Entropy production in the classical Yang-Mills theory as the coherent state dynamics

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in collaboration with
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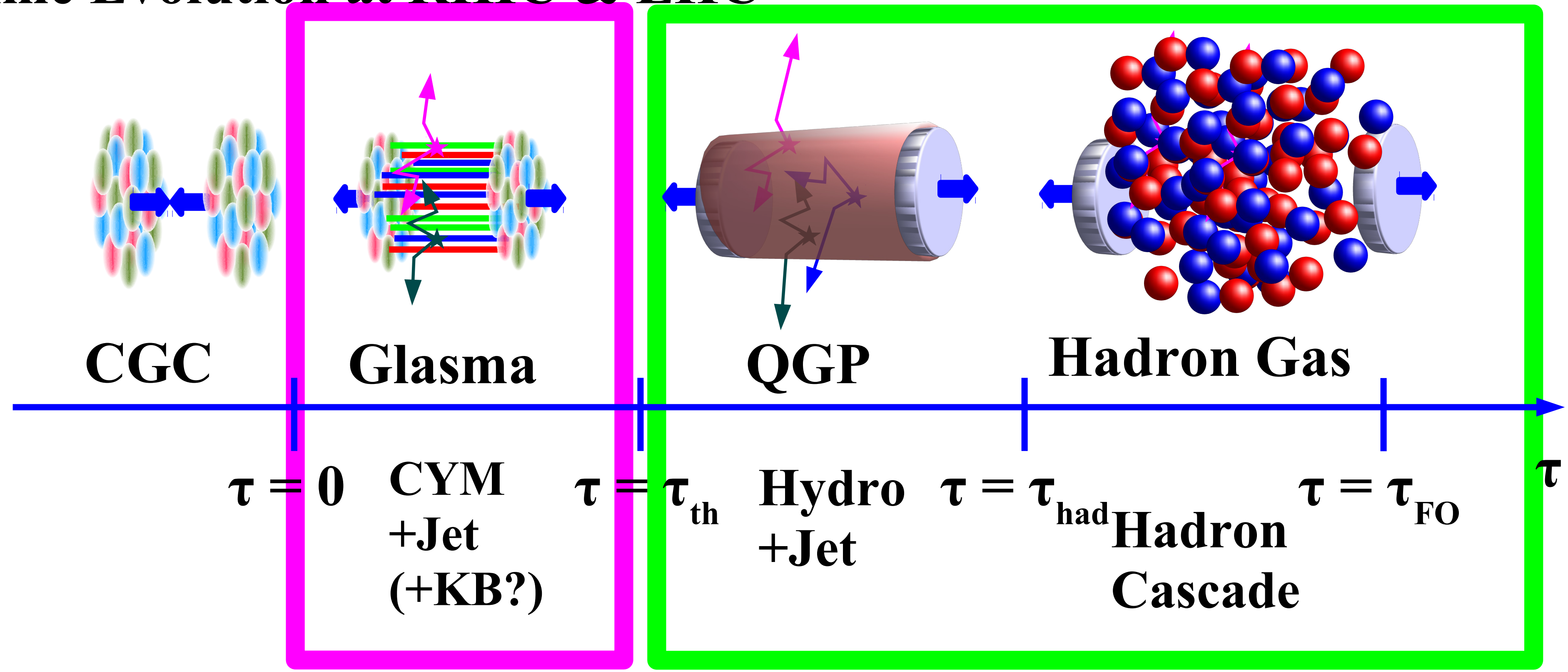
The Approach to Equilibrium in Strongly Interacting Matter
April 2-4, 2014, BNL

Based on the works,

- *Towards a Theory of Entropy Production in the Little and Big Bang*, T. Kunihiro, B. Müller, A. Ohnishi, A. Schäfer, Prog. Theor. Phys. 121 ('09), 555 [arXiv:0809.4831].
- *Chaotic behavior in classical Yang-Mills dynamics*, T. Kunihiro, B. Müller, A. Ohnishi, A. Schäfer, T. T. Takahashi, A. Yamamoto, Phys. Rev. D 82 (2010), 114015 [arXiv:1008.1156].
- *Entropy production in classical Yang-Mills theory from Glasma initial conditions*, H. Iida, T. Kunihiro, B. Müller, A. Ohnishi, A. Schäfer, T. T. Takahashi, Phys. Rev. D 88 (2013), 094006 [arXiv:1304.1807].
- H. Iida, T. Kunihiro, A. Ohnishi, T.T. Takahashi, arXiv:1404.xxxx

Thermalization in High-Energy Heavy-Ion Collisions

- **Time Evolution at RHIC & LHC**



Theor. Challenges

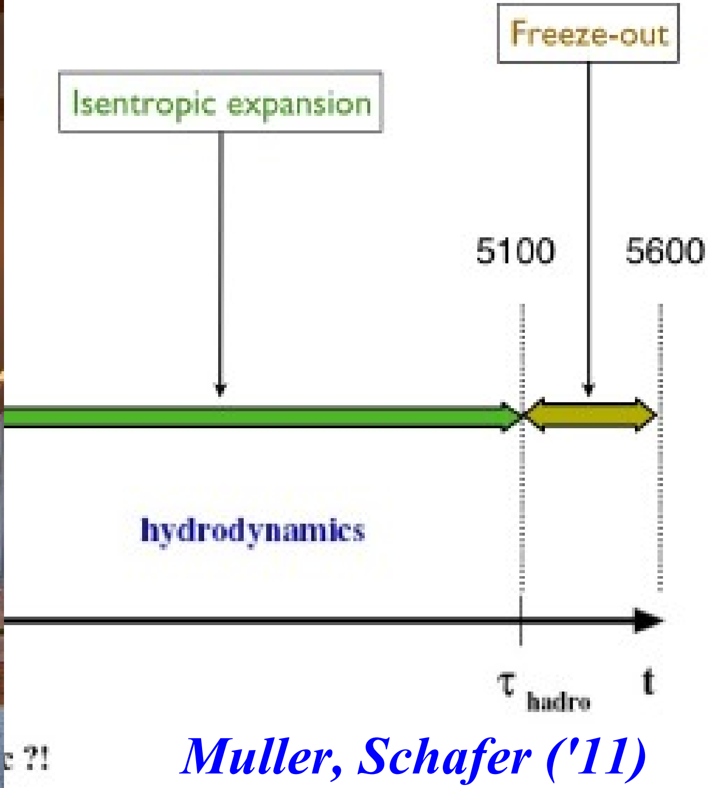
- Thermalization under dynamical classical field
- Theoretically interesting and Phenomenologically important. dN/ dη, init. cond. of hydro.

Phen. Challenges

flow, jet, hard probes
→ hydro., transport coef., E-loss, hadron prop., phase diagram, ...

Entropy Production in Heavy-Ion Collisions

- **We have been working on entropy production in non-equilibrium stage, since the international Molecule-type workshop on "Entropy Production before QGP" (2008.08.01-28)**
  (A. Schafer, R. Fries, B. Mueller, M. Strickland, T. Schafer, M. Natsuume, Y. Nara, T. Hirano, K. Fukushima, T. Kunihiro, AO)



Muller, Schafer ('11)

Entropy production in Classical Yang-Mills dynamics

- **Perturbative estimate of thermalization time is longer than expected from hydrodynamics simulations.**
  → Classical Yang-Mills field is expected to play a role of entropy prod.
  Baier, A.Mueller, Schiff, Son ('01); Chatterjee, Srivastava ('09), Heinz, Kolb ('02)

- **How does CYM field have entropy ? Chaos & Decoherence !**
  - **Entropy from chaoticity**
    - **(Husimi-)Wehrl entropy Kunihiro, B. Müller, AO, Schäfer ('09)**

      $$S_{Wehrl} = -\int \frac{d^n x\, d^n p}{(2\pi)^n} H \log H \quad (H = \text{phase space prob. fn.})$$

    - **Entropy production rate = Kolmogorov-Sinaii entropy**
      B. Muller, Trayanov ('92), Biro, Gong, B. Muller ('94), Bolte, B. Muller ('00). Kunihiro, B. Müller, AO, Schäfer, Takahashi, Yamamoto ('10), Iida, Kunihiro, B.Müller, AO, Schäfer, Takahashi ('13)
  - **Decoherence entropy**
    B.Muller, Schafer ('03, '06), Fries, B. Muller, Schafer ('09), Iida, Kunihiro, AO, Takahashi ('14)
  - **Classical Statistical simulation**
    Berges, Scheffler, Sexty ('08), Epelbaum, Gelis ('13)

Contents

We discuss the CYM entropy and its production rate with emphasis on the decoherence entropy

