# *Auxiliary field Monte-Carlo study of the QCD phase diagram at strong coupling*

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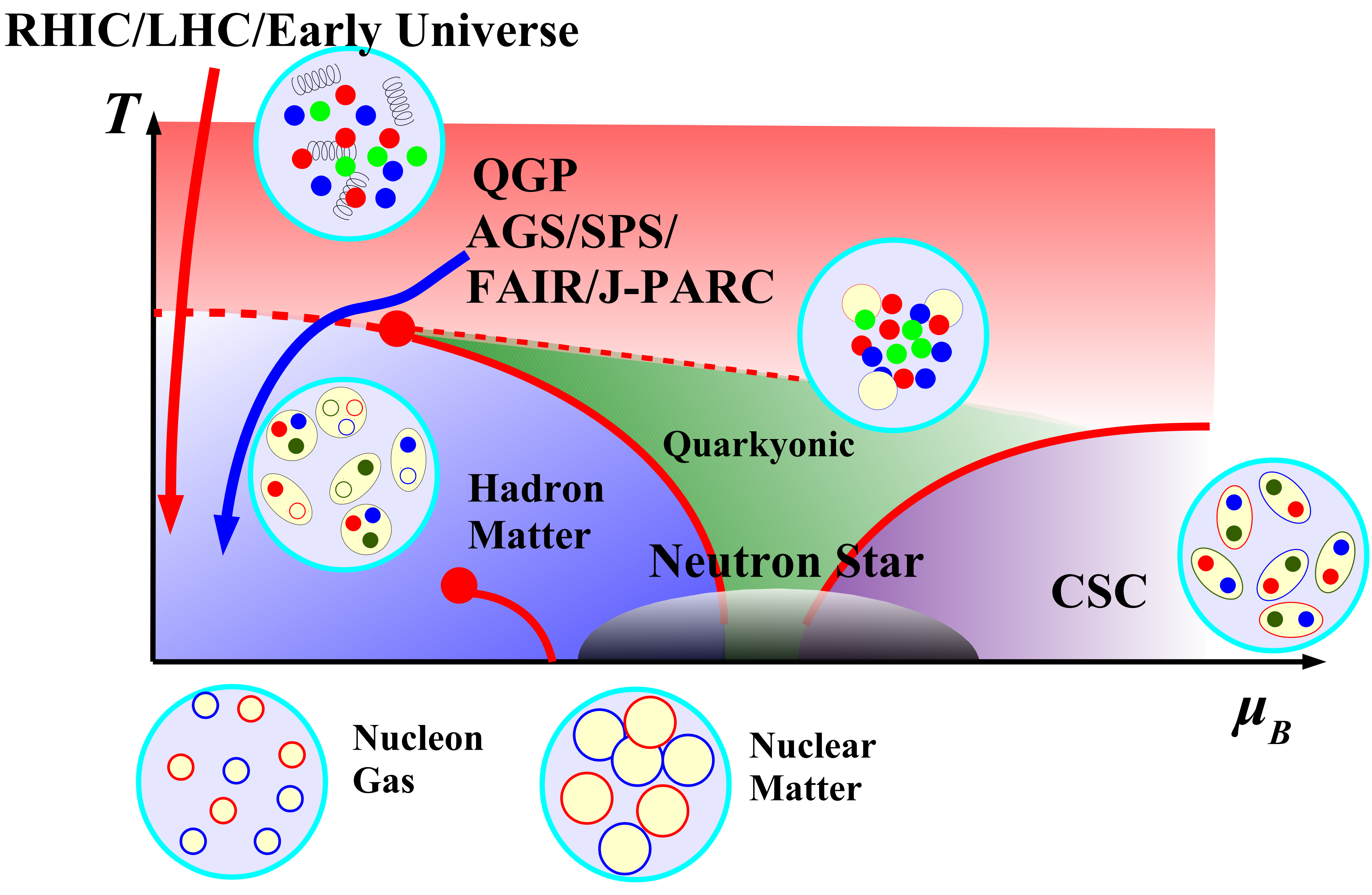
**in collaboration with**

**T.Z.Nakano (YITP/Kyoto U.), T. Ichihara (Kyoto U.)**

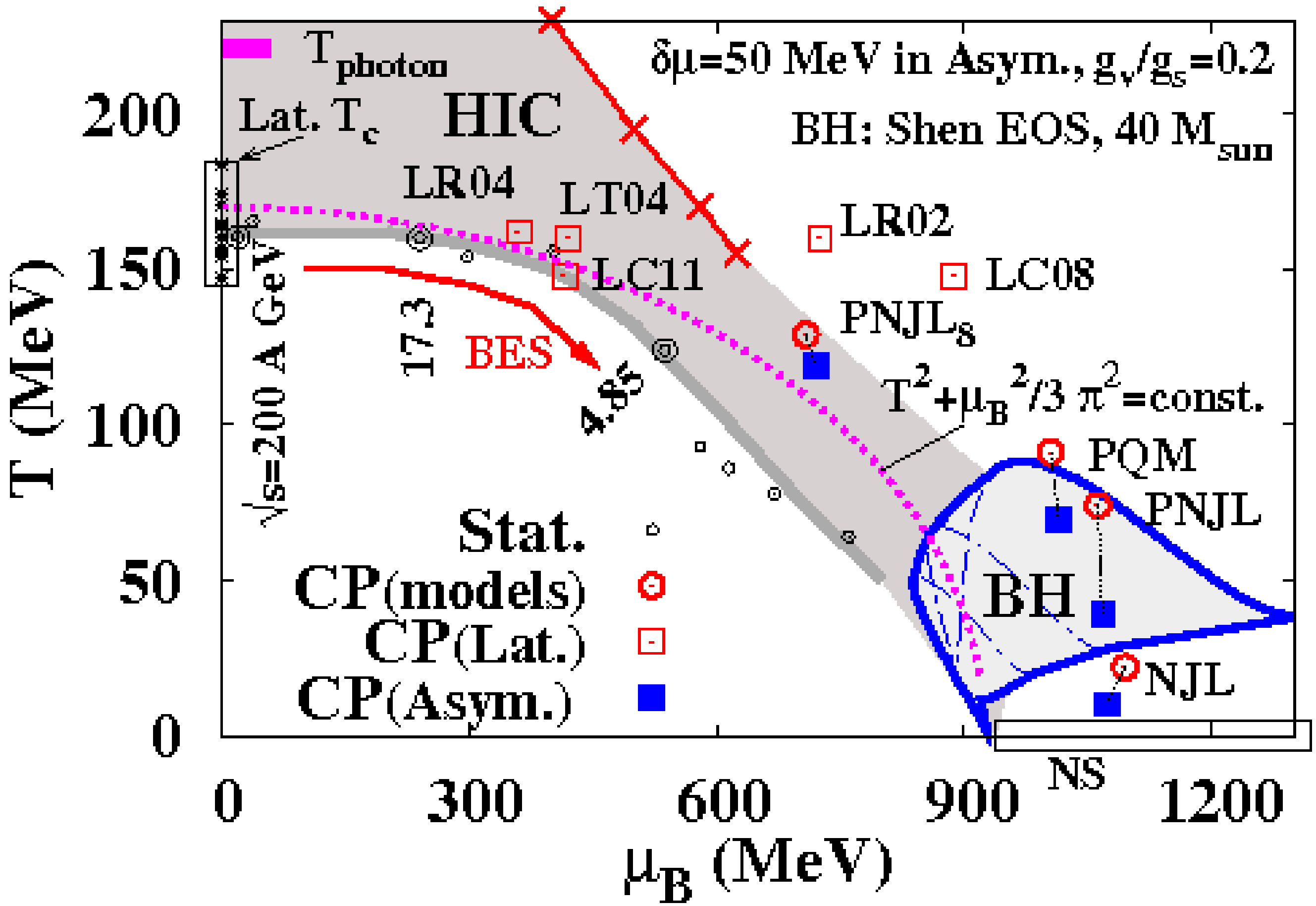
- **Introduction**
- **Auxiliary Field Monte-Carlo treatment of SC-LQCD**
- **Monte-Carlo estimate of the phase boundary**
- **Summary**

