Phase diagram and a sign problem in lattice QCD at strong coupling

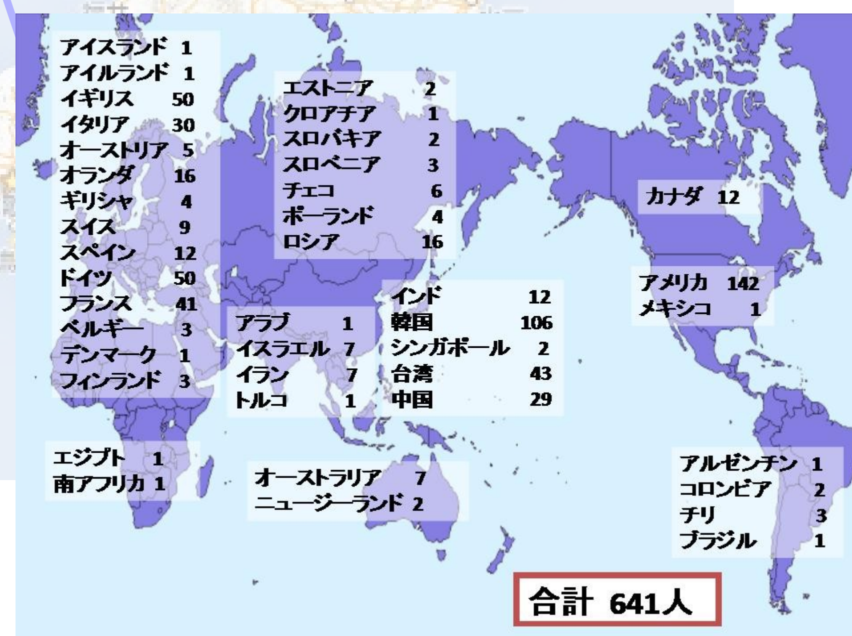
A. Ohnishi (YITP)
in collaboration with
T.Z.Nakano (YITP/Kyoto U.), T. Ichihara (Kyoto U.)

- **Introduction**
  - **Finite density QCD matter: Why and How**
- **Phase diagram in strong coupling lattice QCD**
  - **Strong coupling limit, finite coupling effects, fluctuations**
- **Summary**

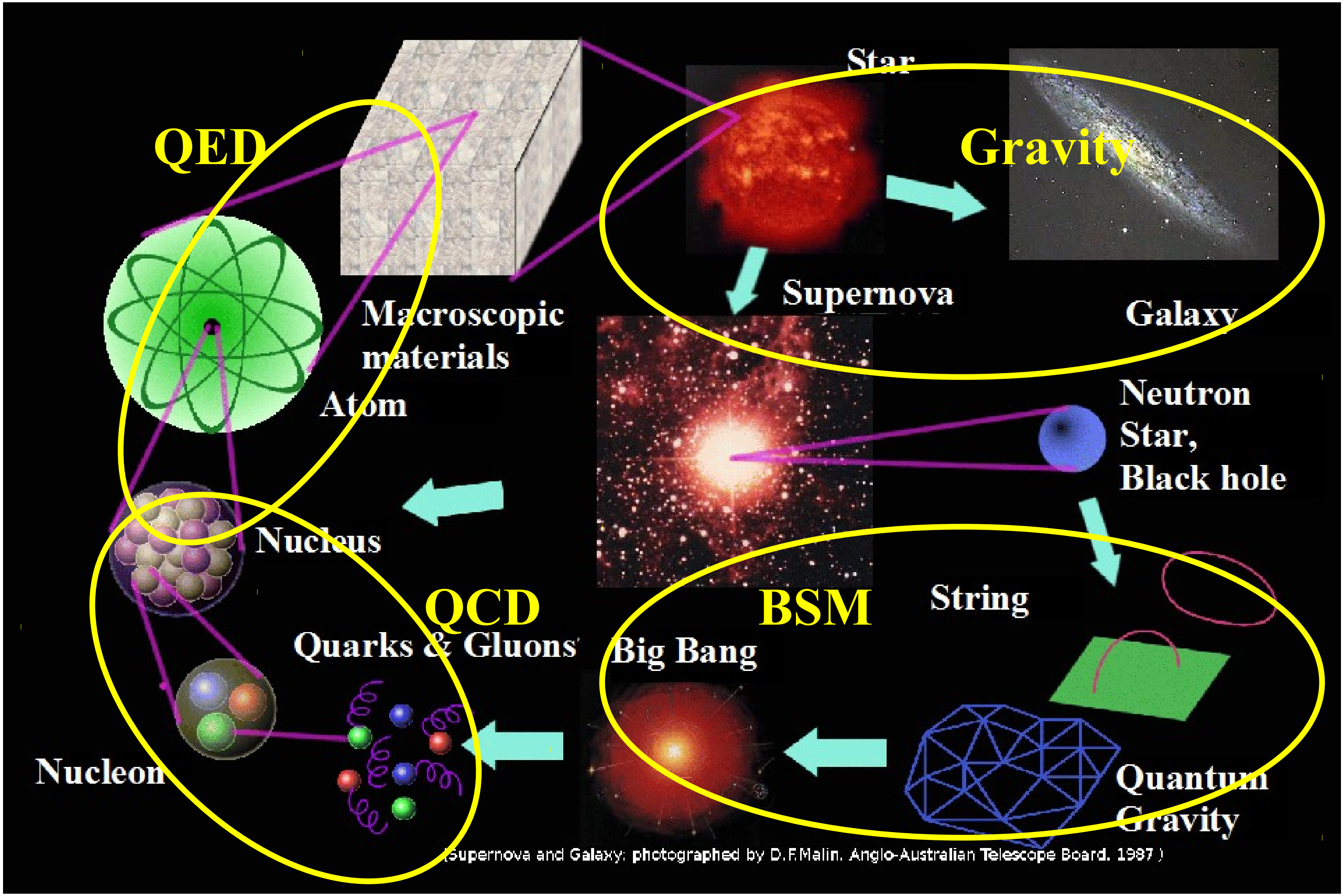
T. Z. Nakano, K. Miura, AO, PRD83(2011),016014
AO, T. Ichihara, T.Z. Nakano, PoS Lattice2012 (2012), 088
T. Ichihara, T. Z. Nakano, AO, PoS Lattice2013 (2013), to appear

Do you know Yukawa Institute ?

- Yukawa Institute for Theoretical Physics, Kyoto University
  - Founded in 1953 (We celebrated 60 year birthday of YITP in Sep.), to memorize Yukawa's Nobel prize (first winner in Japan).
  - Domestic & International Collaboration program
    20-30 domestic workshops, ~ 10 international workshops,
    ~ 1000(?) domestic visitors, 600-700 visitors from abroad
  - Su Houng was the visiting professor in YITP in 2010.



Hierarchy of Matter



We cannot describe nuclei from quarks & gluons yet.
→ Main obstacle in describing our world from SM.

