varepsilon(x)\pi(x)]/2N_c\gamma^2+m_0/\gamma$$

$$X_N=B_N+B_{N-2}(2;N-1)$$

$$B_N=I_N B_{N-1}+B_{N-2}$$

$$B_N=\begin{vmatrix} I_1 & e^{\mu} & 0 & & \\ -e^{-\mu} & I_2 & e^{\mu} & & \\ 0 & -e^{-\mu} & I_3 & e^{\mu} & \\ \vdots & & & \ddots & \\ & & & -e^{-\mu} & I_N \end{vmatrix}$$

Auxiliary Field Monte-Carlo Integral

- **Effective action of Auxiliary Fields**

$$S_{\mathrm{eff}} = \frac{\Omega}{4 N_c \gamma^2} \sum_{k, f_M(\boldsymbol{k})>0} f_M(\boldsymbol{k}) \left[\sigma_k^* \sigma_k + \pi_k^* \pi_k \right]$$
$$- \sum_{\boldsymbol{x}} \log \left[X_N(\boldsymbol{x})^3 - 2 X_N(\boldsymbol{x}) + 2 \cosh(3 N_\tau \mu) \right]$$
$$X_N(\boldsymbol{x}) = X_N[\sigma(\boldsymbol{x}, \tau), \pi(\boldsymbol{x}, \tau)]$$

  - **μ dependence appears only in the log.**
  - **σ_k, π_k have to be generated in momentum space, while X_N requires σ(x) and π(x) → Fourier transf. in each step.**
  - **X_N is complex, and this action has the sign problem. But the sign problem is milder because of the phase cancellation and is less severe at larger μ.**

Let's try at finite μ !

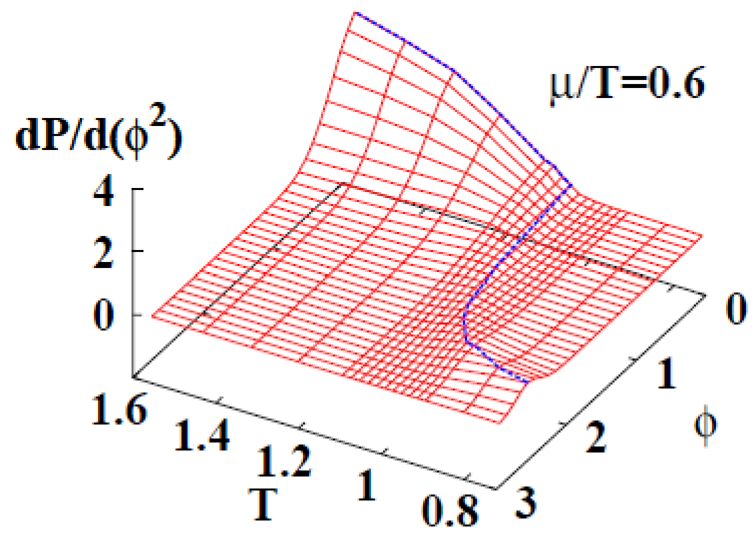
Auxiliary Field Monte-Carlo (σMC) estimate of the phase boundary