***Work in progress***

# *QCD Phase Diagram*



# *QCD phase diagram (Exp. & Theor. Studies)*

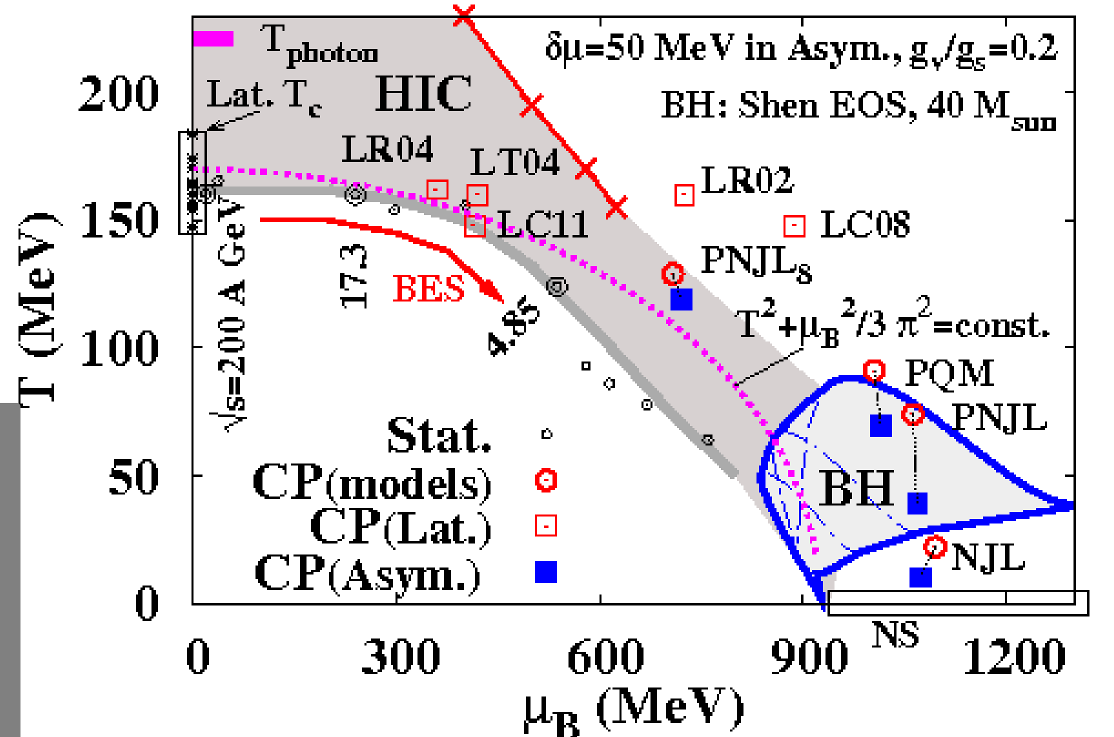



*QCD phase transition is not only an academic problem, but also a subject which would be measured in HIC or Compact Stars*

# *QCD phase diagram (Exp. & Theor. Studies)*

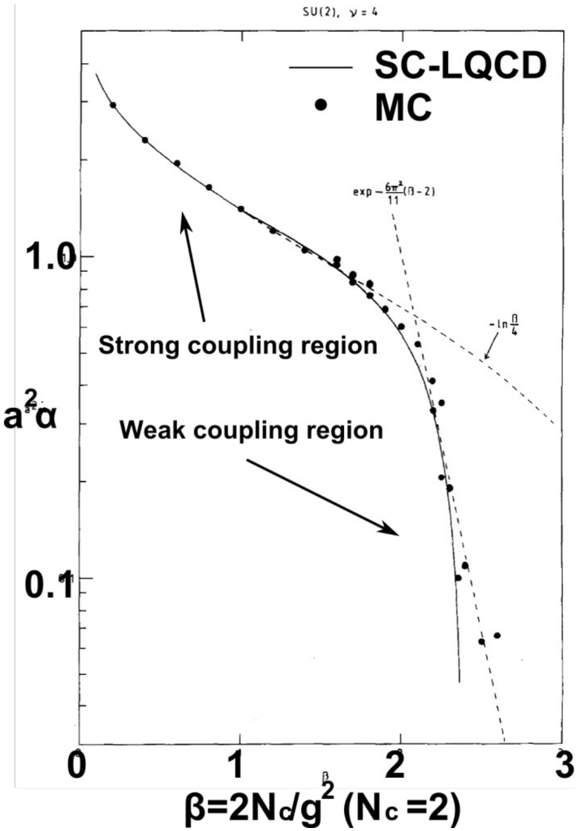
- **Lattice QCD at finite density has the sign problem.**
  **→ Approx. methods and/or Effective model studies are necessary.**
  - **Approximate methods: Taylor exp. (LT04), Imag. μ , Canonical (LC04, 08), Reweighting (LR02, 04), Fugacity exp. (Nagata / Adams), Strong Coupling Lattice QCD**
  - **Effective models:**
    **NJL, PNJL, PQM, ..**

*Direct Sampling in LQCD at finite μ*
*→ Histogram method (Ejiri) or SC-LQCD*



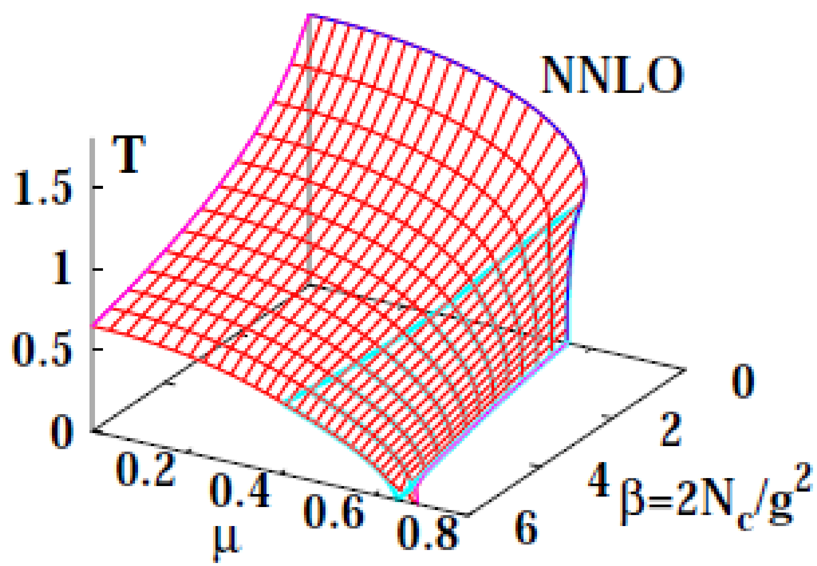
# *Strong Coupling Lattice QCD*

## Pure YM




*Wilson ('74), Creutz ('80),*
*Munster ('80, '81), Lottini,*
*Philipsen, Langelage's ('11)*