- **Introduction**
- **Entropy production from chaotic nature of CYM**
  - **Chaoticity, Lyapunov exponent, and Kolmogorov-Sinaii entropy**
  - **KS entropy from CYM**
    T. Kunihiro, B. Müller, AO, A. Schäfer, T.T. Takahashi, A. Yamamoto, PRD82('10),114015 [arXiv:1008.1156].
    H.Iida, T.Kunihiro, B.Müller, AO, A.Schäfer, T.T.Takahashi, PRD88('13),094006 [arXiv:1304.1807].
- **CYM as a coherent state and decoherence entropy**
  - **Decoherence entropy**
  - **CYM as a coherent state**
  - **Decoherence entropy from CYM dynamics**
    H.Iida, T.Kunihiro, AO, T.T.Takahashi, arXiv:1404.xxxx (to be submitted soon).
- **Summary**

Entropy production from chaotic nature of CYM

Chaoticity, Lyapunov exponent, and KS entropy

- **Entropy in classical dynamics = Wehrl entropy**

$$S = -\int d\Gamma \, H \log H$$

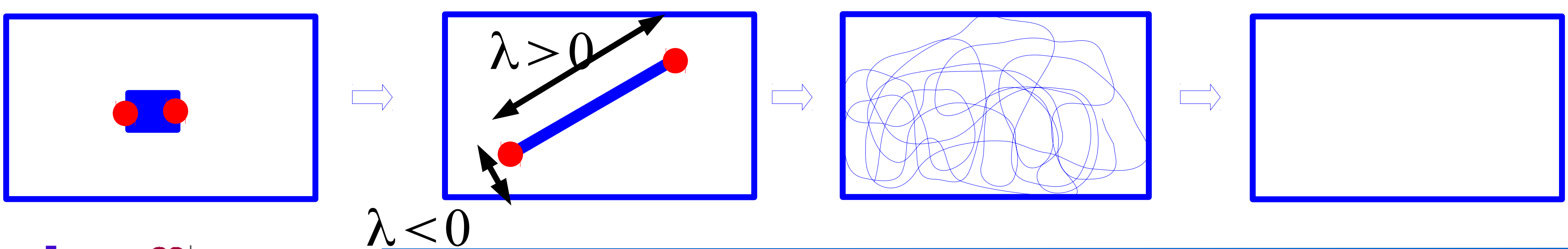
($d\Gamma$= dx dp = phase space, H= phase space dist. fn., e.g. Husimi fn.)

- **Lyapunov exponent and Kolmogorov-Sinaii entropy**

$$\delta X_i(t) = \delta X_i(t_0) \exp[\lambda_i (t - t_0)] \quad (X = (x, p)),$$

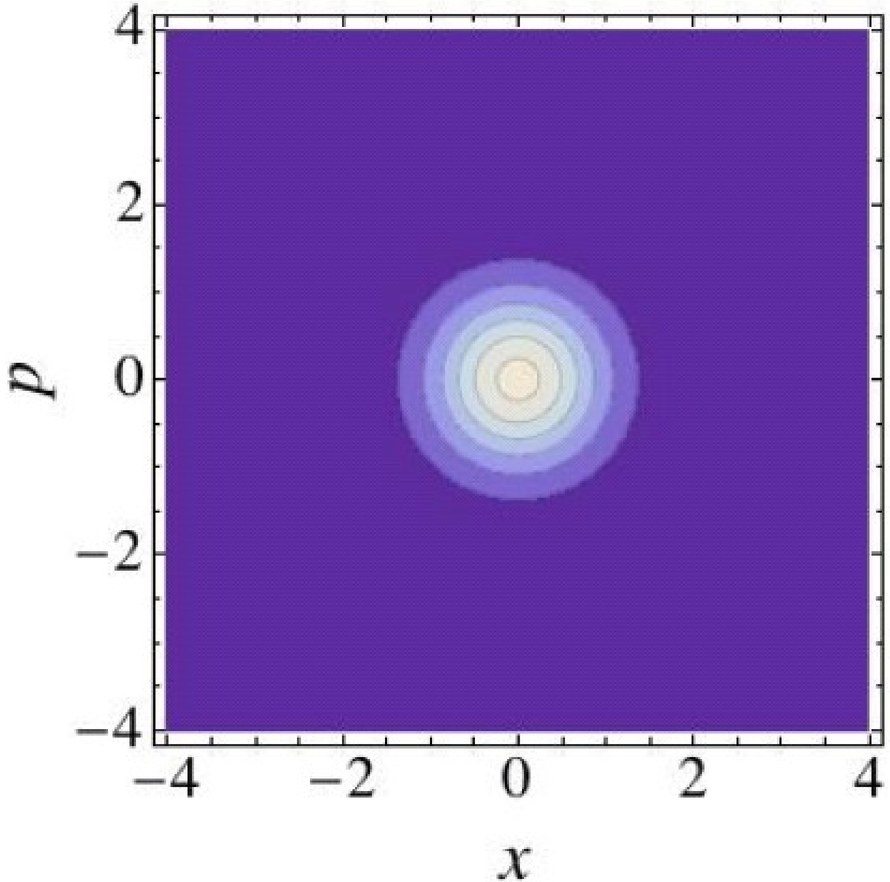
$$dS/dt = S_{\rm KS} \equiv \sum_{i, \lambda_i > 0} \lambda_i$$

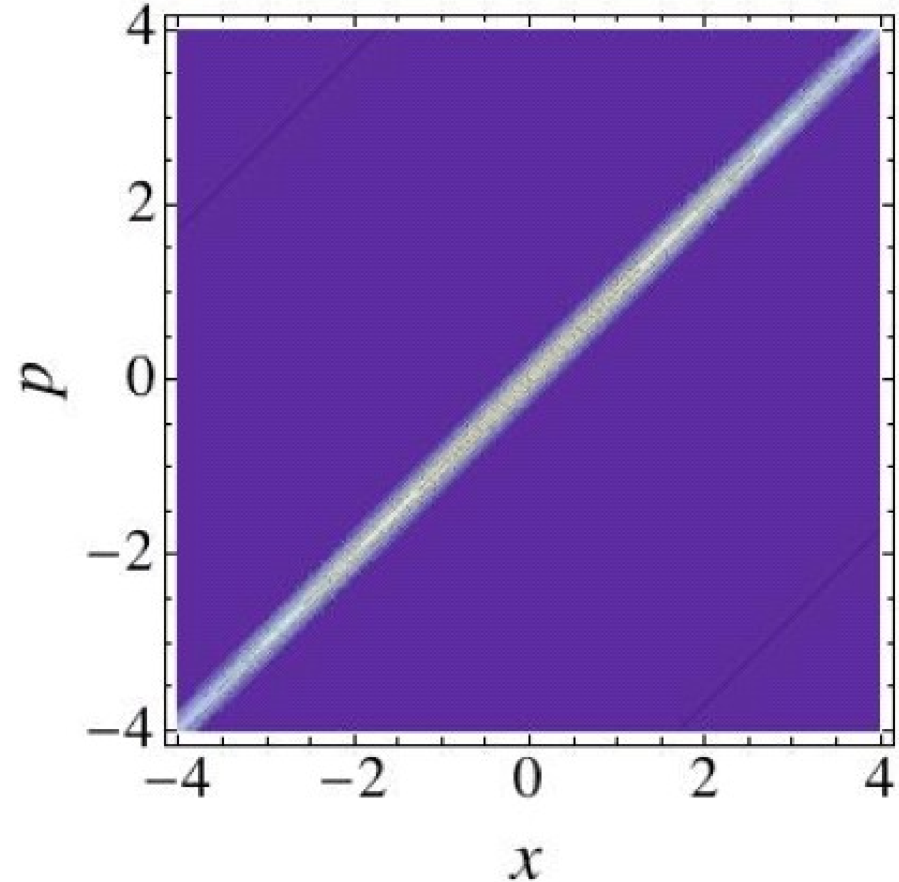
  - **δ X = difference of two trajectories from adjacent initial conditions**
    λ = initial state sensitivity (Lyapunov exponent, measure of chaoticity)
  - **When λ > 0, exponentially growing number of phase space cells are visited**
    → phase space dist. fn. becomes smooth after proper coarse graining
    → entropy production (Kolmogorov-Sinaii entropy)



Evolution of the Wigner Function

- Liouville theorem → conservation of the phase space volume
  - Exponential growth in (x+p/λ), Exponential narrowing in (x-p/λ)