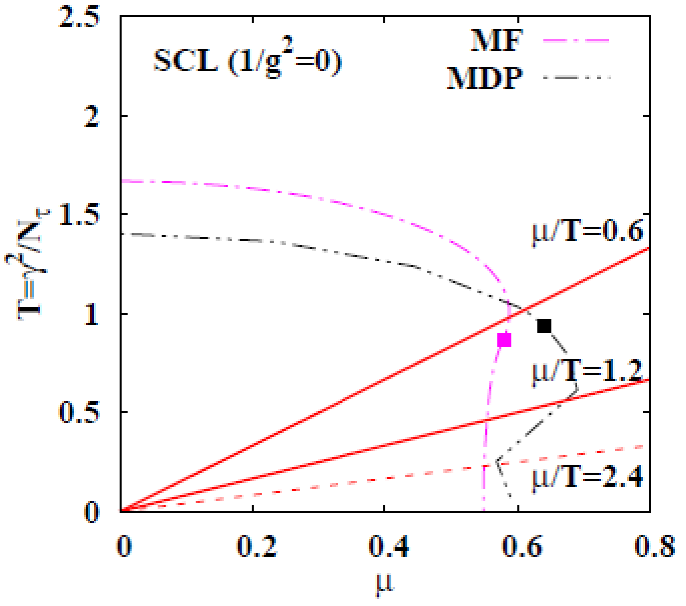
Numerical Calculation

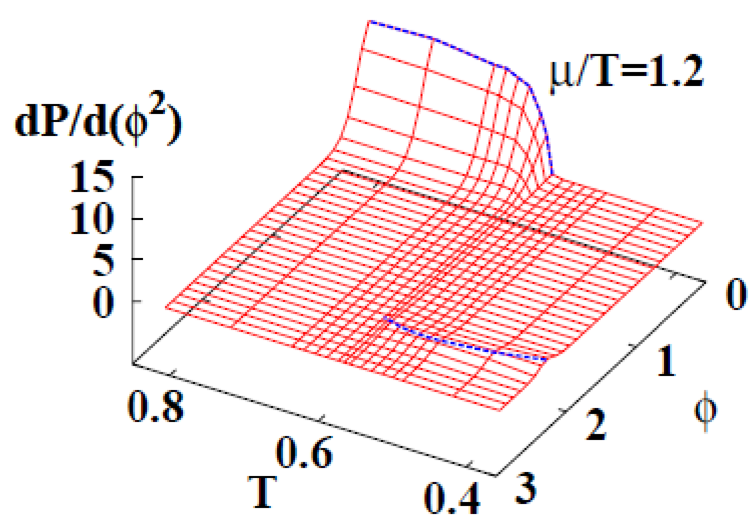
- **4^4 asymmetric lattice + Metropolis sampling of σ_k and π_k.**
- **Metropolis sampling of full configuration (σ_k and π_k) at a time. (efficient for small lattice)**
- **Initial cond. = const. σ**
- **Chiral limit (m=0) simulation → Symmetry in $\sigma \leftrightarrow -\sigma$**
- **Sign problem is not severe ($\langle \cos\theta \rangle \sim (0.9\text{-}1.0)$) in a 4^4 lattice.**
- **Computer: My PC (Core i7)**