## YM+Quarks (MF)



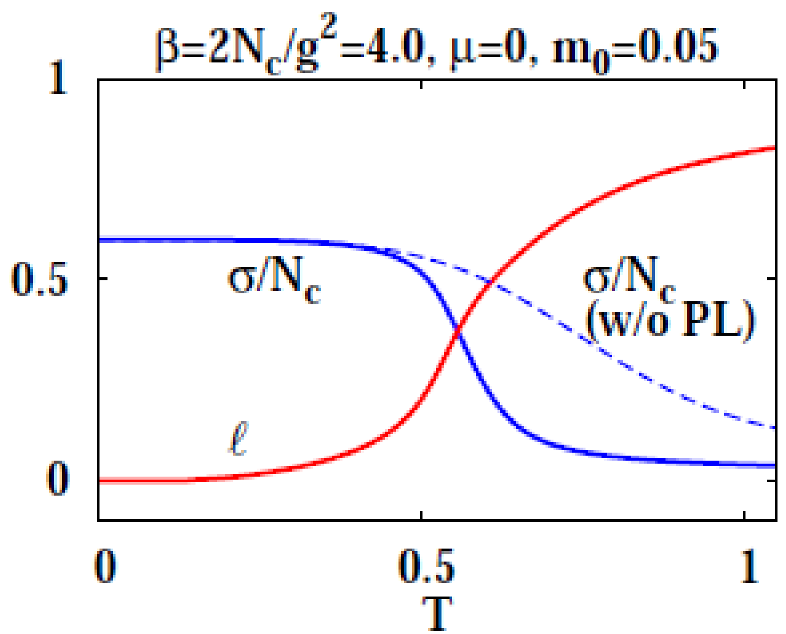

*Kawamoto ('80), Kawamoto, Smit ('81),*
*Damagaard, Hochberg, Kawamoto ('85),*
*Bilic, Karsch, Redlich ('92),*
*Fukushima ('03); Nishida ('03),*
*Kawamoto, Miura, AO, Ohnuma ('07).*
*Miura, Nakano, AO, Kawamoto ('09)*
*Nakano, Miura, AO ('10)*

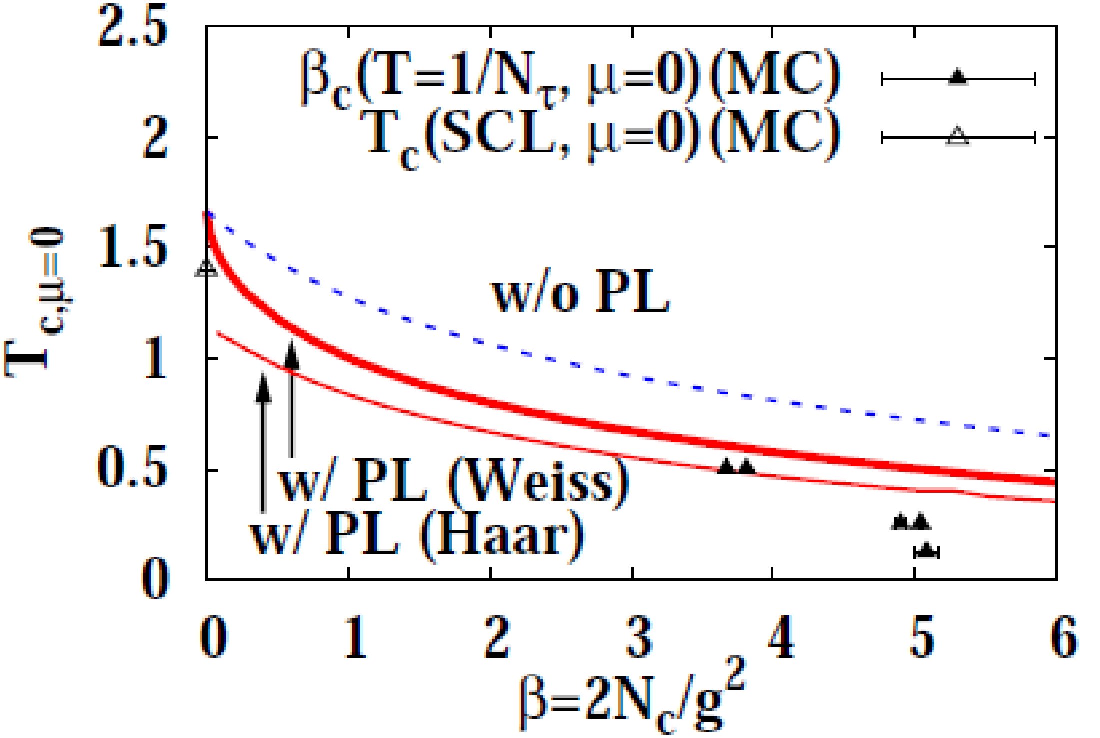
# SC-LQCD with Polyakov Loop Effects at $\mu$=0

*T. Z. Nakano, K. Miura, AO, PRD 83 (2011), 016014 [arXiv:1009.1518 [hep-lat]]*

- **P-SC-LQCD reproduces $T_c(\mu=0)$ in the strong coupling region ( $\beta = 2N_c/g^2 \leq 4$)**

  *MC data: SCL (Karsch et al. (MDP), de Forcrand, Fromm (MDP)), $N_\tau$ =2 (de Forcrand, private), $N_\tau$=4 (Gottlieb et al.('87), Fodor-Katz ('02)), $N_\tau$ =8 (Gavai et al.('90))*