- **QCD**
  - **Fundamental theory of strong interactions**
  - **Non-Abelian gauge thoery (cf. QED)**
  - **Asymptotic freedom at high energies (g → 0 @ large Q), Strong coupling at low energies**
- **Lagrangian**

$$L=\bar{q}(i\gamma^\mu D_\mu - m)q - \frac{1}{2}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}$$

$$D_\mu = \partial_\mu + i g A_\mu \quad (\text{Covariant derivative})$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g [A_\mu, A_\nu] = \frac{-i}{g}[D_\mu, D_\nu] \quad (\text{Field strength})$$

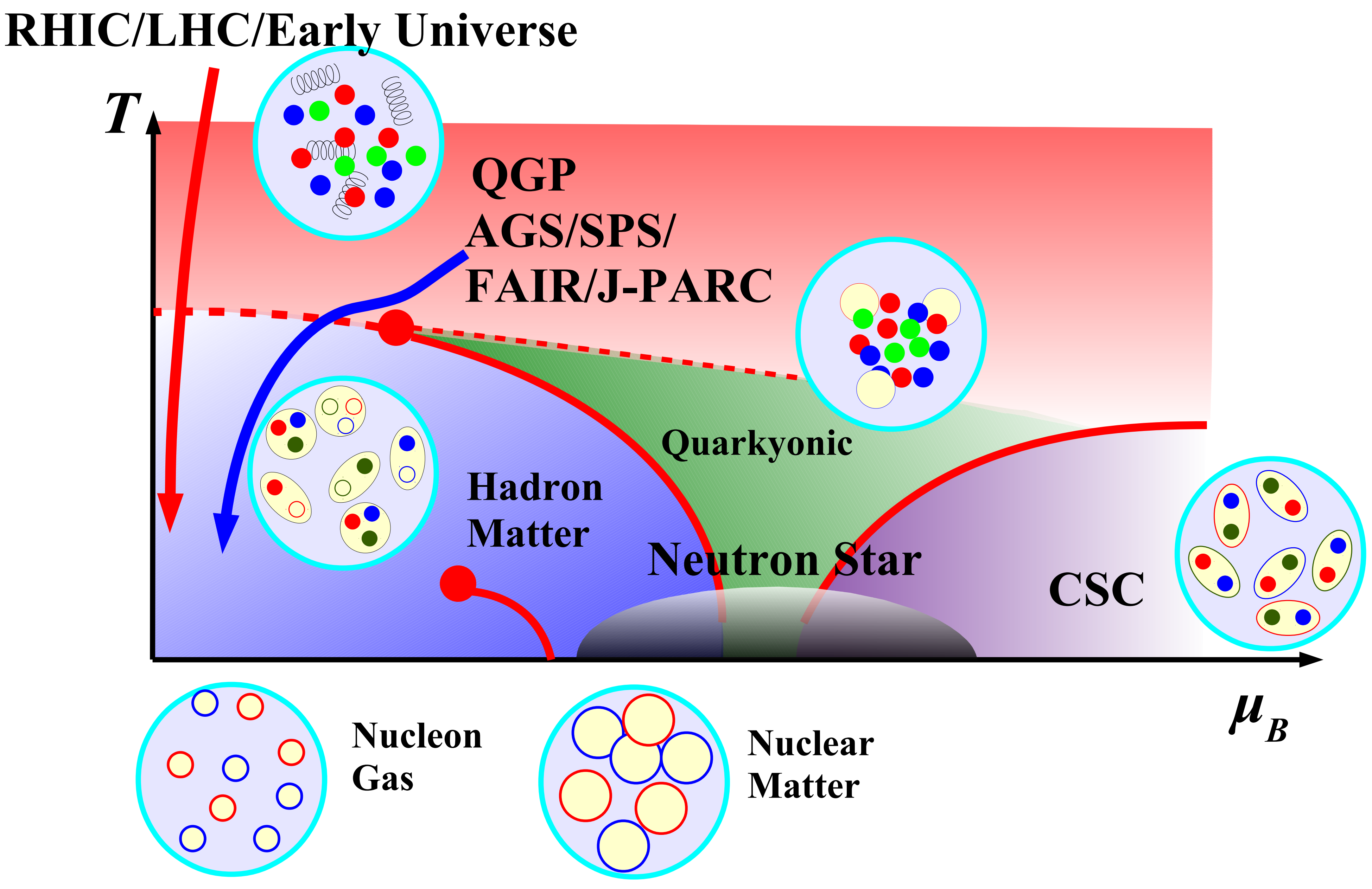
$$A_\mu = A_\mu^a t^a \quad (t^a = \mathrm{SU(3)}\ \text{generator}, [t^a, t^b] = i f_{abc} t^c, \mathrm{tr}\,(t^a t^b) = \frac{1}{2}\delta_{ab})$$

- **Gauge transformation**

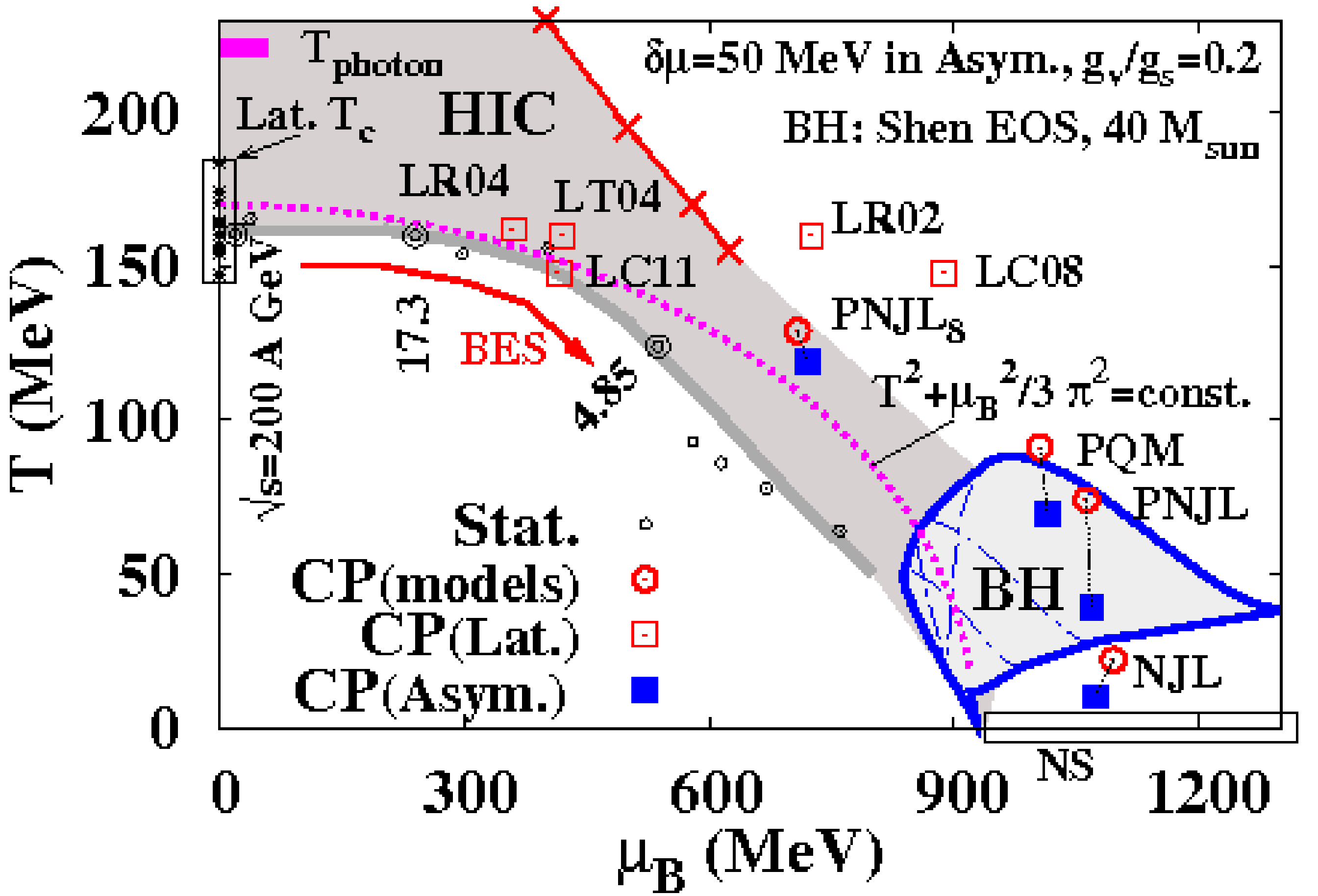
$$q(x) \to V(x) q(x), \quad g A_\mu(x) \to V(x)(g A_\mu(x) - i\partial_\mu) V^+(x)$$

$$F_{\mu\nu}(x) \to V(x) F_{\mu\nu} V^+(x), \quad D_\mu(x) \to V(x) D_\mu(x) V^+(x)$$

QCD Phase Diagram



QCD phase diagram (Exp. & Theor. Studies)



QCD phase transition is not only an academic problem, but also a subject which would be measured in HIC or Compact Stars

AO, PTPS 193('12)1

How can we investigate QCD phase diagram ?