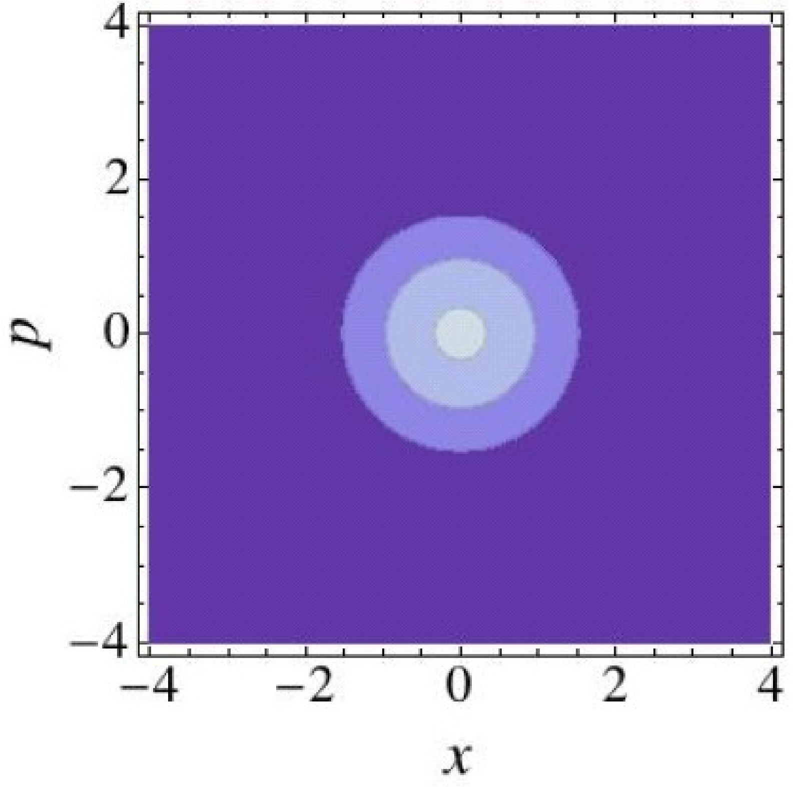
Kunihiro, Müller, AO, Schäfer ('09)

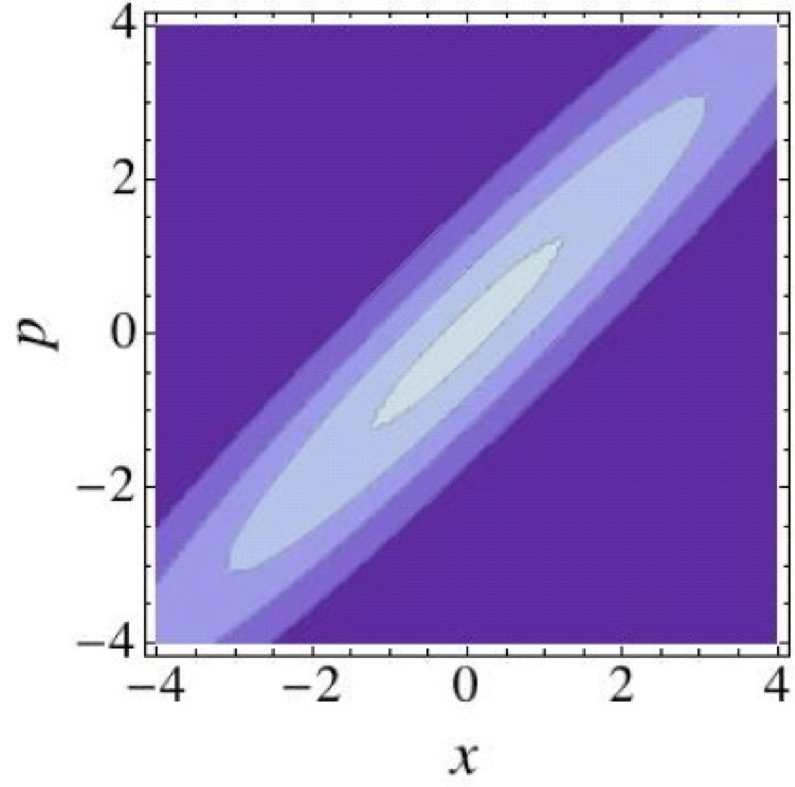
$$\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 - \frac{1}{2}\lambda^2\hat{x}^2$$

λ=1, λt=0, 2

Evolution of the Husimi Function

- **Coherent state broadening of phase space**
  - **Minimum width in (x-p/λ) → phase space dist. func. is smeared !**





Kunihiro, Müller, AO, Schäfer ('09)

$$\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 - \frac{1}{2}\lambda^2\hat{x}^2$$

λ=1, λt=0, 2

Classical Yang-Mills dynamics on the lattice

- **Lattice CYM Hamiltonian in temporal gauge (A_0=0) in the lattice unit**

$$H=\frac{1}{2}\sum_{x,a,i}\left[E_i^a(x)^2+B_i^a(x)^2\right]$$

$$F_{ij}^a(x)=\partial_i A_j^a(x)-\partial_j A_i^a(x)+\sum_{b,c} f^{abc} \boxed{A_i^b(x) A_j^c(x)}=\varepsilon_{ijk} B_k^a(x)$$

Non-linear & coupling

- **Non-compact (A, E) form !**
  - **Demerit: Gauge invariance is not fully satisfied at finite lattice spacing.**
  - **Merit: Easy to consider the coherent state, and conformality is manifest.**
- **Initial conditions ($E_i^a(x)$=0 is assumed here.)**
  - **Random initial condition: $A_i^a(x)$ = random in [-η, η],**
  - **Modulated init. cond.:** $A_i^a(\vec{r}) = \delta_{i2}\left[\epsilon_1 \sin\left(\frac{2x\pi}{N_x}\right) + \epsilon_2 \sin\left(\frac{2nx\pi}{N_x}\right)\sin\left(\frac{2mz\pi}{N_z}\right)\right]$
  - **Constant-A init. cond.** $A_i^a(x)=\sqrt{B/g}\left(\delta_{i2}\delta^{a3}+\delta_{i3}\delta^{a2}\right)$ *Berges et al.('12)*

magnetic field ~ z direction ($\varepsilon_1 >> \varepsilon_2$), w and w/o fluc.

How to obtain Lyapunov exponents

- **EOM of δX → Integral**

$$\dot{X}=\begin{pmatrix}\dot{x}\\ \dot{p}\end{pmatrix}=\begin{pmatrix}H_p\\ -H_x\end{pmatrix} \rightarrow \delta\dot{X}=\begin{pmatrix}H_{px} & H_{pp}\\ -H_{xx} & -H_{xp}\end{pmatrix}\delta X\equiv\tilde{H}\delta X$$

$$\left(H_{px}\equiv\partial^2 H/\partial p\partial x \quad \text{etc}\right)$$

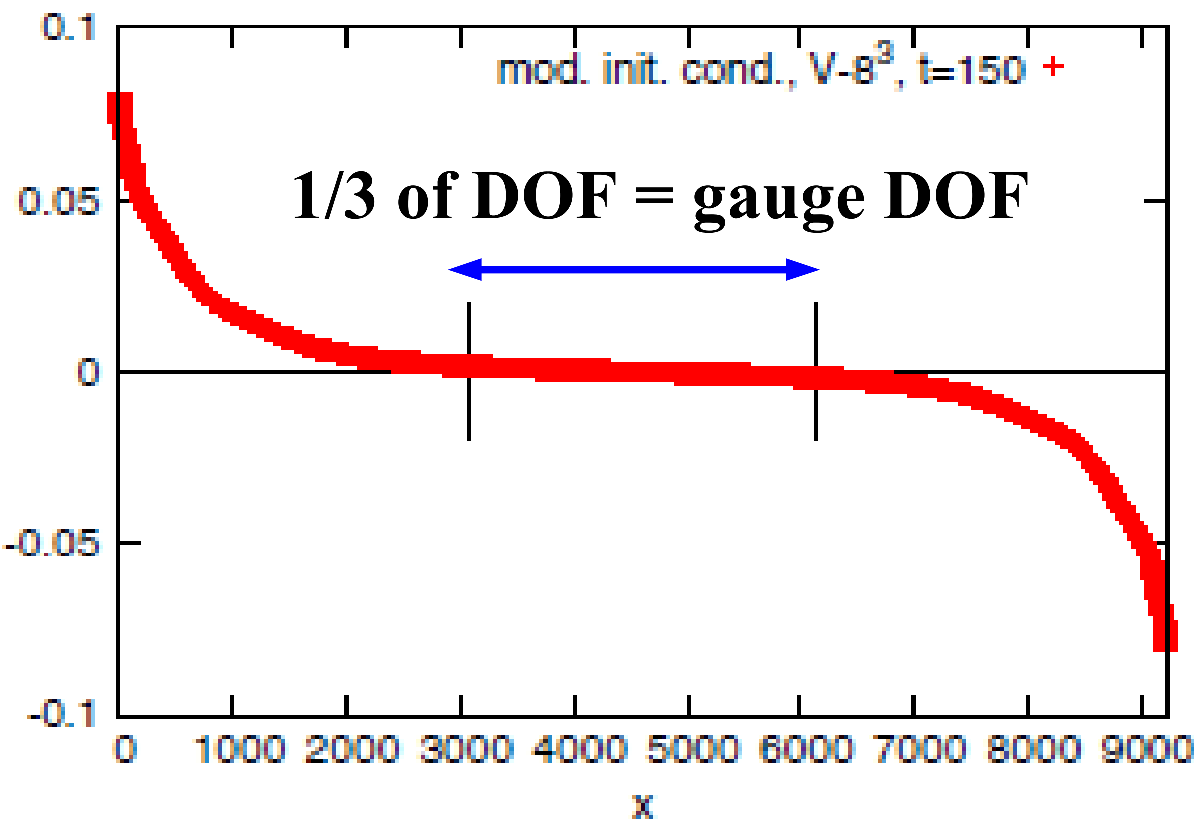
$$\delta X(t)=T\exp\left(\int_0^t dt'\,\tilde{H}(t')\right)\delta X(t=0)\simeq T\prod_{k=1,N}\left(1+\tilde{H}\Delta t\right)\delta X(t=0)$$