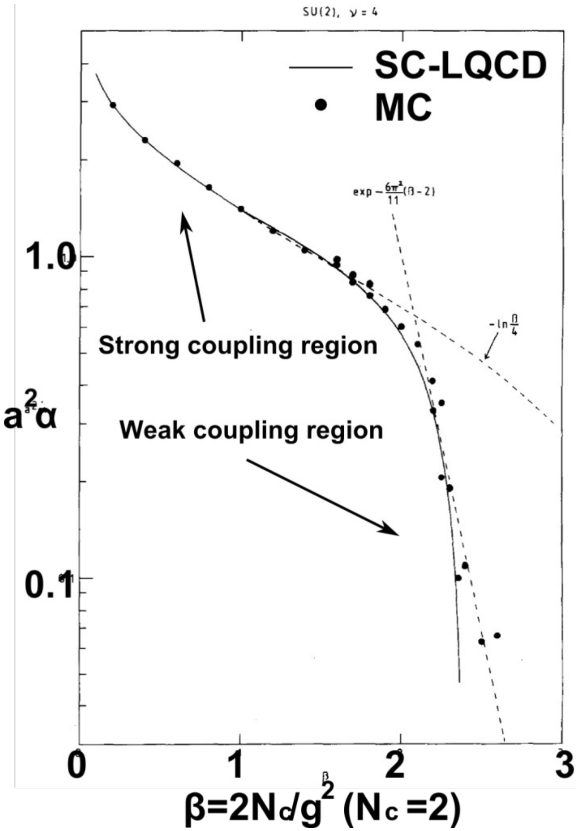



**Lattice Unit**

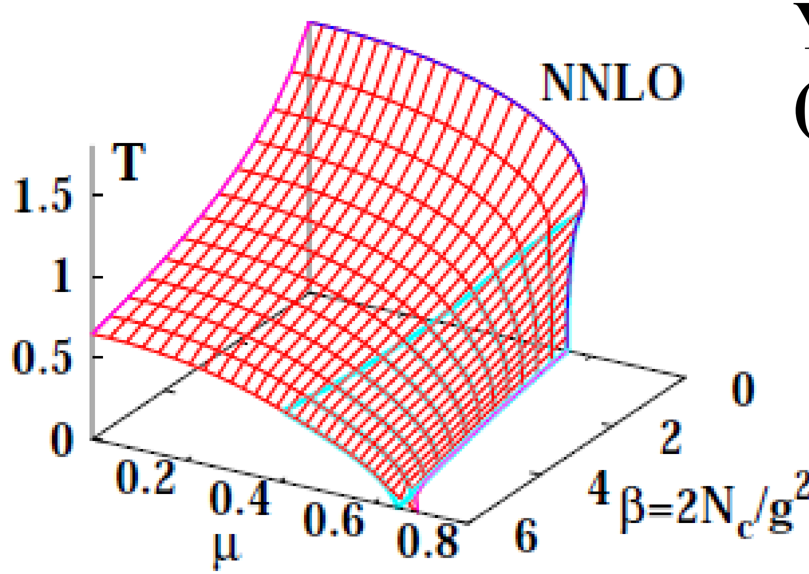
# *Strong Coupling Lattice QCD*

## Pure YM



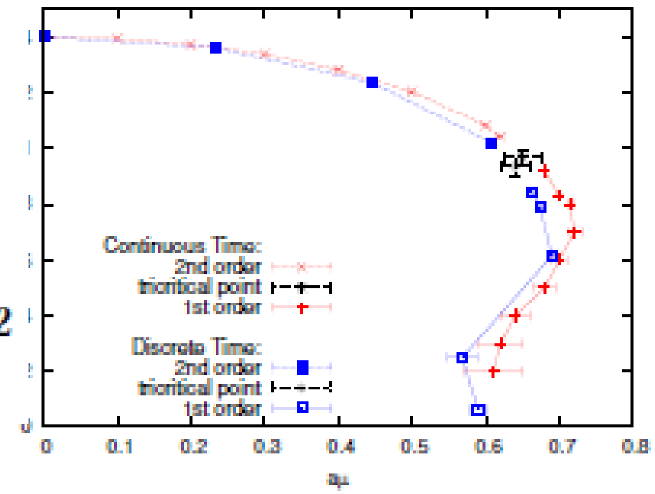

*Wilson ('74), Creutz ('80), Munster ('80, '81), Lottini, Philipsen, Langelage's ('11)*

## YM+Quarks (MF)




*Kawamoto ('80), Kawamoto, Smit ('81), Damagaard, Hochberg, Kawamoto ('85), Bilic, Karsch, Redlich ('92), Fukushima ('03); Nishida ('03), Kawamoto, Miura, AO, Ohnuma ('07). Miura, Nakano, AO, Kawamoto ('09) Nakano, Miura, AO ('10)*

## YM+Q+Fluc. (MDP) (SCL($1/g^2=0$))




*Mutter, Karsch ('89), de Forcrand, Fromm ('10), de Forcrand, Unger ('11)*

**Challenge: YM+Q+Fluc.+Finite Coupling Effects**

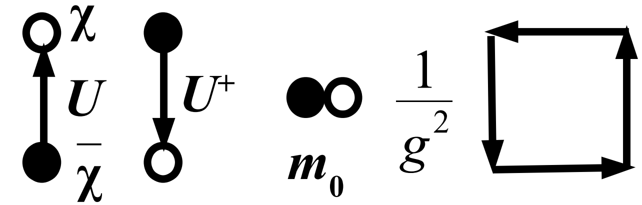
*de Forcrand, Fromm, Langelage, Miura, Philipsen, Unger ('11), AO, Nakano, Ichihara (in prep.)*

# *Auxiliary field effective action in the Strong Coupling Limit*

# Strong Coupling Expansion

- **Lattice QCD action (aniso. lattice, unrooted staggered Fermion)**

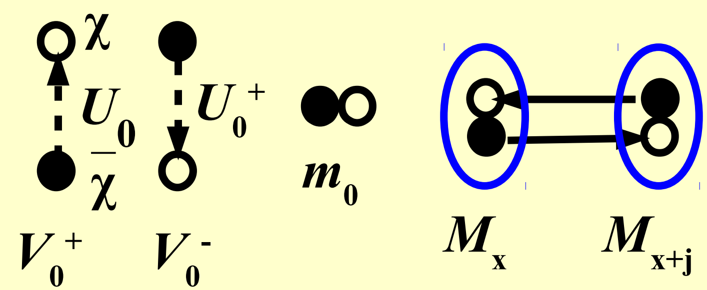
$$S_{LQCD} = S_F + S_G \qquad S_G = -\frac{1}{g^2}\sum_{plaq.} \mathrm{tr}\left[U_P + U_P^{+}\right] f_P$$




$$S_F = \frac{1}{2}\sum_x \left[V^{+}(x) - V^{-}(x)\right] + \frac{1}{2\gamma}\sum_{x,j} \eta_j(x)\left[\bar{\chi}_x U_{x,j}\chi_{x+\hat{j}} - \bar{\chi}_{x+\hat{j}} U_{x,j}^{+}\chi_x\right] + \sum_x \frac{m_0}{\gamma} M_x$$

$$V^{+}(x) = e^{\mu/\gamma^2}\bar{\chi}_x U_{x,0}\chi_{x+\hat{0}}, \quad V^{-}(x) = e^{-\mu/\gamma^2}\bar{\chi}_{x+\hat{0}} U_{x,0}^{+}\chi_x, \quad M_x = \bar{\chi}_x\chi_x, \quad a_\tau = a/\gamma, \quad f_P = 1 \ \text{or}\ 1/\gamma$$

- **Strong coupling expansion (Strong coupling limit)**
  - **Ignore plaquette action ($1/g^2$)**
  - **Integrate out spatial link variables of min. quark number diagrams (1/d expansion)**




$$\int dU\, U_{ab} U_{cd}^{+} = \delta_{ad}\delta_{bc}/N_c$$

$$S_{\mathrm{eff}} = \frac{1}{2}\sum_x \left[V^{+}(x) - V^{-}(x)\right] - \frac{1}{4N_c\gamma^2}\sum_{x,j} M_x M_{x+\hat{j}} + \frac{m_0}{\gamma}\sum_x M_x$$

# *Introduction of Auxiliary Fields*

- **Bosonization of *MM* term (Four Fermi (two-body) interaction)**

$$S_F^{(s)} = -\alpha \sum_{j,x} M_x M_{x+\hat{j}} = -\alpha \sum_{x,y} M_x V_{x,y} M_y \quad \left[ V_{x,y} = \frac{1}{2} \sum_j \left( \delta_{x+\hat{j}\,y} + \delta_{x-\hat{j},y} \right) \right]$$

  - **Meson matrix (*V*) has positive and negative eigen values**

$$f_M(\boldsymbol{k}) = \sum_j \cos k_j \quad , \quad f_M(\bar{\boldsymbol{k}}) = -f_M(\boldsymbol{k}) \quad \left[ \bar{\boldsymbol{k}} = \boldsymbol{k} + (\pi, \pi, \pi) \right]$$

  - **Negative mode = "High" momentum mode**
    **→ Involves a factor exp(i π ($x_1$+$x_2$+$x_3$)) = (-1)**($x_1$+$x_2$+$x_3$)**
    **in coordinate representation**

  - **Bosonization of Negative mode: Extended HS transf.**
    **→ Introducing " *i* " gives rise to the sign problem.**
    *Miura, Nakano, AO (09), Miura, Nakano, AO, Kawamoto (09)*

$$\exp(\alpha A B) = \int d\varphi\, d\phi \exp\left[ -\alpha \left( \varphi^2 - (A+B)\varphi + \phi^2 - i(A-B)\phi \right) \right]$$
$$\approx \exp\left[ -\alpha \left( \bar{\psi}\psi - A\psi - \bar{\psi} B \right) \right]_{\text{stationary}}$$

