Phase diagram at strong coupling

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in collaboration with

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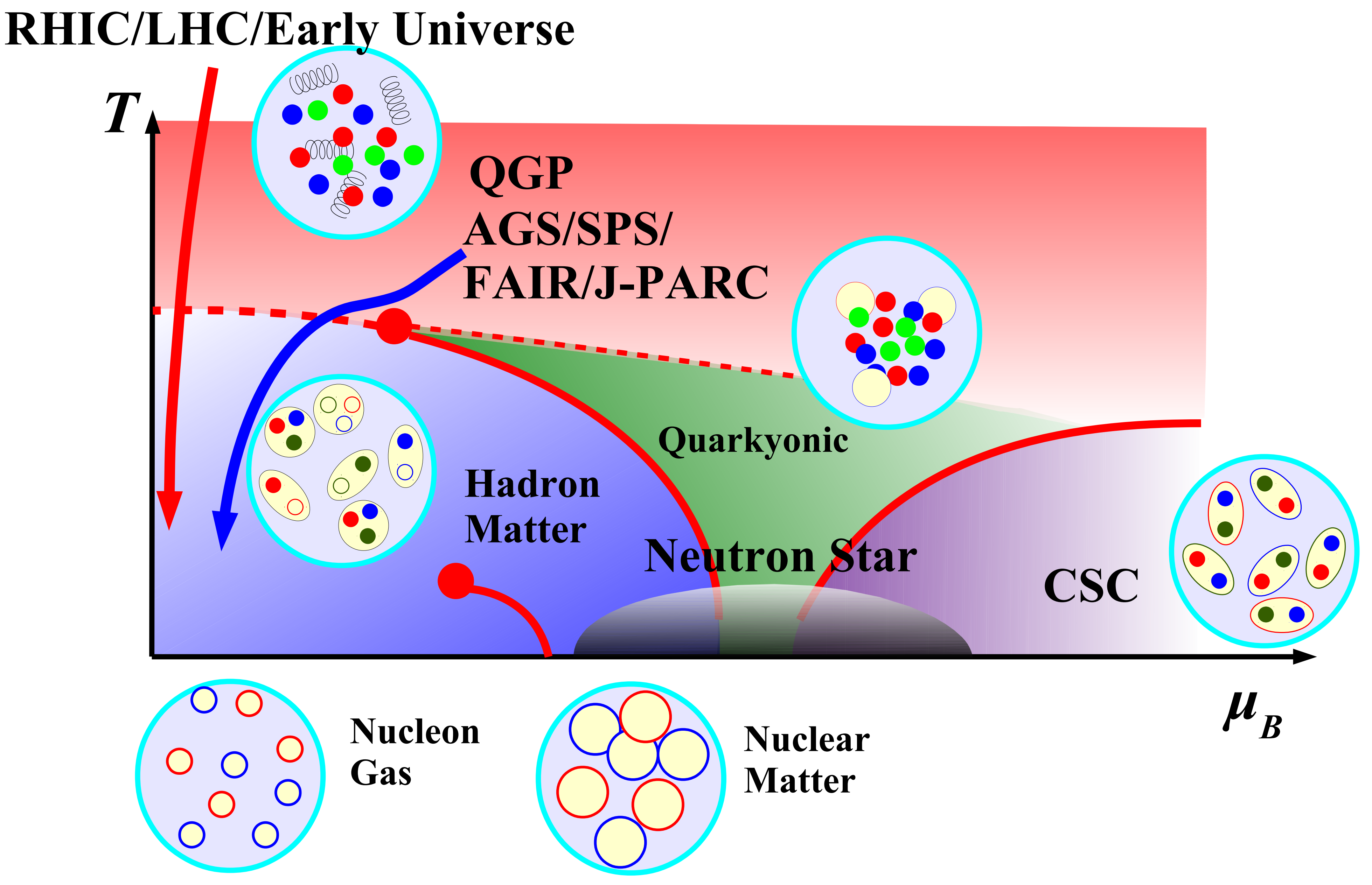
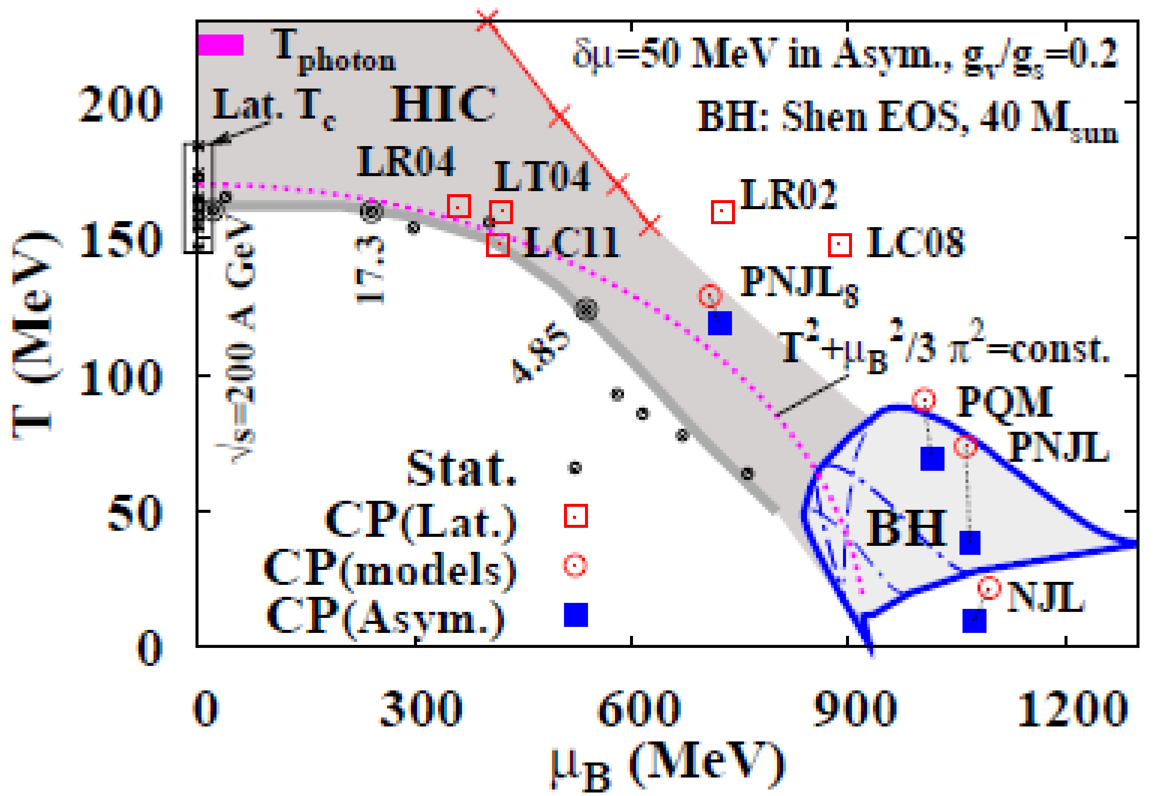
ATHIC2014 OSAKA