$$=U(0,t)\,\delta X(t=0)$$

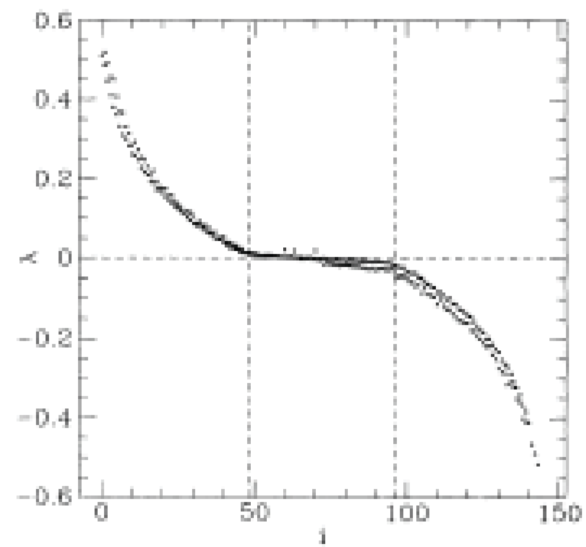
  - **Diagonalizing U and the eigen value becomes λt.**
  - **Matrix size = 3 (xyz) x (N_c^2-1) x L^3 x 2 (A,E)**

Typical Lyapunov spectrum

- **Sum of all Lyapunov exponent = 0 (Liouville theorem)**
- **1/3 Positive, 1/3 negative, and 1/3 zero (or pure imag.)**



Iida, Kunihiro, B.Müller, AO, Schäfer, Takahashi ('13)



cf) Lyapunov spectrum ($V=2^3$)
Gong, Phys.Rev.D49, 2642 (1994).

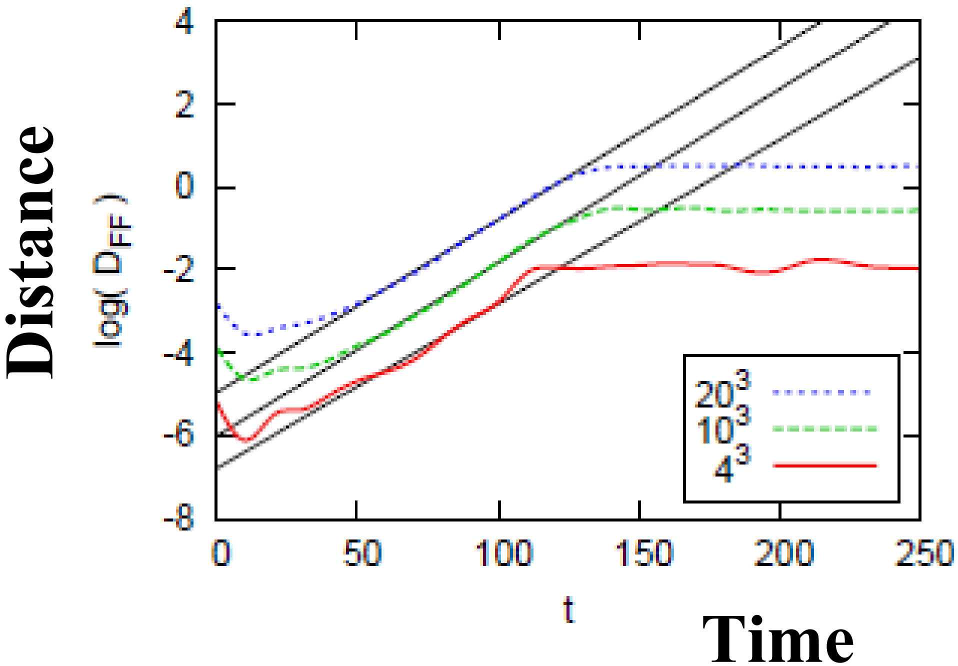
KS entropy in CYM from random init. cond.

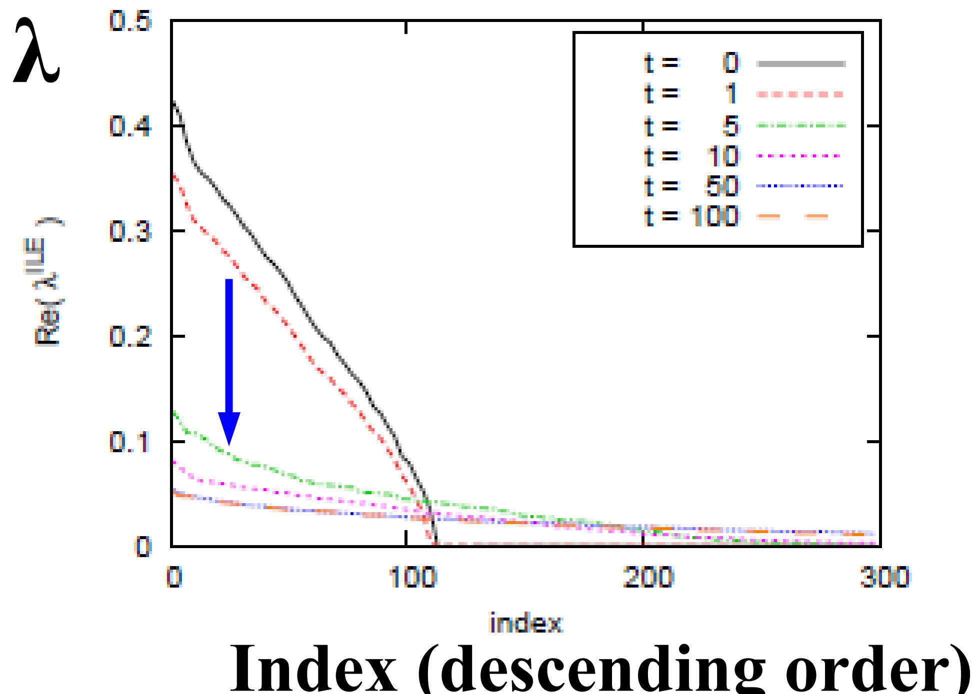
Kunihiro, Müller, AO, Schäfer, Takahashi, Yamamoto ('10)

- **Evolution of distance from adjacent init. cond.**
  - **Exponential growth of distance → Instability or Chaoticity**

$$D_{FF} = \sqrt{\sum_x \{\sum_{a,i,j} F^a_{ij}(x)^2 - \sum_{a,i,j} F'^a_{ij}(x)^2\}^2}$$

- **Lyapunov exponent distribution**
  - **Rapid spread of positive LEs and macroscopic # of positive LEs → Chaoticity**