- Negative

# *Phase cancellation mechanism in σMC*

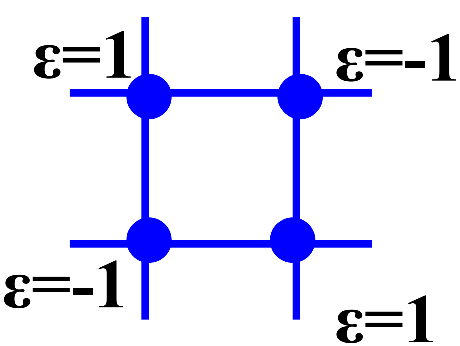
- **Bosonized effective action**

$$S_{\rm eff}=\frac{1}{2}\sum_{x,y}\bar{\chi}_x D_{x,y}\chi_y+\frac{\Omega}{4N_c\gamma^2}\sum_{k,f_M(\boldsymbol{k})>0} f_M(\boldsymbol{k})\left[\sigma_k^*\sigma_k+\pi_k^*\pi_k\right]$$

$$D_{x,y}=\delta_{x+\hat{0},y}\delta_{\boldsymbol{x},\boldsymbol{y}}e^{\mu/\gamma^2}U_{x,0}-\delta_{x,y+\hat{0}}\delta_{\boldsymbol{x},\boldsymbol{y}}e^{-\mu/\gamma^2}U^+_{y,0}+2\left[\Sigma_x+\frac{m_0}{\gamma}\right]\delta_{x,y}\ ,\quad \Sigma_x=\frac{\sigma_x+i\varepsilon_x\pi_x}{2N_c\gamma^2}$$

$$\sigma(x)=\sum_{k,f_M(\boldsymbol{k})>0} f_M(\boldsymbol{k})e^{ikx}\sigma_k,\quad \pi(x)=\sum_{k,f_M(\boldsymbol{k})>0} f_M(\boldsymbol{k})e^{ikx}\pi_k$$

- **Fermion matrix is spatially separated**
  **→ Fermion det at each point**
- **Imaginary part (π) involves**
  **$\varepsilon=(-1)^{x0+x1+x2+x3}=\exp(i\,\pi(x_0+x_1+x_2+x_3))$**
  **→ Phase cancellation of nearest neighbor spatial site det for π field having low k**



# Auxiliary Field Monte-Carlo Integral

- **Effective action of Auxiliary Fields**

$$S_{\mathrm{eff}}=\frac{\Omega}{4N_c\gamma^2}\sum_{k,\,f_M(\boldsymbol{k})>0} f_M(\boldsymbol{k})\left[\sigma_k^*\sigma_k+\pi_k^*\pi_k\right]$$
$$-\sum_{\boldsymbol{x}}\log\left[X_N(\boldsymbol{x})^3-2X_N(\boldsymbol{x})+2\cosh(3N_\tau\mu)\right]$$
$$X_N(\boldsymbol{x})=X_N[\sigma(\boldsymbol{x},\tau),\pi(\boldsymbol{x},\tau)]$$

  - **$\mu$ dependence appears only in the log.**
  - **$\sigma_k$, $\pi_k$ have to be generated in momentum space, while $X_N$ requires $\sigma(x)$ and $\pi(x)$ → Fourier transf. in each step.**
  - **$X_N$ is complex, and this action has the sign problem.**
    **But the sign problem is milder because of the phase cancellation and is less severe at larger $\mu$.**

***Let's try at finite $\mu$ !***

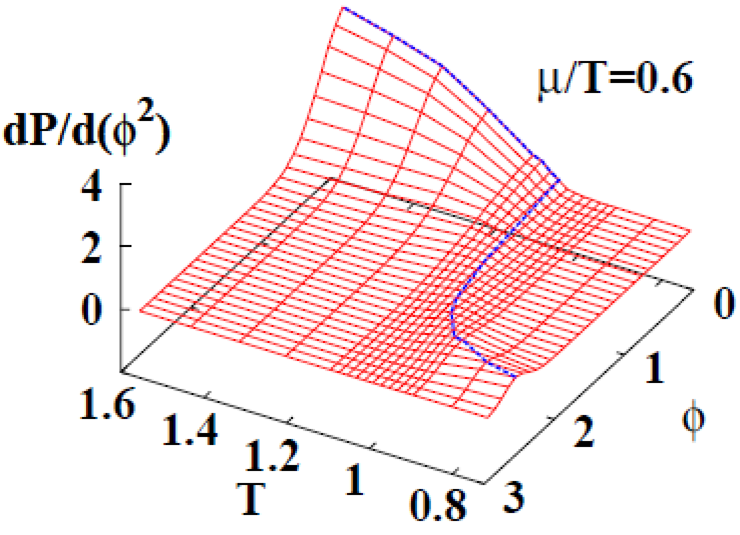
# *Auxiliary Field Monte-Carlo (σMC) estimate of the phase boundary*

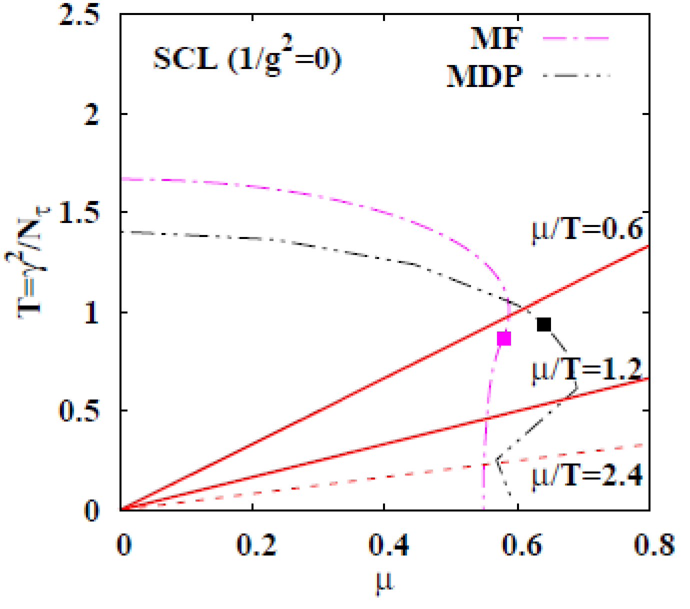
# *Numerical Calculation*

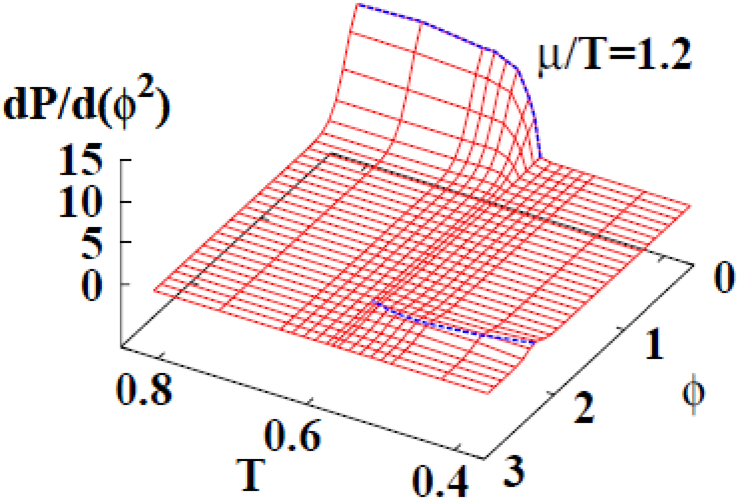
- **$4^4$ asymmetric lattice + Metropolis sampling of $\sigma_k$ and $\pi_k$.**
- **Metropolis sampling of full configuration ($\sigma_k$ and $\pi_k$) at a time. (efficient for small lattice)**
- **Initial cond. = const. $\sigma$**
- **Chiral limit (m=0) simulation → Symmetry in $\sigma \leftrightarrow -\sigma$**
- **Sign problem is not severe ($\langle \cos\theta \rangle \sim (0.9\text{-}1.0)$) in a $4^4$ lattice.**
- **Computer: My PC (Core i7)**

# *Results (1): σ distribution*

- **Fixed μ/T simulation: μ/T= 0 ~ 2.4**
- **Low μ region: Second order (Single peak: finite σ → zero)**
- **High μ region: First order (Dist. func. has two peaks)**