Results (1): σ distribution

- **Fixed μ/T simulation: μ/T= 0 ~ 2.4**
- **Low μ region: Second order (Single peak: finite σ → zero)**
- **High μ region: First order (Dist. func. has two peaks)**







Results (2): Susceptibility and Quark density

- **Weight factor <cos θ>**

$$\langle \cos\theta \rangle = Z / Z_{abs}$$

$$Z = \int D\sigma_k D\pi_k \exp(-S_{eff})$$

$$= \int D\sigma_k D\pi_k \exp(-\mathrm{Re}\, S_{eff}) e^{i\theta}$$

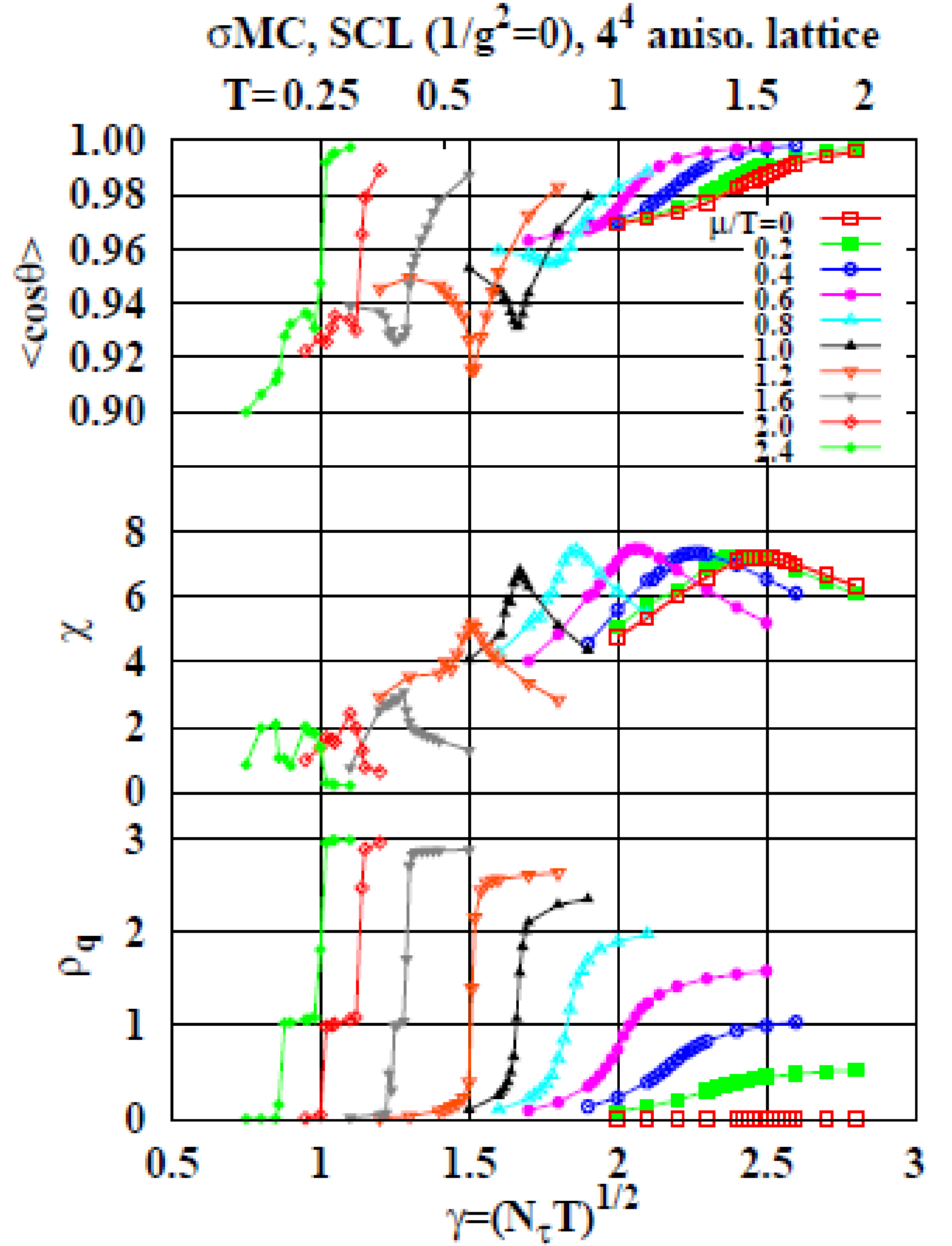
$$Z_{abs} = \int D\sigma_k \pi_k \exp(-\mathrm{Re}\, S_{eff})$$