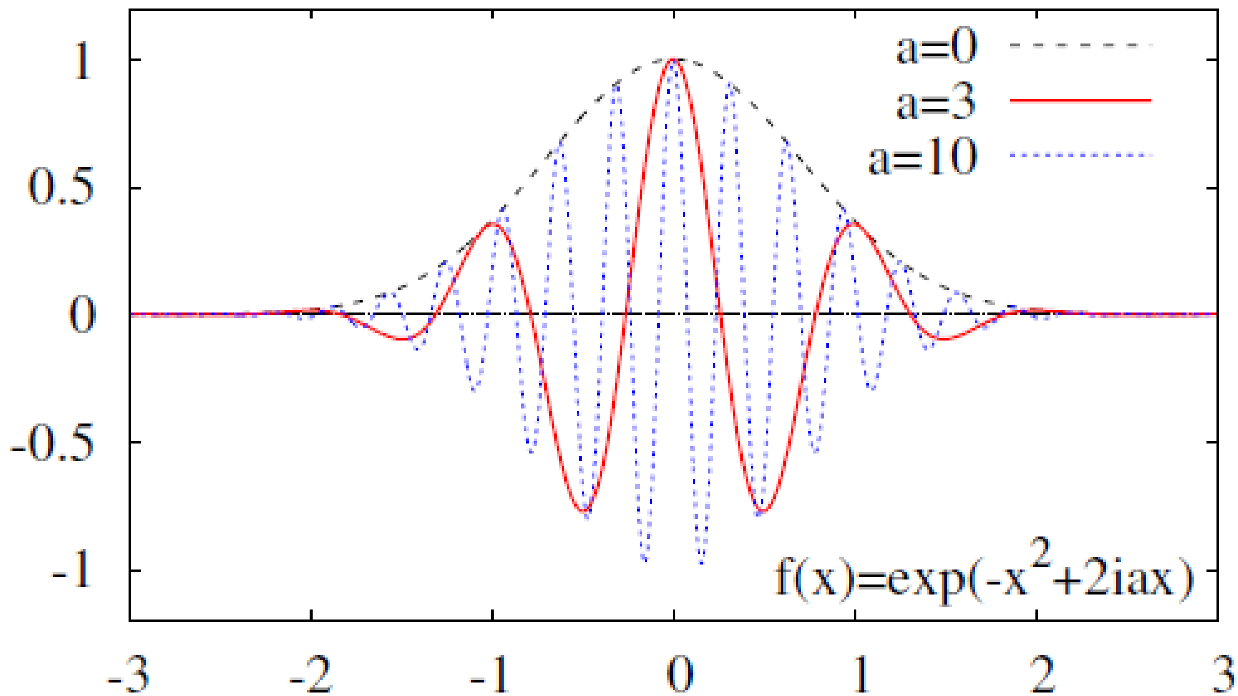
- **Non-pert. & ab initio approach**
  = Monte-Carlo simulation of lattice QCD
  but lattice QCD at finite μ has the sign problem.

Sign Problem

- **Monte-Carlo integral of oscillating function**

$$Z=\int dx \exp(-x^2+2iax)=\sqrt{\pi}\exp(-a^2)$$

$$\langle O\rangle=\frac{1}{Z}\int dx\, O(x)\, e^{-x^2+2iax}$$

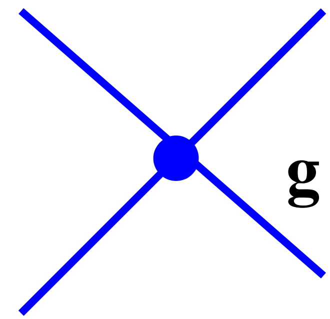


Easy problem for human is not necessarily easy for computers.

Sign Problem (cont.)

- **Generic problem in quantum many-body problems**
  - **Example: Euclid action of interacting Fermions**

$$S=\sum_{x,y}\bar{\psi}_x D_{x,y}\psi_y+g\sum_x(\bar{\psi}\psi)_x(\bar{\psi}\psi)_x$$



  - **Bosonization and MC integral (g>0 → repulsive)**

$$\exp(-g M_x M_x)=\int d\sigma_x \exp(-g\sigma_x^2-2ig\sigma_x M_x) \quad (M_x=(\bar{\psi}\psi)_x)$$

$$Z=\int D[\psi,\bar{\psi},\sigma]\exp\left[-\bar{\psi}(D+2ig\sigma)\psi-g\sum_x\sigma_x^2\right]$$

$$=\int D[\sigma]\ \mathrm{Det}(D+2ig\sigma)\exp\left[-g\sum_x\sigma_x^2\right]$$

We frequently encounter complex Fermion matrix which leads to complex weight in MC integral.
E.g. Solving 2-dim Hubbard model is still problematic.

Partition function & Euclidean Action

- **Partition function**

$$Z=\sum_n \exp(-E_n/T)=\sum_n \langle n|\exp[-\hat{H}/T]|n\rangle$$

$$=\sum_n \langle n|\exp[-i\hat{H}(t_f-t_i)]|n\rangle_{t_f-t_i=-i/T}=\int D\phi \exp(-S_E[\phi])$$

$$S_E[\phi]=\int_0^\beta d\tau\, d^3x\, L_E(\phi,\partial_i\phi,\partial_\tau\phi)\ |_{\phi(x,\beta)=\phi(x,0)}$$

$$L_E(\phi,\partial_i\phi,\partial_\tau\phi)=-L(\phi,\partial_i\phi,i\partial_t\phi)$$

$$t=-i\tau,\quad \partial_\tau=-i\partial_t,\ \beta=1/T$$

$$iS=i\int_0^{-i\beta}dt\int d^3x\,L=\int_0^\beta d\tau\, d^3x\, L=-\int_0^\beta d\tau\, d^3x\, L_E$$

  - **Partition function = Sum of imaginary time evolved amplitudes**

- **Euclidean Action: Example**

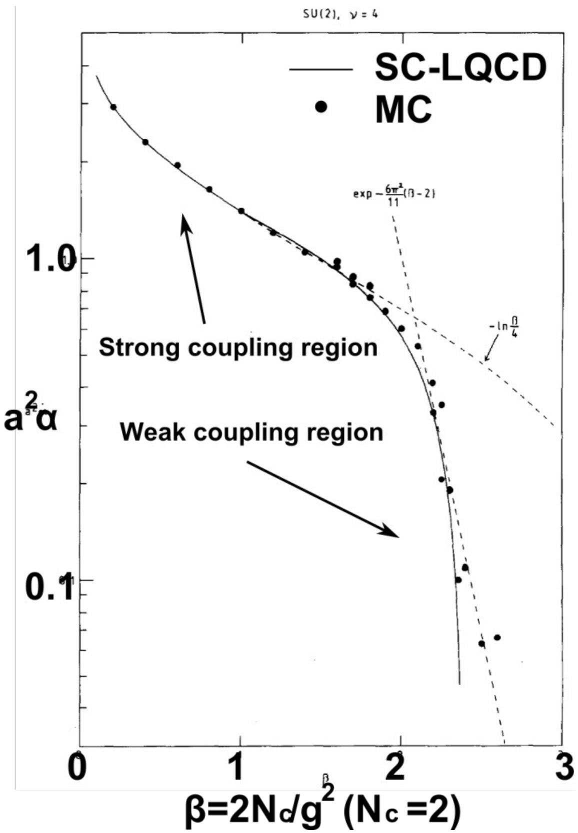
$$L=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2-U(\phi) \longrightarrow L_E=\frac{1}{2}\partial_\mu\phi\partial_\mu\phi+\frac{1}{2}m^2\phi^2+U(\phi)$$

How can we investigate QCD phase diagram ?