# Results (2): Susceptibility and Quark density

- **Weight factor <cos θ>**

$$\langle \cos\theta \rangle = Z / Z_{abs}$$

$$Z = \int D\sigma_k D\pi_k \exp(-S_{eff}) = \int D\sigma_k D\pi_k \exp(-\mathrm{Re}\, S_{eff}) e^{i\theta}$$

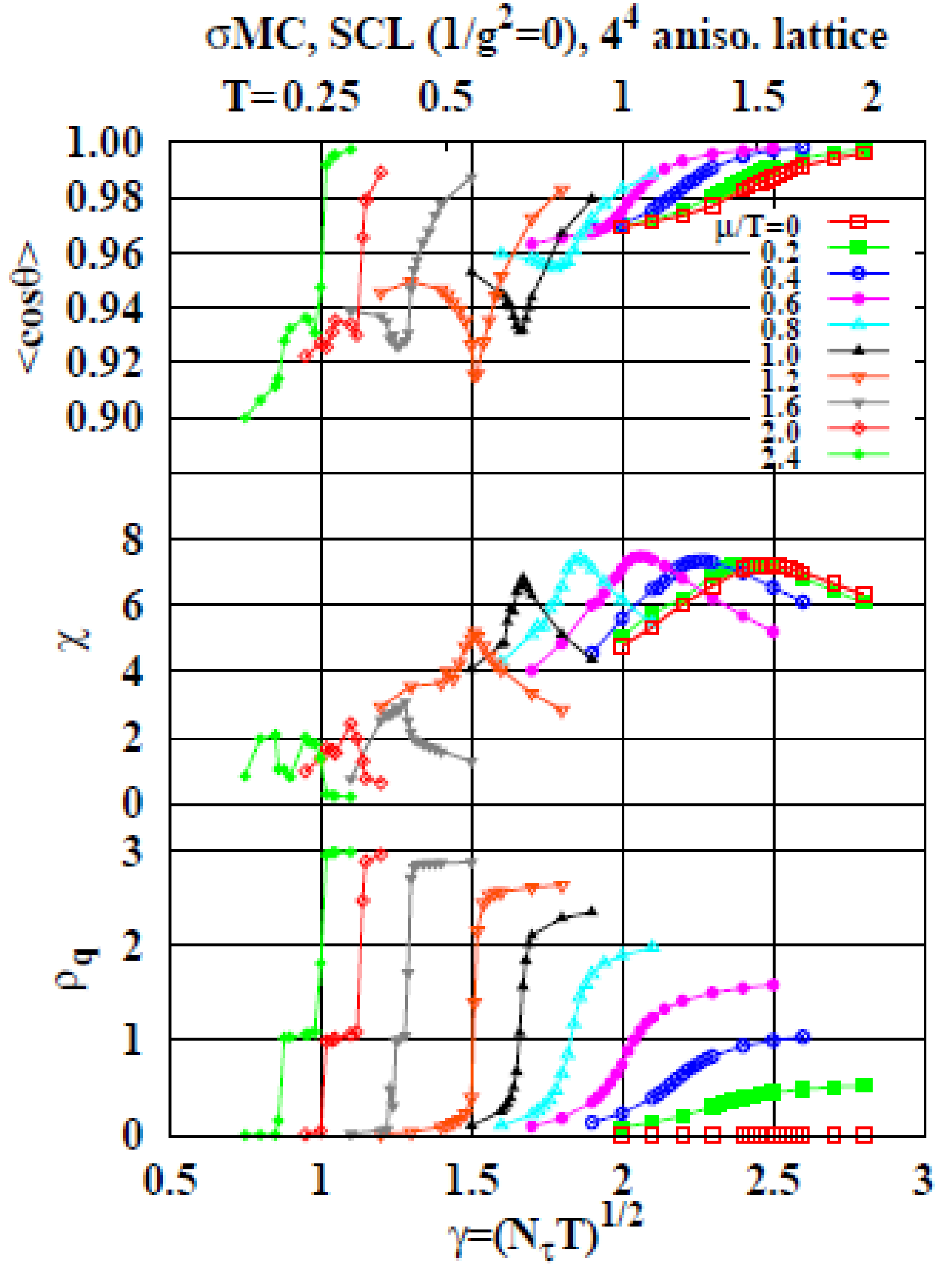
$$Z_{abs} = \int D\sigma_k D\pi_k \exp\left(-\mathrm{Re}\, S_{eff}\right)$$

- **Chiral susceptibility**

$$\chi = -\frac{1}{L^3} \frac{\partial^2 \log Z}{\partial m_0^2}$$

- **Quark number density**

$$\rho_q = -\frac{1}{L^3} \frac{\gamma^2}{N_\tau} \frac{\partial \log Z}{\partial \mu}$$

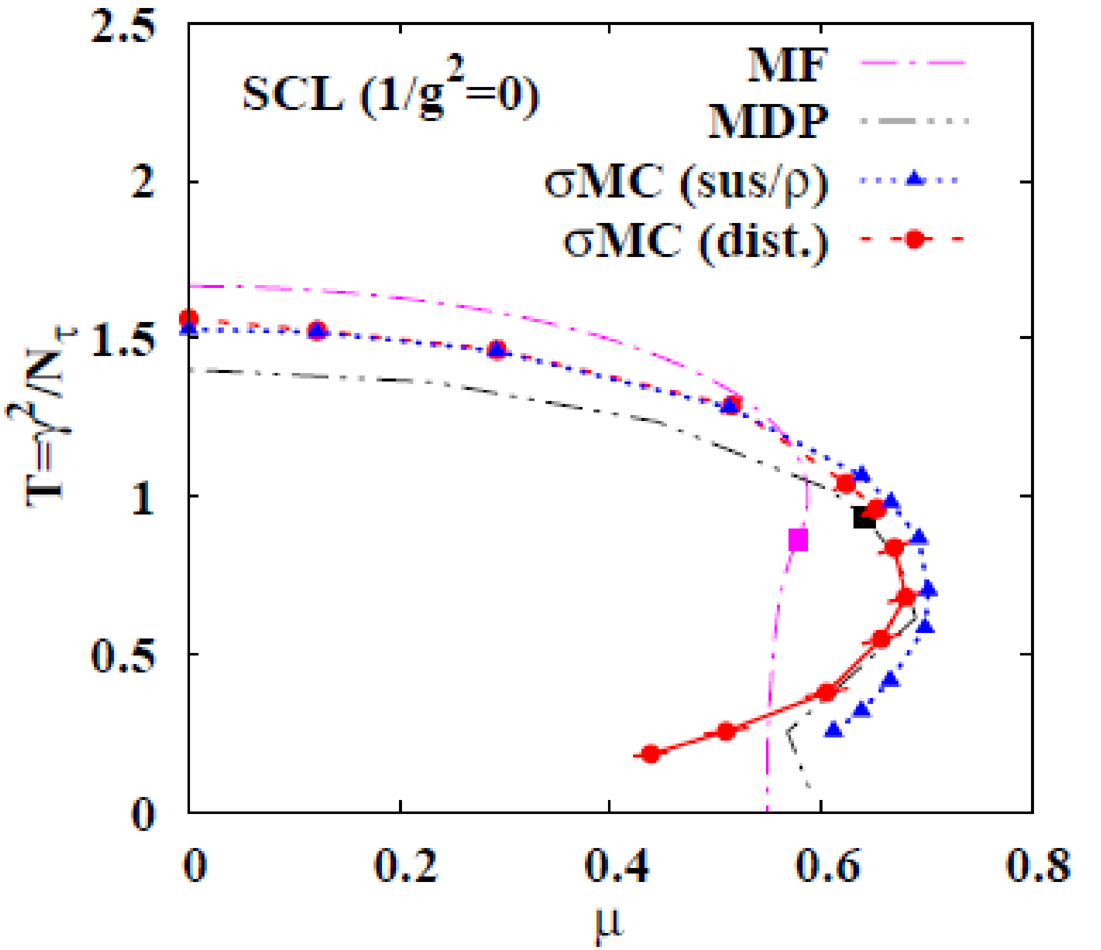


# *Results (3): Phase diagram*

- By taking $T = \gamma^2/N_\tau$ , $\gamma$ dep. of the phase boundary becomes small. *Bilic et al. ('92)*
- Definitions of phase boundar
  - $\phi^2 = \sigma^2 + \pi^2$ dist. peak: finite or zero (red curve)
  - Chiral susceptibility peak (blue)
- Fluctuation effect
  - Reduction of $T_c$ at $\mu$=0
  - Enlarged hadron phase at medium T