- **Chiral susceptibility**

$$\chi = -\frac{1}{L^3} \frac{\partial^2 \log Z}{\partial m_0^2}$$

- **Quark number density**

$$\rho_q = -\frac{1}{L^3} \frac{\gamma^2}{N_\tau} \frac{\partial \log Z}{\partial \mu}$$

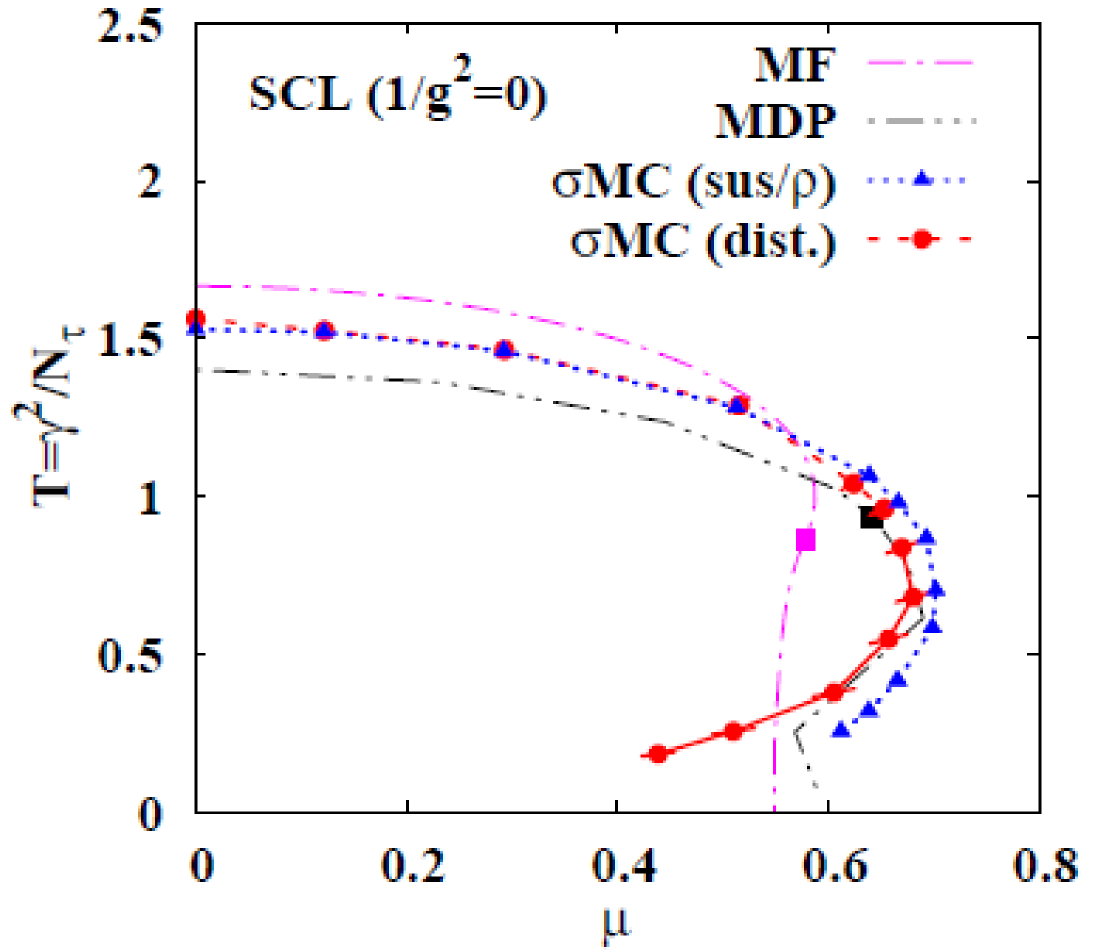


Results (3): Phase diagram

- By taking $T = \gamma^2/N_\tau$, γ dep. of the phase boundary becomes small. *Bilic et al. ('92)*
- Definitions of phase boundar
  - $\phi^2 = \sigma^2 + \pi^2$ dist. peak: finite or zero (red curve)
  - Chiral susceptibility peak (blue)
- Fluctuation effect
  - Reduction of T_c at μ=0
  - Enlarged hadron phase at medium T

→ Consistent with MDP

de Forcrand, Fromm ('09); de Forcrand, Unger ('11)



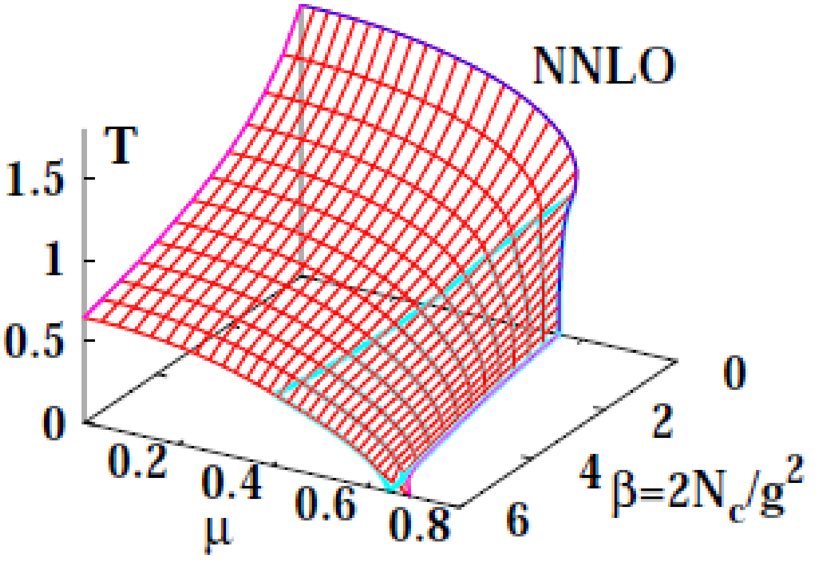
Clausius-Clapeyron Relation

- **First order phase boundary → two phases coexist**

$$P_h = P_q \quad \rightarrow \quad dP_h = dP_q \quad \rightarrow \quad \frac{d\mu}{dT} = -\frac{s_q - s_h}{\rho_q - \rho_h}$$

$$dP_h = \rho_h d\mu + s_h dT, \quad dP_q = \rho_q d\mu + s_q dT$$

- **Continuum theory**
  → Quark matter has larger entropy and density (dμ /dT < 0)
- **Strong coupling lattice**
  - **SCL: Quark density is larger than half-filling, and “Quark hole” carries entropy → dμ/dT > 0**
  - **NLO, NNLO → dμ/dT < 0**



AO, Miura, Nakano, Kawamoto ('09)

Summary

- **We have proposed an auxiliary field MC method (σMC), to simulate the SCL quark-U_0 action (LO in strong coupling ($1/g^0$) and 1/d ($1/d^0$) expansion) without further approximation.**
  c.f. Determinantal MC by Abe, Seki
- **Sign problem is mild in small lattice (<cos θ> ~ (0.9-1) for 4^4), because of the phase cancellation coming from nearest neighbor interaction.**
- **Phase boundary is moderately modified from MF results by fluctuations, if $T = \gamma^2/N_\tau$ and $\mu = \gamma^2\mu_0$ scaling is adopted.**
- **σMC results are compatible with MDP results, while the shift of T_c at μ=0 is around half (LO in 1/d expansion in σMC).**
- **Extension to NLO SC-LQCD is straightforward.**
- **To do: Larger lattice, finite coupling, different Fermion, higher 1/d terms including baryonic action and chiral Polyakov coupling.**

Thank you