- **Non-pert. & ab initio approach**
  = Monte-Carlo simulation of lattice QCD
  but lattice QCD at finite μ has the sign problem.
- **Effective model and/or Approximations are necessary.**
  - **Effective models:**
    Nambu-Jona-Lasinio model,
    Polyakov loop extended NJL model, …
    but model dependence is large.
  - **Approximate methods:**
    Taylor expansion in μ/T, Imag. μ , Canonical ensemble,
    Re-weighting, Fugacity expansion, Histogram method,
    Complex Langevin approach, Strong coupling lattice QCD

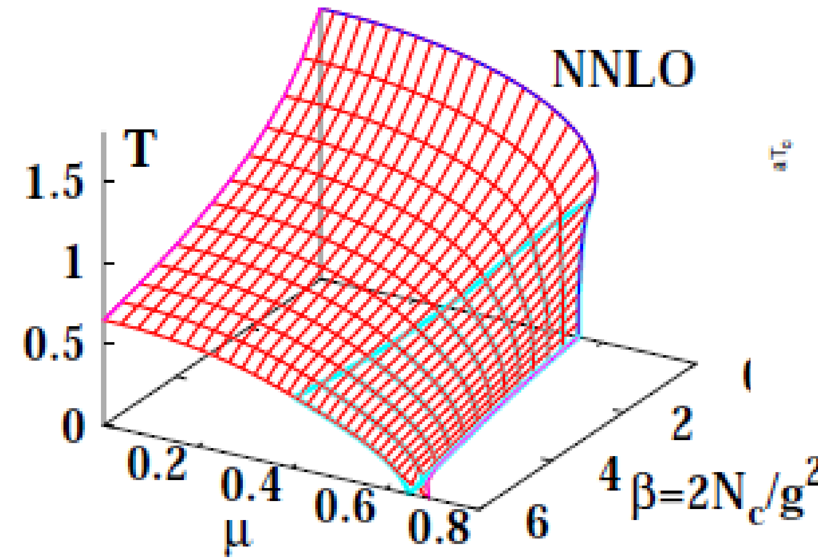
Strong Coupling Lattice QCD

Pure YM



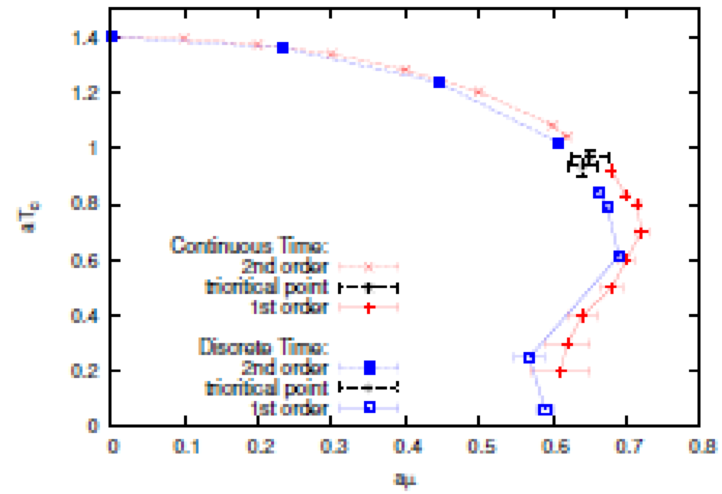
Wilson ('74), Creutz ('80),
Munster ('80, '81), Lottini,
Philipsen, Langelage's ('11)

YM+Quarks (MF)



Kawamoto ('80), Kawamoto, Smit ('81),
Damagaard, Hochberg, Kawamoto ('85),
Bilic, Karsch, Redlich ('92),
Fukushima ('03); Nishida ('03),
Kawamoto, Miura, AO, Ohnuma ('07).
Miura, Nakano, AO, Kawamoto ('09)
Nakano, Miura, AO ('10)

YM+Q+Fluc. (MDP)
(SCL($1/g^2=0$))



Mutter, Karsch ('89),
de Forcrand, Fromm ('10),
de Forcrand, Unger ('11)

How can we investigate QCD phase diagram ?

- **Non-pert. & ab initio approach**
  = Monte-Carlo simulation of lattice QCD
  but lattice QCD at finite μ has the sign problem.

- **Effective model and/or Approximations are necessary.**

  - **Effective models:**
    Nambu-Jona-Lasinio model,
    Polyakov loop extended NJL model, …
    but model dependence is large.

  - **Approximate methods:**
    Taylor expansion in μ/T, Imag. μ , Canonical ensemble,
    Re-weighting, Fugacity expansion, Histogram method,
    Complex Langevin approach, Strong coupling lattice QCD

We discuss the QCD phase diagram
in strong coupling lattice QCD (SC-LQCD),
and examine how the sign problem can be weakened.

Strong coupling lattice QCD

Disclaimer:

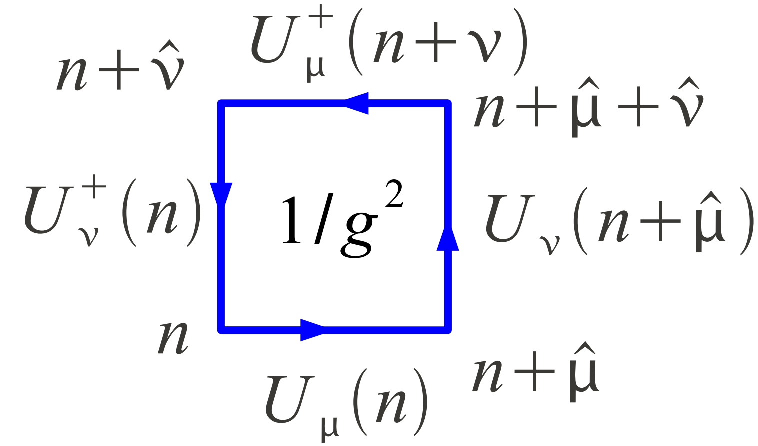
Lattice Unit (a=1),

Staggered Fermion (Nf=4 in the cont. limit),

Lattice QCD action

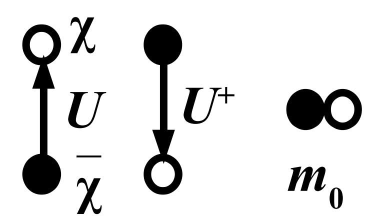
- **Gluon field → Link variables** $U_\mu(x) \simeq \exp(i g A_\mu)$
- **Gluon action → Plaquette action**

$$S_G = \frac{2N_c}{g^2} \sum_{plaq.} \left[1 - \frac{1}{N_c} \mathrm{Re}\,\mathrm{tr}\, U_{\mu\nu}(n) \right]$$



  - **Loop → surface integral of "rotation" $F_{\mu\nu}$ in the U(1) case.**

- **Quark kinetic term (staggered fermion)**



$$S_F = \frac{1}{2} \sum_x \left[\bar{\chi}_x e^{\mu} U_{0,x} \chi_{x+\hat{0}} - \bar{\chi}_{x+\hat{0}} e^{-\mu} U_{0,x}^+ \chi_x \right]$$

$$+ \frac{1}{2} \sum_{x,j} \eta_j(x) \left[\bar{\chi}_x U_{x,j} \chi_{x+\hat{j}} - \bar{\chi}_{x+\hat{j}} U_{x,j}^+ \chi_x \right] + \sum_x m_0 M_x$$

$$= S_F^{(t)} + S_F^{(s)} + \sum_x m_0 M_x$$

Quantum Chromodynamics (QCD)

- **QCD**
  - **Fundamental theory of strong interactions**
  - **Non-Abelian gauge thoery (cf. QED)**
  - **Asymptotic freedom at high energies (g → 0 @ large Q), Strong coupling at low energies**
- **Lagrangian**

$$L=\bar{q}\left(i\gamma^{\mu}D_{\mu}-m\right)q-\frac{1}{2}\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}$$

$$D_{\mu}=\partial_{\mu}+igA_{\mu}\quad(\text{Covariant derivative})$$

$$F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}-ig[A_{\mu},A_{\nu}]=\frac{-i}{g}[D_{\mu},D_{\nu}]\quad(\text{Field strength})$$

$$A_{\mu}=A_{\mu}^{a}t^{a}\quad\left(t^{a}=\text{SU(3) generator},[t^{a},t^{b}]=if_{abc}t^{c},\mathrm{tr}\left(t^{a}t^{b}\right)=\frac{1}{2}\delta_{ab}\right)$$