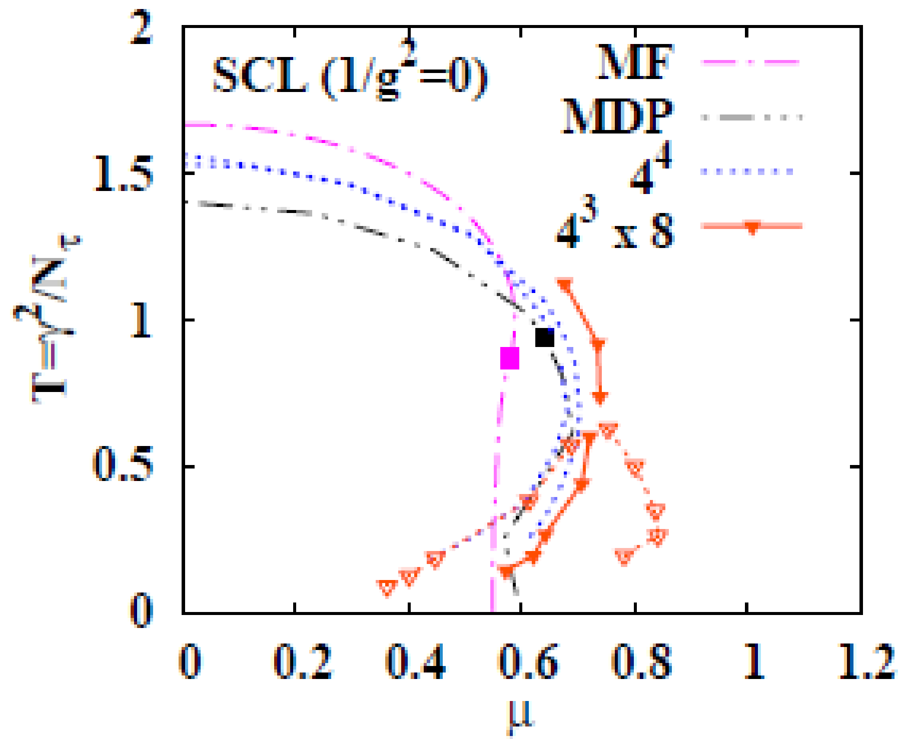
→ Consistent with MDP

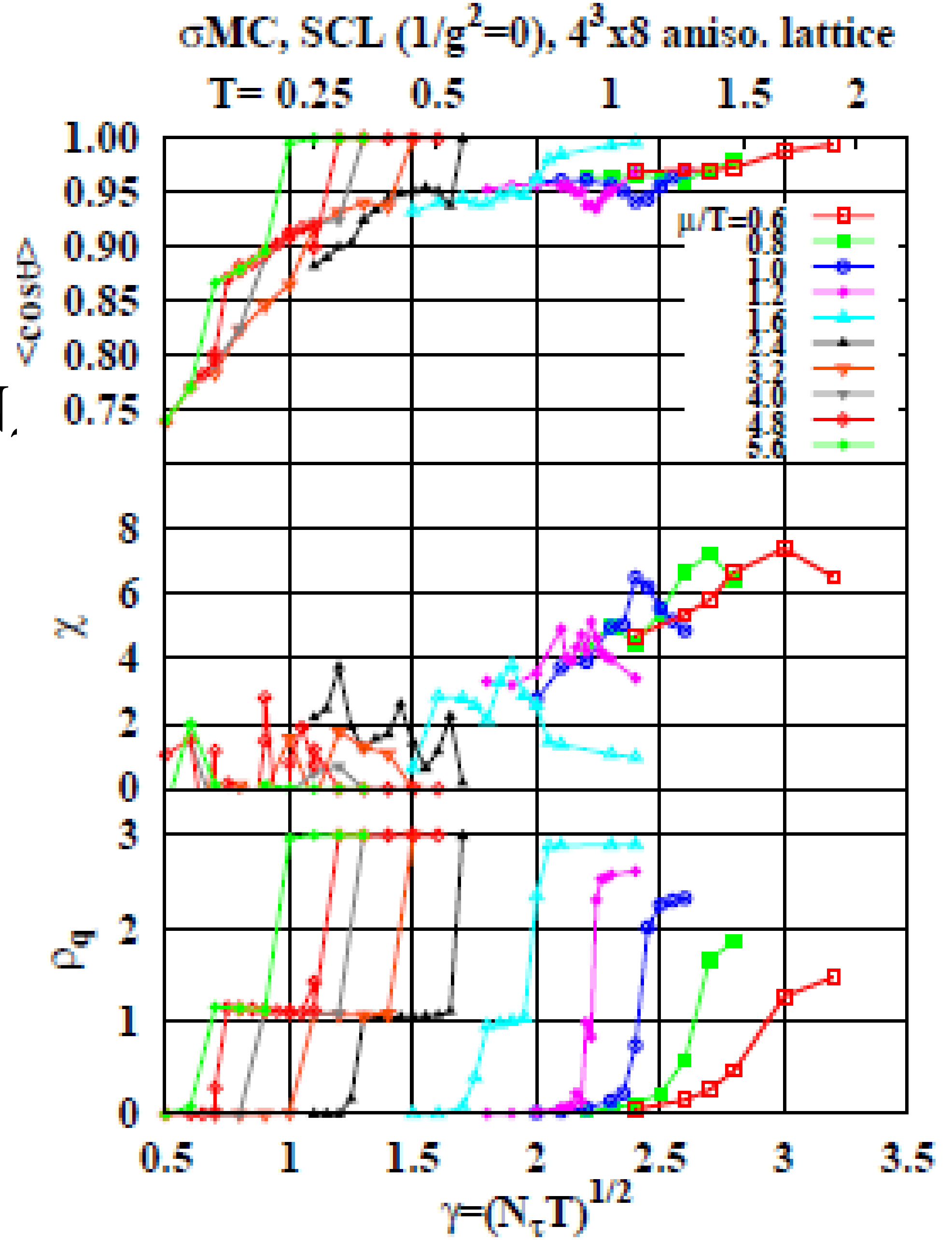
*de Forcrand, Fromm ('09); de Forcrand, Unger ('11)*



# Results (4): Larger Lattice

- Can we go to a larger lattice ?
- dμ/dT > 0 at low T.
  → How about in a larger lattice ?
- Suggestion by de Forcrand
  → low T behavior is sensitive to $N_\tau$
  → $4^3$ x 8



# Summary

- We have proposed an auxiliary field MC method (σMC) in SC-LQCD.
  - To simulate the SCL quark-$U_0$ action (LO in strong coupling ($1/g^0$) and $1/d$ ($1/d^0$) expansion) without further approximation.
    *c.f. Determinantal MC by Abe, Seki*
  - Sign problem is mild in small lattice (<cos θ> ~ (0.9-1) for $4^4$), because of the phase cancellation coming from nearest neighbor interaction.
  - Extension to NLO SC-LQCD is straightforward.
- Phase boundary is obtained and found to be compatible with recent MDP results.
  - Phase boundary is moderately modified from MF results by fluctuations, if $T = \gamma^2/N_\tau$ and $\mu = \gamma^2\mu_0$ scaling is adopted.
  - σMC results are compatible with MDP results, while the shift of $T_c$ at $\mu=0$ is around half (LO in $1/d$ expansion in σMC).