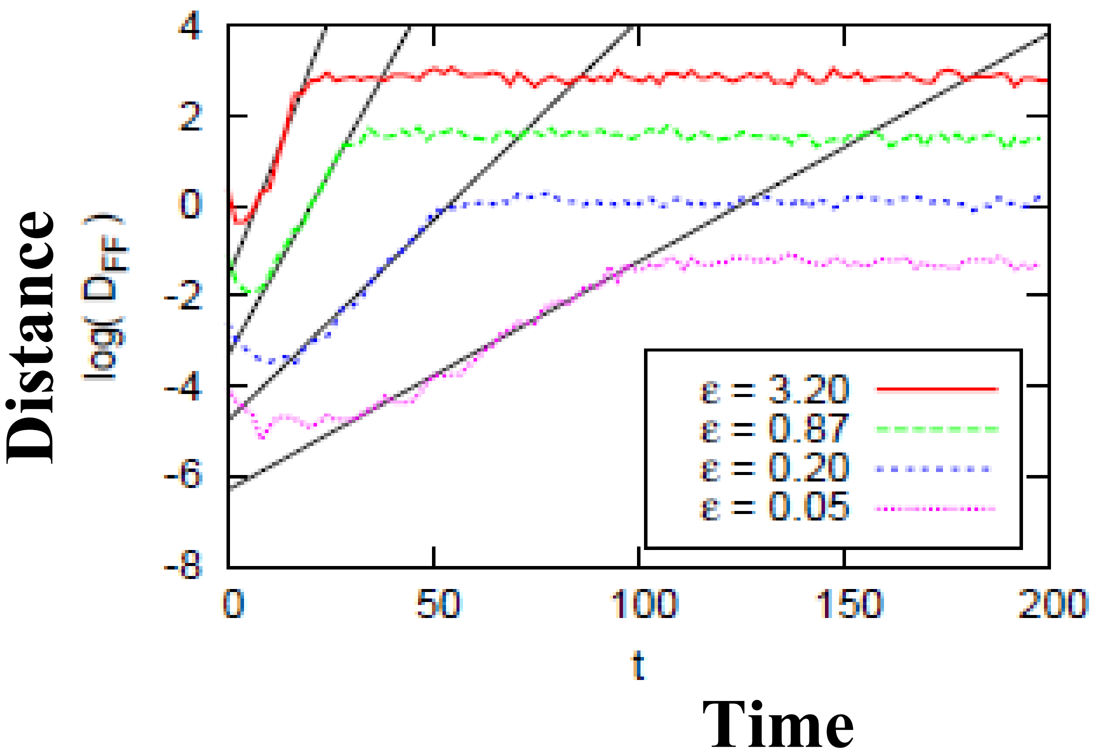


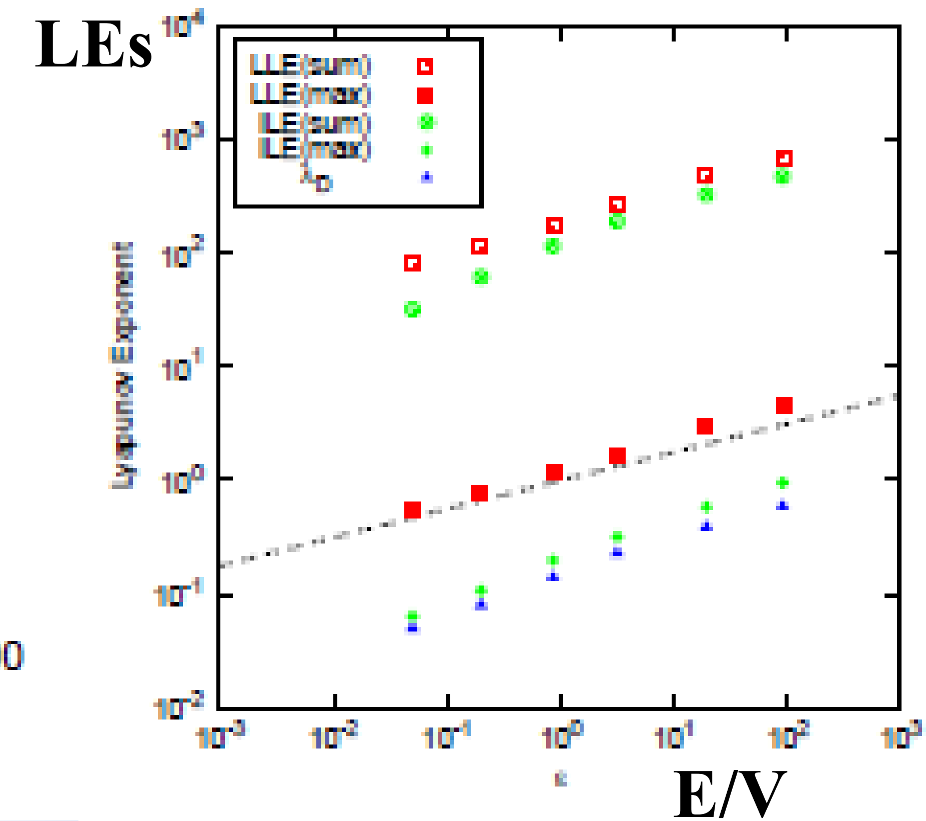
KS entropy in CYM from random init. cond.

Kunihiro, Müller, AO, Schäfer, Takahashi, Yamamoto ('10)

- **Energy density dependence**
  - **Larger energy → Larger Lyapunov exp.**
  - **CYM is conformal → $\lambda \propto \varepsilon^{1/4}$ (ε = E/V)**

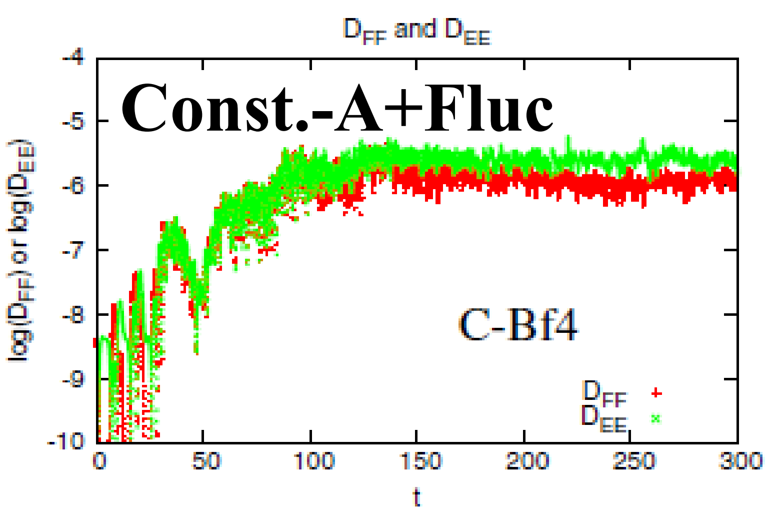
$$\frac{dS}{dt} = S_{KS} \sim c_{KS}\,\varepsilon^{1/4}\ , \quad c_{KS} \sim 2 \quad \text{(Lattice unit)}$$

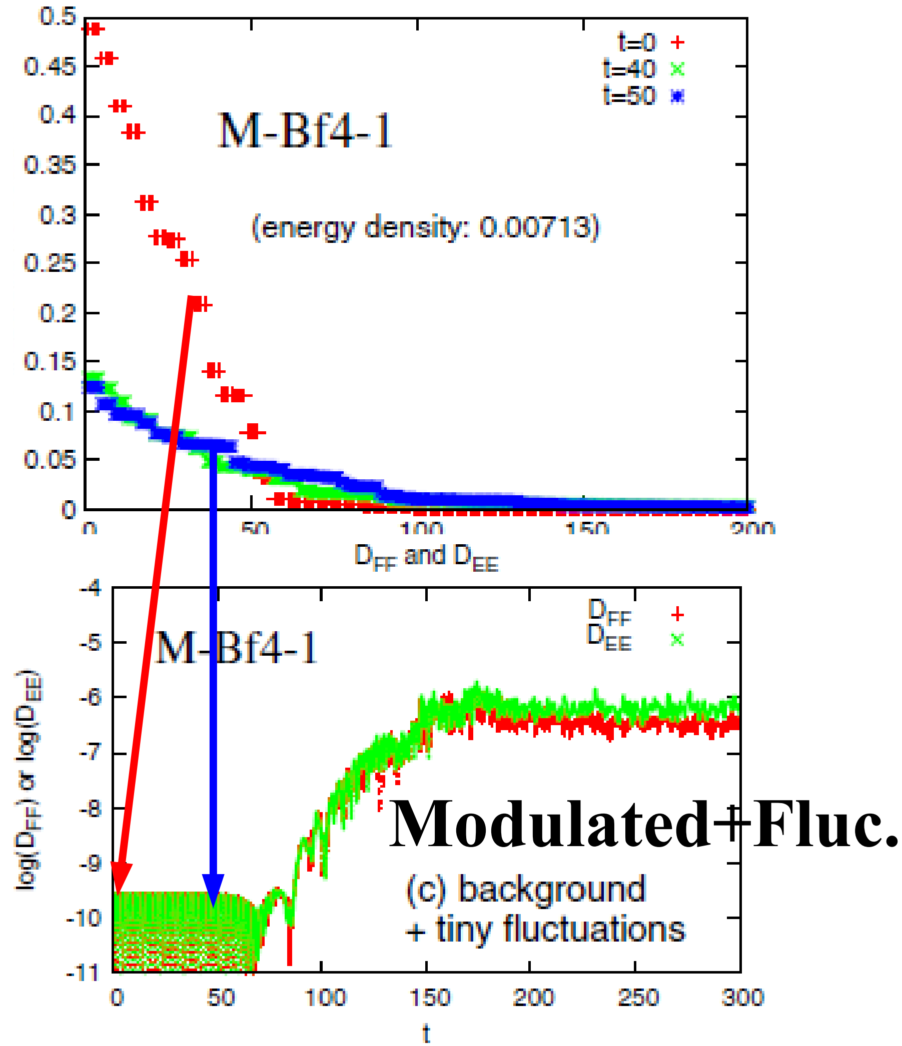




KS entropy in CYM from glasma-like init. cond.