- **Gauge transformation**

$$q(x)\rightarrow V(x)q(x),\quad gA_{\mu}(x)\rightarrow V(x)\left(gA_{\mu}(x)-i\partial_{\mu}\right)V^{+}(x)$$

$$F_{\mu\nu}(x)\rightarrow V(x)F_{\mu\nu}V^{+}(x),\quad D_{\mu}(x)\rightarrow V(x)D_{\mu}(x)V^{+}(x)$$

Link integral → Area Law

- **One-link integral**

$$\int dU\, U_{ab} U^{+}_{cd} = \frac{1}{N_c} \delta_{ad} \delta_{bc}$$

- **Wilson loop in pure Yang-Mills theory**

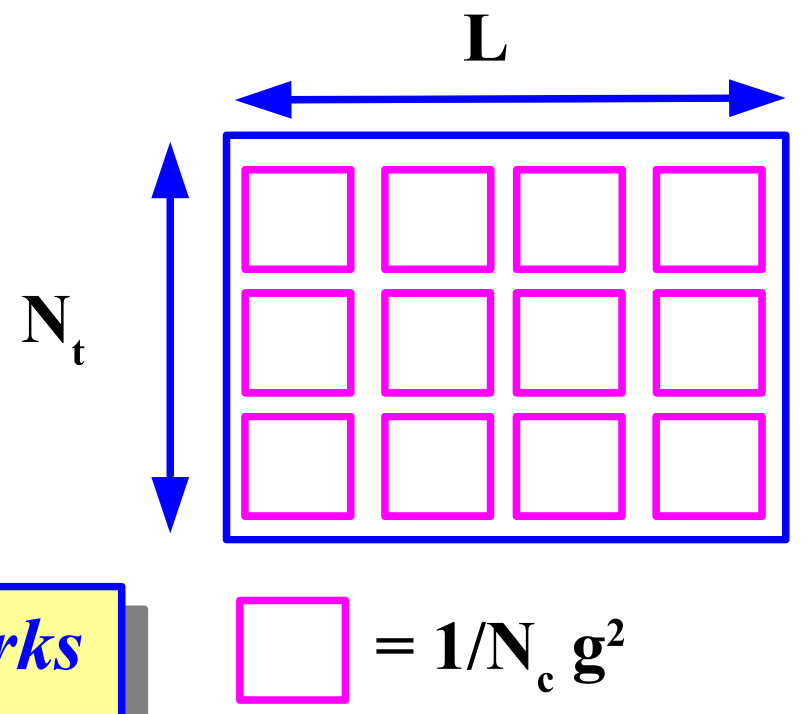
$$\langle W(C = L \times N_\tau) \rangle = \frac{1}{Z} \int DU\, W(C) \exp\left[\frac{1}{g^2} \sum_P \mathrm{tr}\left(U_P + U_P^{+}\right)\right]$$

$$= \exp\left(-V(L) N_\tau\right)$$

in the strong coupling limit

$$\langle W(C) \rangle = N \left(\frac{1}{g^2 N}\right)^{L N_\tau}$$

$$\to V(L) = L \log\left(g^2 N\right)$$



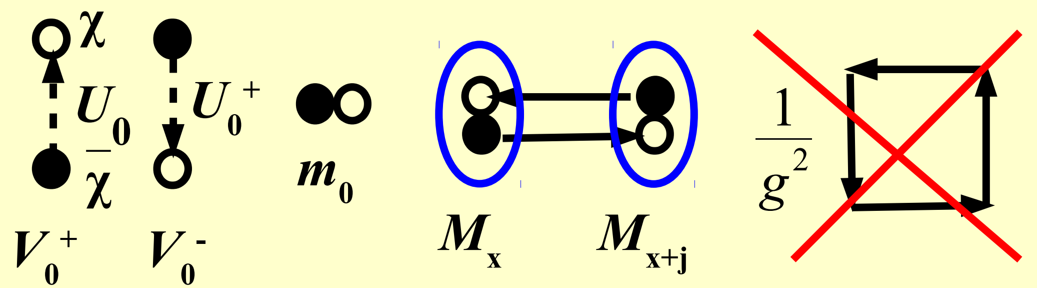
Linear potential between heavy-quarks
→ Confinement (Wilson, 1974)

Link integral → Effective action

- **Effective action in the strong coupling limit (SCL)**
  - **Ignore plaquette action ($1/g^2$)**
    → We can integrate each link independently !
  - **Integrate out *spatial* link variables of min. quark number diagrams (1/d expansion)**

$$S_{\mathrm{eff}} = S_F^{(t)} - \frac{1}{4N_c}\sum_{x,j} M_x M_{x+\hat{j}} + m_0 \sum_x M_x \quad \left(M_x = \bar{\chi}_x \chi_x\right)$$

Damgaard, Kawamoto, Shigemoto ('84)

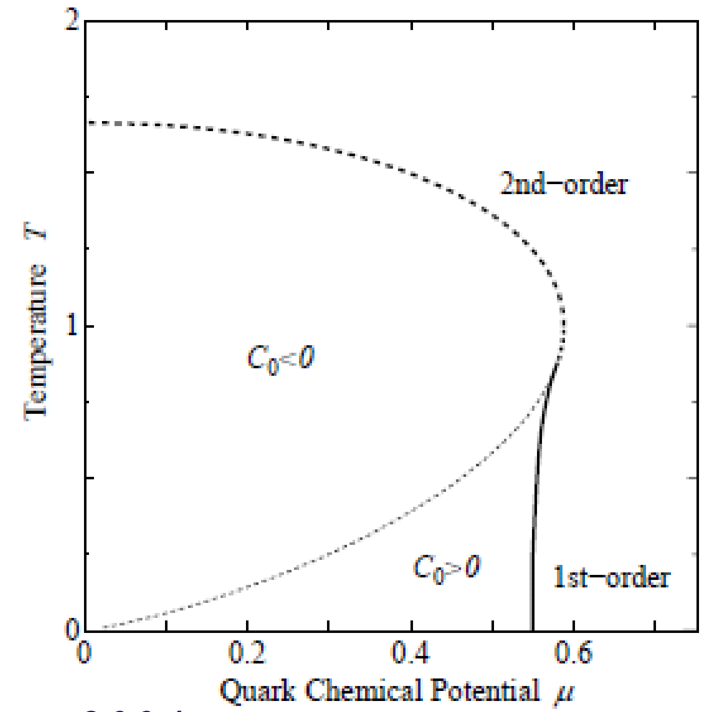


$$\int dU\, U_{ab} U^{+}_{cd} = \delta_{ad}\delta_{bc}/N_c$$

Lattice QCD in SCL
→ Fermion action
with nearest neighbor
four Fermi interaction

Phase diagram in SC-LQCD (mean field)

- "Standard" simple procedure in Fermion many-body problem
  - Bosonize interaction term (Hubbard-Stratonovich transformation)
  - Mean field approximation (constant auxiliary field)
  - Fermion & temporal link integral
    Damgaard, Kawamoto, Shigemoto ('84); Faldt, Petersson ('86); Bilic, Karsch, Redlich ('92); Fukushima ('04); Nishida ('04)