# *Future work*

- **To do:**
  - **Larger lattice ($8^4$, $16^3$x8, ...)**
  - **Finite coupling effects (NLO, NNLO, Polyakov loop, ...)**
  - **Higher 1/d terms including baryonic action**
  - **Polyakov coupling (back reaction)**
  - **Unrooted staggered fermion corresponds to 4 flavors (tates) in continuum**
    **→ Different Fermion (e.g. staggered-Wilson fermion).**

***Thank you***

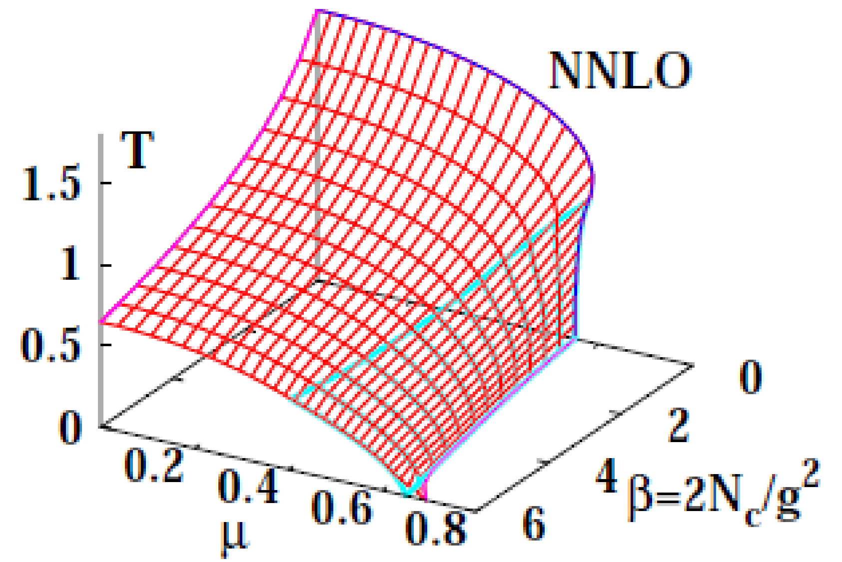
# Clausius-Clapeyron Relation

- **First order phase boundary → two phases coexist**

$$P_h = P_q \;\rightarrow\; dP_h = dP_q \;\rightarrow\; \frac{d\mu}{dT} = -\frac{s_q - s_h}{\rho_q - \rho_h}$$

$$dP_h = \rho_h d\mu + s_h dT\,,\quad dP_q = \rho_q d\mu + s_q dT$$

- **Continuum theory**
  **→ Quark matter has larger entropy and density (dμ /dT < 0)**
- **Strong coupling lattice**
  - **SCL: Quark density is larger than half-filling, and "Quark hole" carries entropy → dμ/dT > 0**
  - **NLO, NNLO → dμ/dT < 0**




*AO, Miura, Nakano, Kawamoto ('09)*

# Introduction of Auxiliary Fields

$$S^{(s)} = -\frac{1}{4N_c\gamma^2}\sum_{x,j} M_x M_{x+\hat{j}} = -\frac{L^3}{4N_c\gamma^2}\sum_{\tau,\mathbf{k}} f_M(\mathbf{k})\,\tilde{M}_{\mathbf{k}}(\tau)\,\tilde{M}_{-\mathbf{k}}(\tau)$$

$$= \frac{L^3}{4N_c\gamma^2}\sum_{\tau,\mathbf{k},f_M(\mathbf{k})>0} f_M(\mathbf{k})\Big[\varphi_{\mathbf{k}}(\tau)^2 + \phi_{\mathbf{k}}(\tau)^2 + \varphi_{\mathbf{k}}(\tilde{M}_{\mathbf{k}} + \tilde{M}_{-\mathbf{k}}) - i\phi_{\mathbf{k}}(\tilde{M}_{\mathbf{k}} - \tilde{M}_{-\mathbf{k}})$$

$$+ \varphi_{\bar{\mathbf{k}}}(\tau)^2 + \phi_{\bar{\mathbf{k}}}(\tau)^2 + i\varphi_{\bar{\mathbf{k}}}(\tilde{M}_{\bar{\mathbf{k}}} + \tilde{M}_{-\bar{\mathbf{k}}}) + \phi_{\bar{\mathbf{k}}}(\tilde{M}_{\bar{\mathbf{k}}} - \tilde{M}_{-\bar{\mathbf{k}}})\Big]$$

$$= \frac{\Omega}{2N_c\gamma^2}\sum_{k,f_M(\mathbf{k})>0} f_M(\mathbf{k})\left[\sigma_k^*\sigma_k + \pi_k^*\pi_k\right] + \frac{1}{2N_c\gamma^2}\sum_x M_x\left[\sigma(x) + i\varepsilon(x)\pi(x)\right]$$

$$\Omega = L^3 N_\tau$$

$$\sigma(x) = \sum_{k,f_M(\mathbf{k})>0} f_M(\mathbf{k})e^{ikx}\sigma_k\ ,\quad \pi(x) = \sum_{k,f_M(\mathbf{k})>0} f_M(\mathbf{k})e^{ikx}\pi_k$$

$$\sigma_k = \varphi_k + i\phi_k\ ,\ \pi_k = \varphi_{\bar{k}} + i\phi_{\bar{k}}$$

$$V_{x,y} = \frac{1}{2}\sum_j\left(\delta_{x+\hat{j},y} + \delta_{x-\hat{j},y}\right)\ ,\quad f_M(\mathbf{k}) = \sum_j \cos k_j\ ,\quad \bar{\mathbf{k}} = \mathbf{k} + (\pi,\pi,\pi)$$

# Fermion Determinant

*Faldt, Petersson, 1986*

- **Fermion action is separated to each spatial point and bi-linear → Determinant of Nτ x Nc matrix**

$$\exp(-V_{\rm eff}/T) = \int dU_0 \begin{vmatrix} I_1 & e^{\mu} & 0 & & e^{-\mu} U^{+} \\ -e^{-\mu} & I_2 & e^{\mu} & & \\ 0 & -e^{-\mu} & I_3 & e^{\mu} & \\ \vdots & & & \ddots & \\ -e^{\mu} U & & & -e^{-\mu} & I_N \end{vmatrix}$$

**Nc x Nτ**

$$= \int dU_0 \det\left[ X_N[\sigma] \otimes \mathbf{1}_c + e^{-\mu/T} U^{+} + (-1)^{N_\tau} e^{\mu/T} U \right]$$

**Nc**

$$= X_N^3 - 2 X_N + 2\cosh(3 N_\tau \mu)$$

$$I_\tau/2 = [\sigma(x) + i\varepsilon(x)\pi(x)]/2 N_c \gamma^2 + m_0/\gamma$$

$$X_N = B_N + B_{N-2}(2;N-1)$$

$$B_N = I_N B_{N-1} + B_{N-2}$$

$$B_N = \begin{vmatrix} I_1 & e^{\mu} & 0 & & \\ -e^{-\mu} & I_2 & e^{\mu} & & \\ 0 & -e^{-\mu} & I_3 & e^{\mu} & \\ \vdots & & & \ddots & \\ & & & -e^{-\mu} & I_N \end{vmatrix}$$