- **Instability under strong color-magnetic field**
  Nielsen, Olesen ('78), Fujii, Itakura ('08), Berges, Scheffler, Schlichting, Sexty ('12)
  - **No chaotic behavior is observed with sine waves and constant-A w/o fluctuations.**
  - **Small fluctuations activate instability and chaoticity.**
  - **Chaoticity emerges after instability spreads to many modes.**





CYM as a coherent state and Decoherence entropy

Decoherence Entropy

Muller, Schafer ('03,'06), Fries, Muller, Schafer ('09), Iida, Kunihiro, AO, Takahashi ('14)

- **Coherent State**

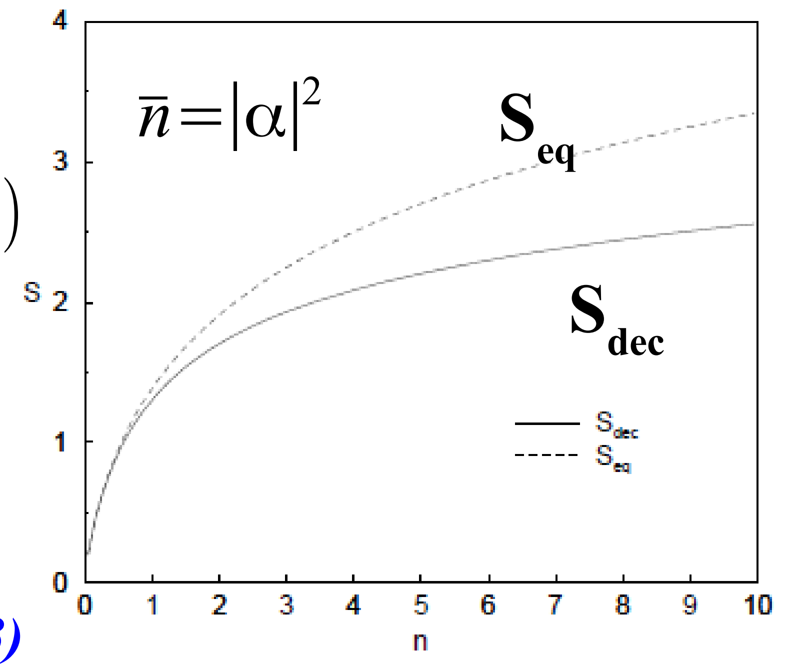
$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

$$|\alpha\rangle = N\exp(\alpha\hat{a}^{+})|0\rangle = \exp(-|\alpha|^2/2)\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

  - **n-quanta states are coherently superposed in a coherent state.**
  - **When this coherence is broken, entropy is generated (decoherence entropy)**

$$P_n = \frac{|\alpha|^{2n}}{n!}\exp(-|\alpha|^2) \quad \text{(Poisson dist.)}$$

$$\rightarrow \quad S_{\text{dec}} = -\sum_{n=0}^{\infty} P_n \log P_n > 0$$



Muller, Schafer ('03)

CYM as a Coherent State

Muller, Schafer ('03,'06), Fries, Muller, Schafer ('09), Iida, Kunihiro, AO, Takahashi ('14)

- **What kind of state does the CYM correspond to ?**
  → Natural guess = Coherent State

$$| \mathrm{CYM} \rangle \simeq \prod_{\boldsymbol{k},a,i} |\alpha_{\boldsymbol{k}ai}\rangle$$

- **Decoherence entropy from CYM**

$$S_{\mathrm{dec}} = -\sum_{\boldsymbol{k},a,i} \sum_{n} P_n(\alpha_{\boldsymbol{k}ai}) \log P_n(\alpha_{\boldsymbol{k}ai})$$

$$\alpha_{\boldsymbol{k}ai} = \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} \left[\omega_{\boldsymbol{k}} A_{ai}(\boldsymbol{k},t) + i E_{ai}(\boldsymbol{k},t) \right], \quad \omega_{\boldsymbol{k}} = \sqrt{\sin^2 k_x + \sin^2 k_y + \sin^2 k_z}$$

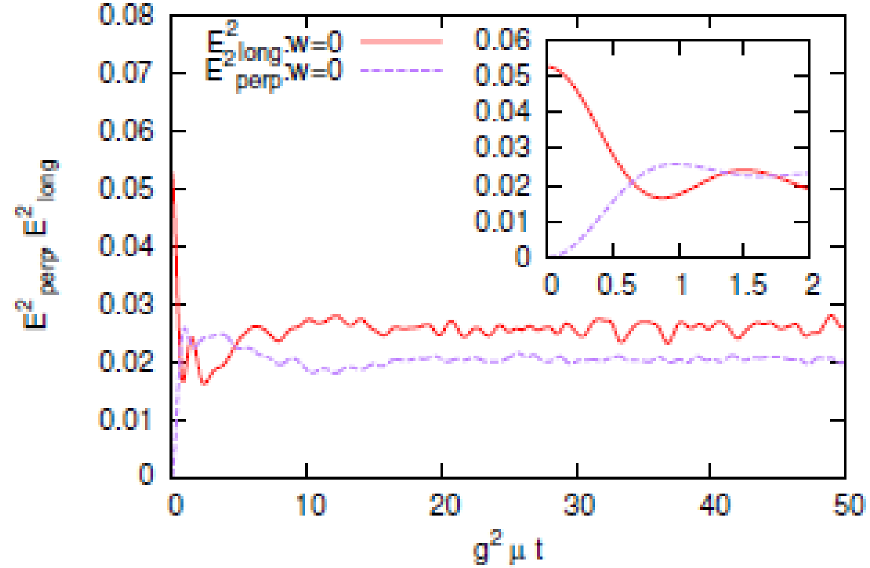
- **Is the above assignment unique ?**
  - **Coherent state in each "coherent domain"** *Fries, Muller, Schafer ('09)*
  - **Deviation from Poisson dist. with coupled oscillator**
    Glauber ('66), Gelis, Venugopalan ('06)

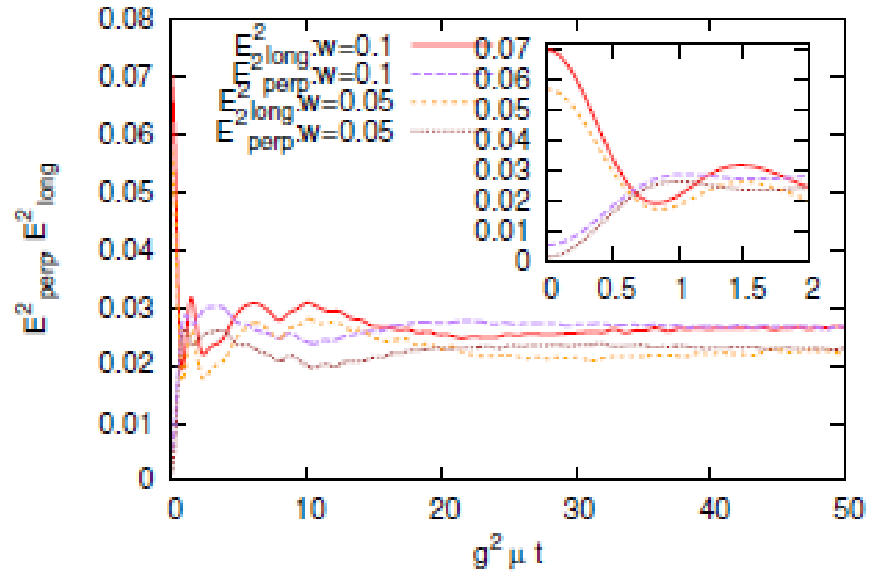
Initial Condition and Time Evolution

- **"Glasma-like" init. cond.**
  - **MV model (boost inv.)**
    + Longitudinal fluctuations

    → Bx,y, Ex,y, Bη, Eη
    McLerran, Venugopalan ('94), Romatschke, Venugopalan ('06), Fukushima, Gelis ('12)
  - **Non-expanding geometry is assumed, Substitute Bη and Eη in MV model into Bz and Ez at t=0.**
- **Time-evolution**
  - **Short time behavior of E^2 does not depend on the fluctuation strength. (and similar to expanding geo. results.)**
    E.g. Lappi, McLerran ('06)
  - **Long-time behavior:**
    Earlier "isotropization" in perp. and long. directions of E^2.