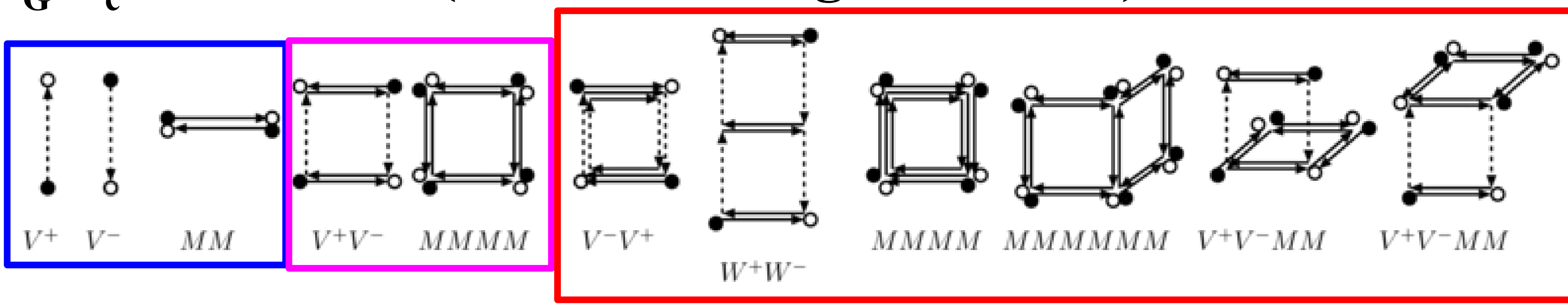
Fukushima, 2004

Finite Coupling Effects

- **Effective Action with finite coupling corrections**
  Integral of $\exp(-S_G)$ over spatial links with $\exp(-S_F)$ weight → S_{eff}

$$S_{\text{eff}} = S_{\text{SCL}} - \log\langle \exp(-S_G)\rangle = S_{\text{SCL}} - \sum_{n=1} \frac{(-1)^n}{n!}\langle S_G^n \rangle_c$$

$\langle S_G^n \rangle_c$ =Cumulant (connected diagram contr.) *c.f. R.Kubo('62)*



$$S_{\text{eff}} = \frac{1}{2}\sum_x (V_x^+ - V_x^-) - \frac{b_\sigma}{2d}\sum_{x,j>0}[MM]_{j,x}$$

SCL (Kawamoto-Smit, '81)

$$+\frac{1}{2}\frac{\beta_\tau}{2d}\sum_{x,j>0}[V^+V^- + V^-V^+]_{j,x} - \frac{1}{2}\frac{\beta_s}{d(d-1)}\sum_{x,j>0,k>0,k\neq j}[MMMM]_{jk,x}$$

NLO (Faldt-Petersson, '86)

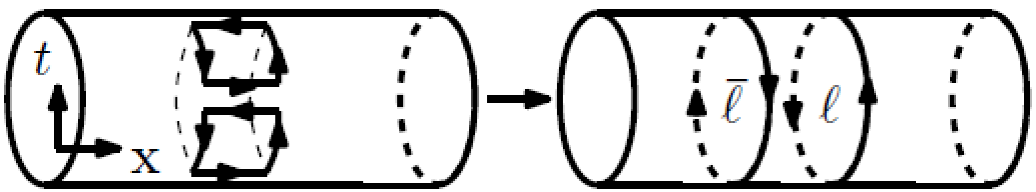
$$-\frac{\beta_{\tau\tau}}{2d}\sum_{x,j>0}[W^+W^- + W^-W^+]_{j,x} - \frac{\beta_{ss}}{4d(d-1)(d-2)}\sum_{\substack{x,j>0,|k|>0,|l|>0\\ |k|\neq j,|l|\neq j,|l|\neq|k|}}[MMMM]_{jk,x}[MM]_{j,x+\hat{l}}$$

$$+\frac{\beta_{\tau s}}{8d(d-1)}\sum_{x,j>0,|k|\neq j}[V^+V^- + V^-V^+]_{j,x}\left([MM]_{j,x+\hat{k}} + [MM]_{j,x+\hat{k}+\hat{0}}\right)$$

NNLO (Nakano, Miura, AO, '09)

Polyakov loop effects in SC-LQCD

- **Polyakov Loop**



$$P=\frac{1}{N_c}\,\mathrm{tr}\,L\,,\qquad L=T\exp\left[-i\int_0^{\beta}dx_4\,A_4\right]=T\prod_{\tau=1}^{N_\tau}U_0(\tau,\boldsymbol{x})$$

- **Order parameter of the deconfinement transition in the heavy quark mass limit.**
  A.M. Polyakov, PLB72('78),477; L. Susskind, PRD20('79)2610; B. Svetitsky, Phys.Rept.132('86),1.
- **Interplay between PL and χ cond. is known to be important in effective models**
  A. Gocksch, M. Ogilvie, PRD31(85)877; K. Fukushima, PLB591('04),277.

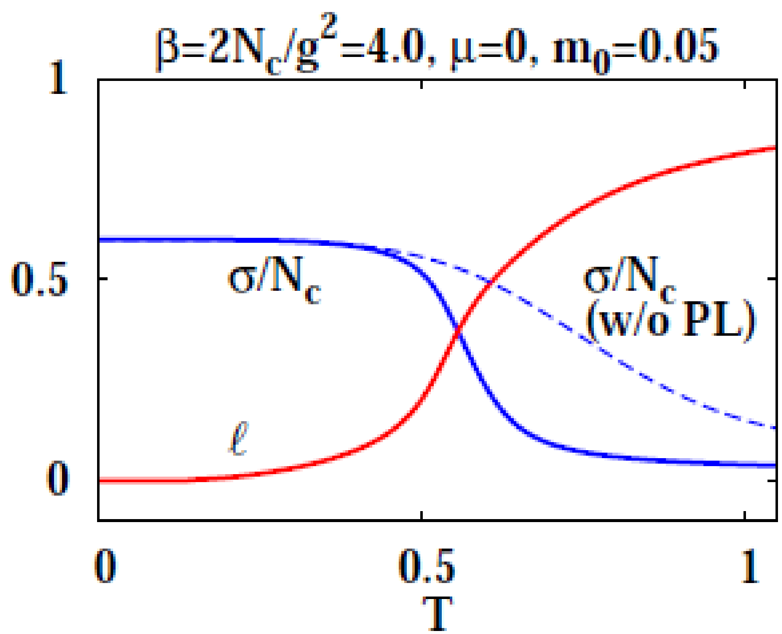
Polyakov loop appears in higer-order of $1/g^2$, but definitely affect QCD phase transition.

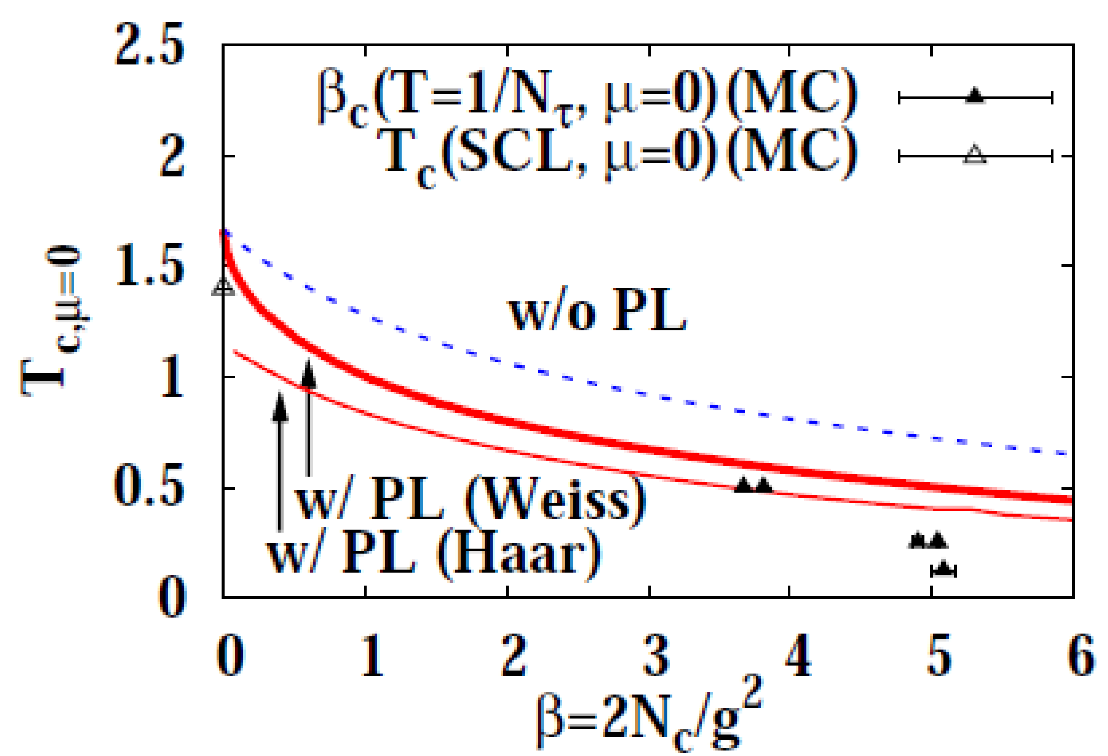
SC-LQCD with Polyakov Loop Effects at μ=0