20^3 lattice





Iida, Kunihiro, AO, Takahashi ('14)

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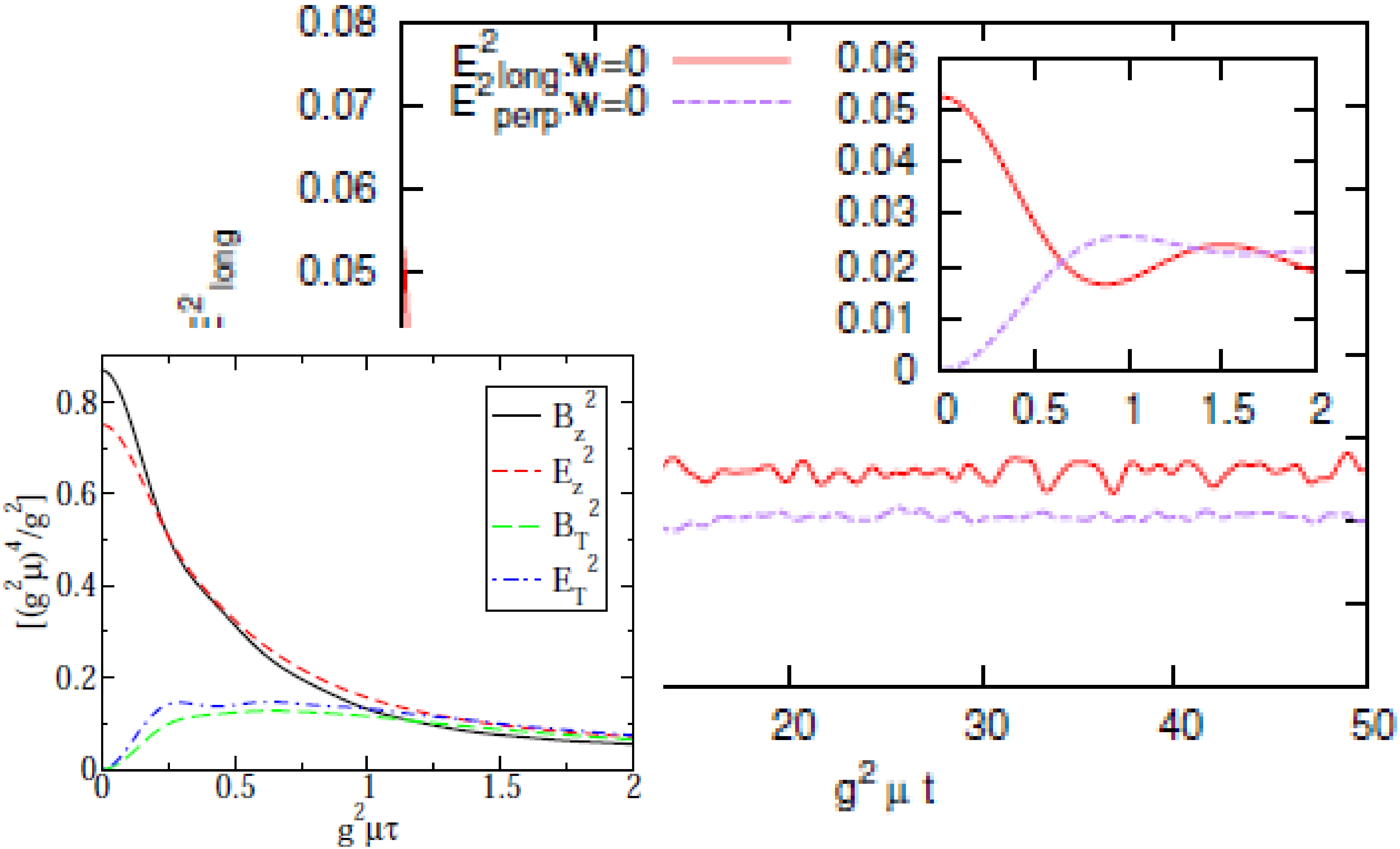


Lappi, McLerran ('06)

Iida, Kunihiro, AO, Takahashi ('14)

Physical Scale

- **Rough estimate: $L^2 \sim \pi R_{Au}^2 \rightarrow 1/a = g^2\mu \sim 0.32$ GeV ($a \sim 0.63$ fm)**
- **Thermal energy estimate** *Kunihiro et al.('10), Muller,Schafer('11)*

$$\varepsilon_{\mathrm{CYM}} = 2(N_c^2-1)\frac{T}{a^3}, \quad \varepsilon_{\mathrm{SB}} = 2(N_c^2-1)\frac{\pi^2}{30}T^4$$

  - **CYM energy density should not exceed Stefan-Boltzmann energy density in equilibrium.**

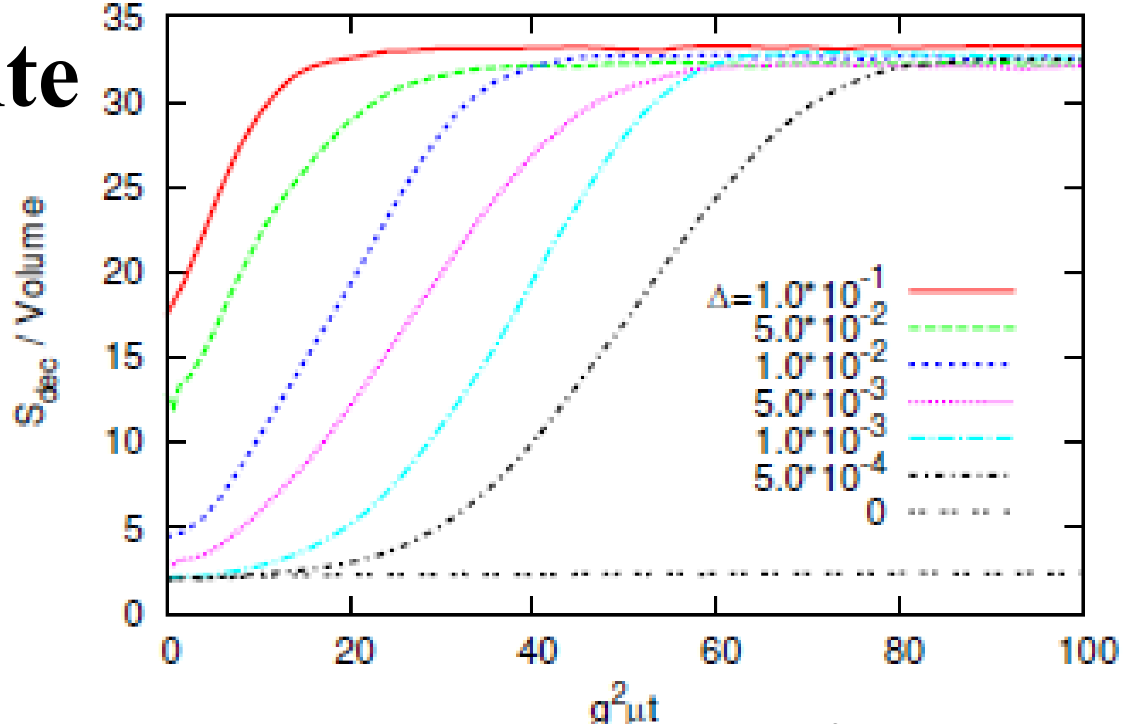
$$\varepsilon_{\mathrm{CYM}} < \varepsilon_{\mathrm{SB}}$$
$$\rightarrow \quad a > \frac{1}{T}\left(\frac{30}{\pi^2}\right)^{1/3} \sim \frac{1.4}{T}$$

This estimate gives
$a > 0.8$ fm (T=350 MeV),
$a > 0.6$ fm (T=500 MeV).

Decoherence Entropy of CYM

- **How about the decoherence entropy ?**
  - **$< \delta E^2>/<E^2> \sim 0.1$ (Δ=0.05) and 0.3 (Δ=0.1)**
  - **$S_{dec} \sim 2.3$ (Δ=0) and 33 (Δ=0.05, 0.1)**
  - **Entropy from initial state fluc. and chaoticity**
  - **No long. fluc. results in 2D (pz=0 mode) entropy, while 3D entropy is realized with finite long. fluc. (non-zero Δ).**

S_{dec}/site



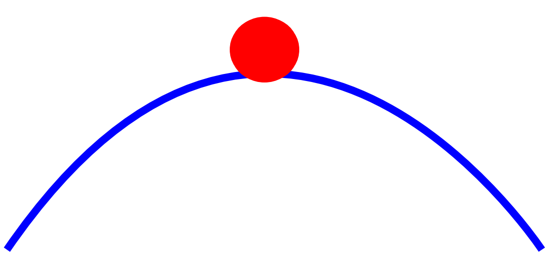
Time

Decoherence Entropy Production Rate

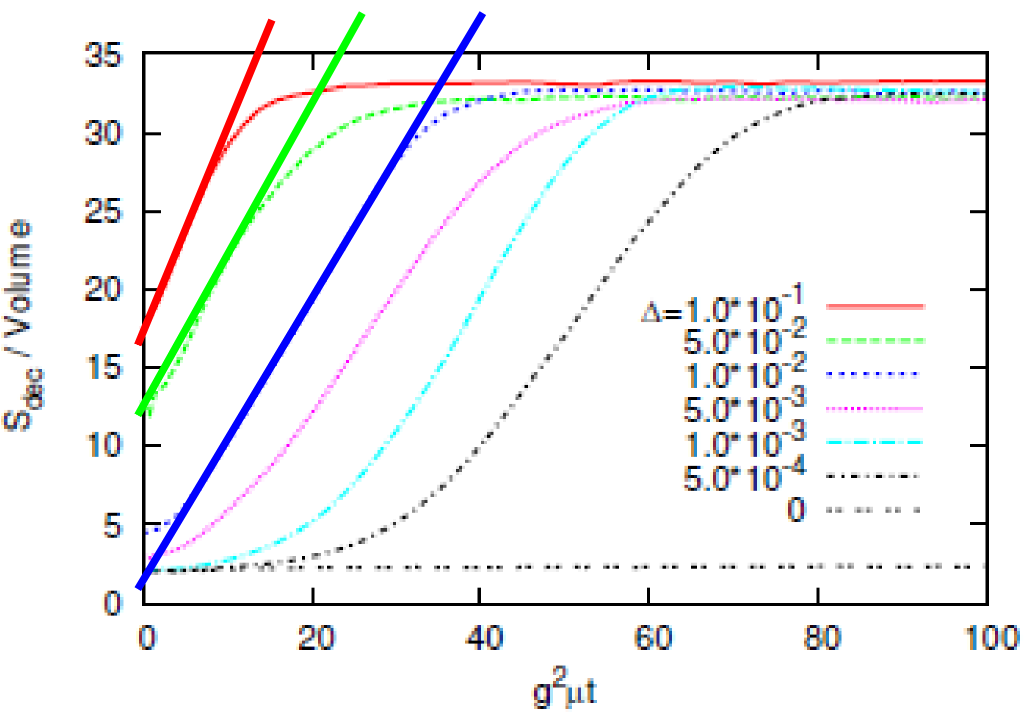
- **Decoherence entropy growth rate should be compared with KS entropy**
  - **$dS_{dec}/dt \sim 0.88$ ($\Delta=0.01$), 1.05 ($\Delta=0.05$), 1.36 ($\Delta=0.1$)**
  - **KS entropy estimate: $S_{KS} \sim c_{KS}\, \varepsilon^{1/4}$, $c_{KS} \sim 2$ (conformal chaotic value)**
  - **Energy density: $\varepsilon = 0.17$ ($\Delta=0.01$), 0.18 ($\Delta=0.05$), 0.21 ($\Delta=0.1$)**
    $\rightarrow c_{KS} = dS^{dec}/dt/\varepsilon^{1/4} = 1.4$ ($\Delta=0.01$), 1.6 ($\Delta=0.05$), 2.0 ($\Delta=0.1$)

$$\frac{1}{S_{\rm KS}} \frac{dS_{\rm dec}}{dt} \sim (0.7-1.0)$$

  - **KS entropy**
    = Potentially realized
    growth rate



$\Delta=0$: unstable
but stationary



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- **Thermal energy estimate** *Kunihiro et al.('10), Muller,Schafer('11)*

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  - **CYM energy density should not exceed Stefan-Boltzmann energy density in equilibrium.**

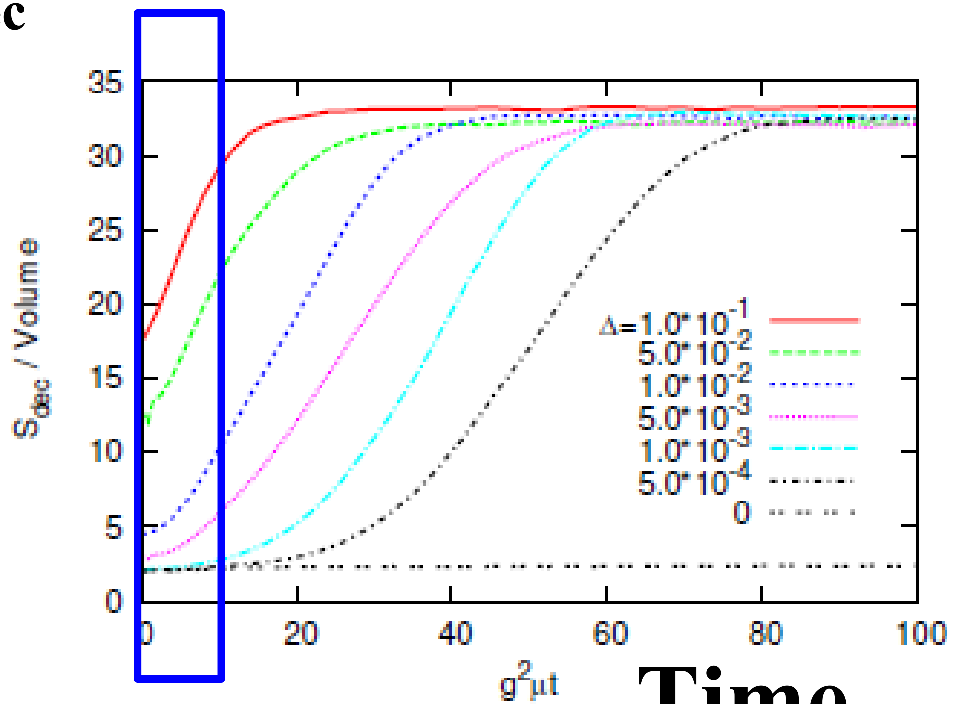
$$\varepsilon_{\rm CYM}<\varepsilon_{\rm SB}$$

$$\rightarrow\quad a>\frac{1}{T}\left(\frac{30}{\pi^2}\right)^{1/3}\sim\frac{1.4}{T}$$

This estimate gives
a > 0.8 fm (T=350 MeV),
a > 0.6 fm (T=500 MeV).

- **First 6 fm/c corresponds to lattice time $g^2\mu t < 10$!**

S_{dec}/site **relevant to HIC**



Time

Summary

- **We have evaluated the entropy from classical Yang-Mills field using**
  - **Kolmogorov-Sinai entropy (as a growth rate),**
  - **Decoherence entropy.**
- **Entropy could be produced even before classical Yang-Mills field decays into particles.**
  - **Suggested scenario: Fluctuation → Realization of instability & Spread to many modes → Chaoticity**
  - **Rough estimate of entropy production rate in non-expanding CYM**

$$\frac{dS}{dt} = S_{KS} \sim c_{KS}\,\varepsilon^{1/4} \quad , \quad c_{KS} \sim 2 \quad \text{(Lattice unit)}$$

  - **Decoherence entropy grows at about the c_{KS} = (1-2) rate, and saturates, and it is sensitive to longitudinal fluctuations.**
  - **For the initial stage entropy, both the time-evolution and the initial entropy value would be important.**

Future works

- **Decoherence entropy in expanding glasma with realistic fluctuation strength E.g. Epelbaum, Gelis ('13)**
- **Estimate of decoherence time during CYM evolution and Defining coherent domain in CYM c.f. Fries, Muller, Schafer ('09)**
- **Coupling to particle DOF**
  - **Fluctuation in classical statistical simulation ~ particles ?**
  - **CYM + gluon test particles Dumitru, Nara, Strickland ('07)**
  - **2PI formalism of CYM and gluon propagator c.f. Nishiyama, AO ('10, w/o CYM), Hatta, Nishiyama ('11), Cassing ('09)**

Thank you for your attention !