T. Z. Nakano, K. Miura, AO, PRD 83 (2011), 016014 [arXiv:1009.1518 [hep-lat]]

- **P-SC-LQCD reproduces $T_c(\mu=0)$ in the strong coupling region ($\beta = 2N_c/g^2 \le 4$)**

MC data: SCL (Karsch et al. (MDP), de Forcrand, Fromm (MDP)), N_τ =2 (de Forcrand, private), N_τ=4 (Gottlieb et al.('87), Fodor-Katz ('02)), N_τ =8 (Gavai et al.('90))

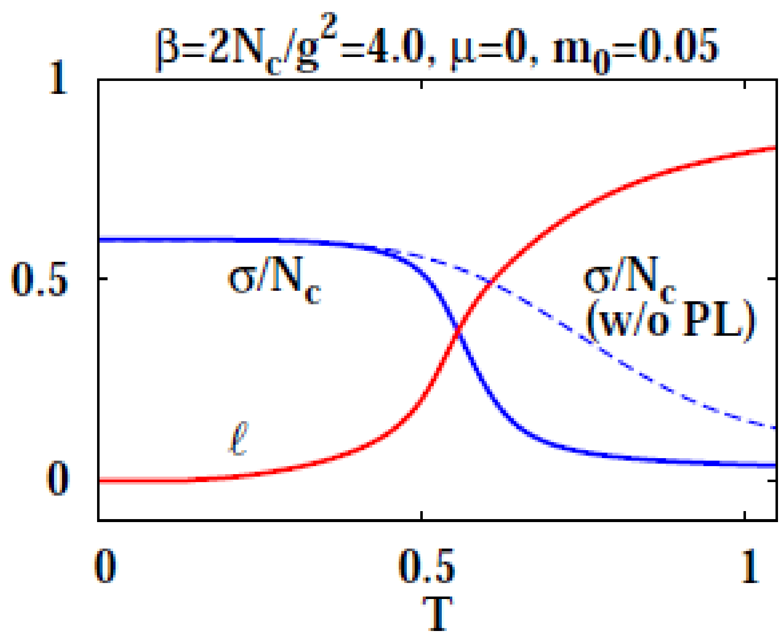


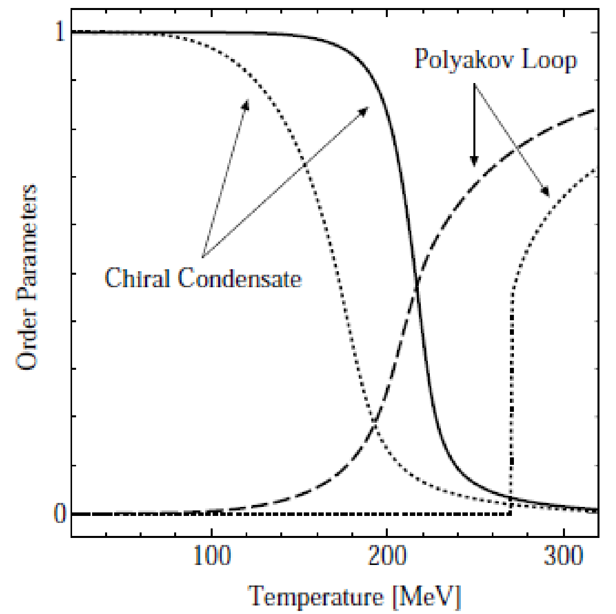


Lattice Unit

Polyakov loop effects on Tc

- **Comparison of Polyakov loop in SC-LQCD and PNJL**
  - **SC-LQCD: Tc decreases with Polyakov loop (Polyakov loop deconfines hadrons)**
  - **PNJL: Tc increases with Polyakov loop (Polyakov loop confine quarks)**





Fukushima ('04)

Beyond the mean field approximation

- **Constant auxiliary field → Fluctuating auxiliary field**

$$S_{\mathrm{eff}} = S_F^{(t)} + \sum_x m_x M_x + \frac{L^3}{4N_c} \sum_{\boldsymbol{k},\tau} f(\boldsymbol{k}) \left[|\sigma_{\boldsymbol{k},\tau}|^2 + |\pi_{\boldsymbol{k},\tau}|^2 \right]$$

$$m_x = m_0 + \frac{1}{4N_c} \sum_j \left((\sigma + i\varepsilon\pi)_{x+\hat{j}} + (\sigma + i\varepsilon\pi)_{x-\hat{j}} \right)$$

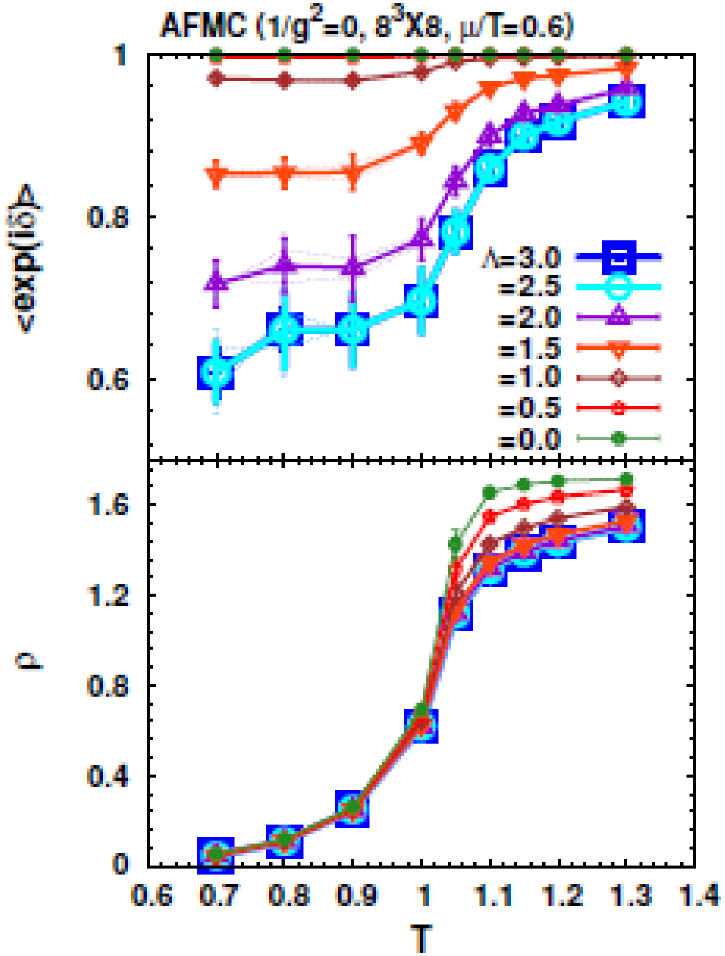
$$f(\boldsymbol{k}) = \sum_j \cos k_j \quad <, \qquad \varepsilon = (-1)^{x_0 + x_1 + x_2 + x_3}$$

  - **Auxiliary fields can be integrated out using MC technique (Auxiliary Field Monte-Carlo (AFMC) method)**
    - **Another method: Monomer-Dimer-Polymer simulation**
      Mutter, Karsch ('89), de Forcrand, Fromm ('10), de Forcrand, Unger ('11)
  - **Bosonization of "repulsive" mode: Extended HS transf.**
    → Introducing " *i* " leads to the complex Fermion determinant.
    Miura, Nakano, AO (09), Miura, Nakano, AO, Kawamoto (09)
  - **Weight cancellation from low momentum modes is small, due to the ε factor.**

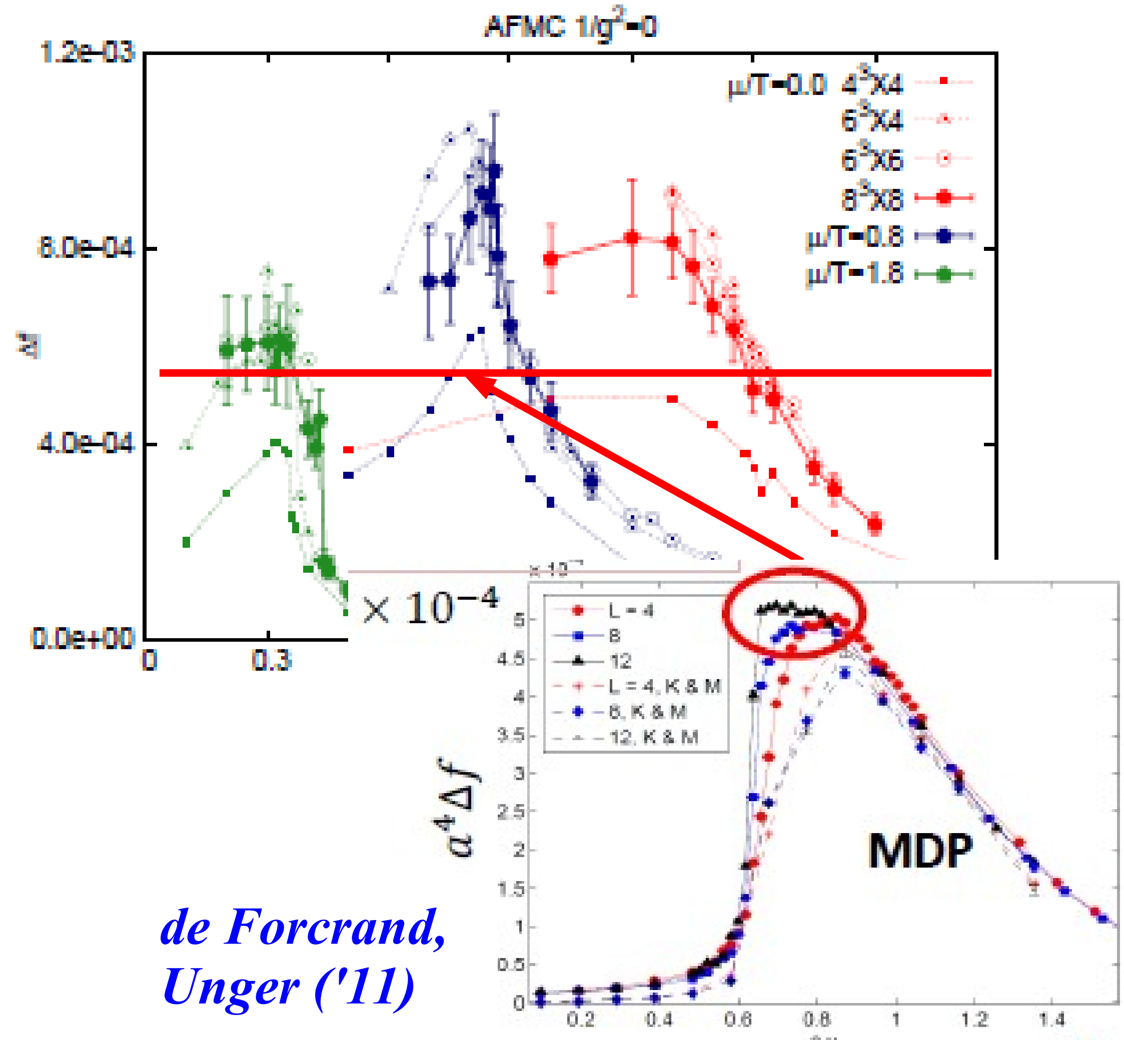
How serious is the weight cancellation ?

- **Statistical weight cancellation in AFMC**

$$\langle \exp(i\delta) \rangle \equiv \exp(-\Omega \Delta f) \ , \quad \Omega = \text{space-time volume}$$



cut-off $\sum_j \sin^2 k_j > \Lambda$



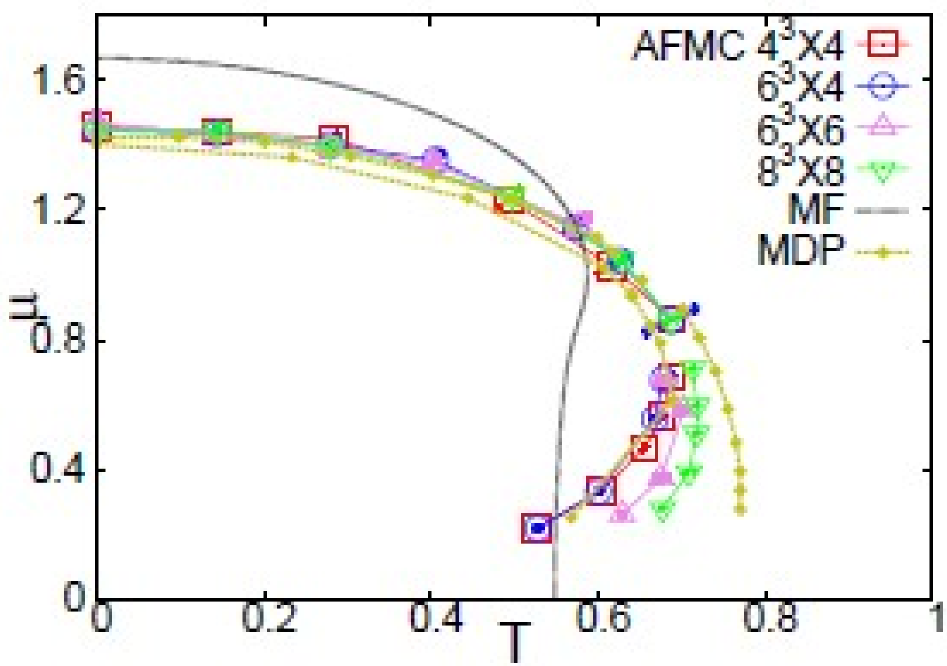
de Forcrand, Unger ('11)

Ichihara, Nakano, AO, PoS Lattice2013 (2013), to appear

Phase diagram

- **AFMC phase diagram**
  - **Reduction of T_c at μ=0 and enlarged hadron phase at medium T compared with the mean field results.**
  - **Quantitatively consistent with MDP simulation, if extrapolated to Nτ → ∞**
    de Forcrand, Fromm ('09); de Forcrand, Unger ('11)
  - **Spatial size dependence is small.**

→ Final answer to the phase boundary in the strong coupling limit !



Ichihara, Nakano, AO, PoS Lattice2013 (2013), to appear

Summary

- **QCD to nuclei is the main missing link to describe our world starting from the standard model. The sign problem in Monte-Carlo simulation of lattice QCD.**
- **Strong coupling lattice QCD is a promising tool in finite density lattice QCD.**
  - **Strong coupling limit + finite coupling correction + Polyakov loop → MC results of Tc is roughly reproduced.**
- **Sign problem is solved in the strong coupling limit !**
  - **Two independent methods show the same phase boundary, and the spatial size dependence is small. (Monomer-dimer-polymer simulation, Auxiliary field MC)**
- **Challenge**
  - **Finite coupling + Fluctuations (Exact integral), Different type of Fermion, 1/d expansion.**

Thank you

Chiral Angle Fixing

How can we simulate correct thermodynamic chiral limit ?