QCD Phase Diagram



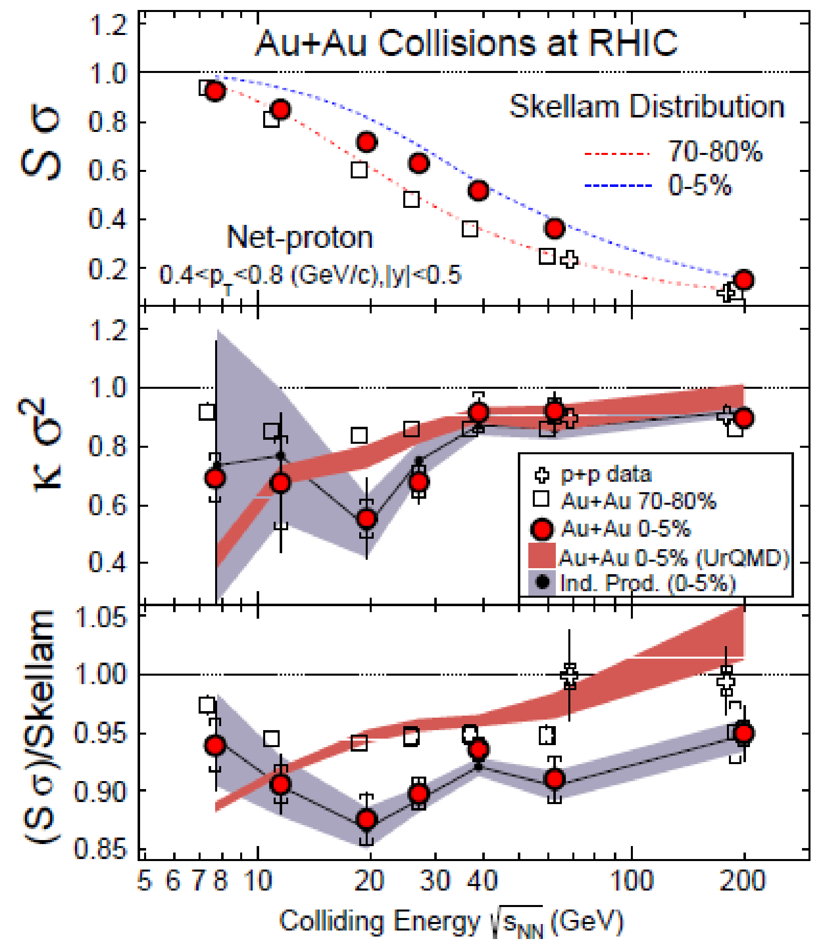
QCD Phase Diagram in HIC

- **Chemical freeze-out line reach the T-axis in 2000 (RHIC started !)**
- **Hint of CP signal is clearly shown in 2014.**



(Stephanov plot)

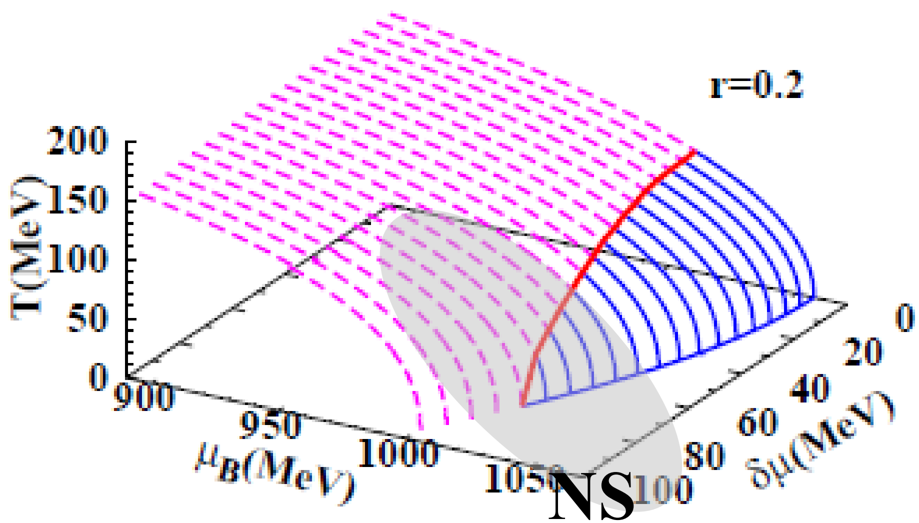
AO, PTP Supp. 193 ('12), 1.



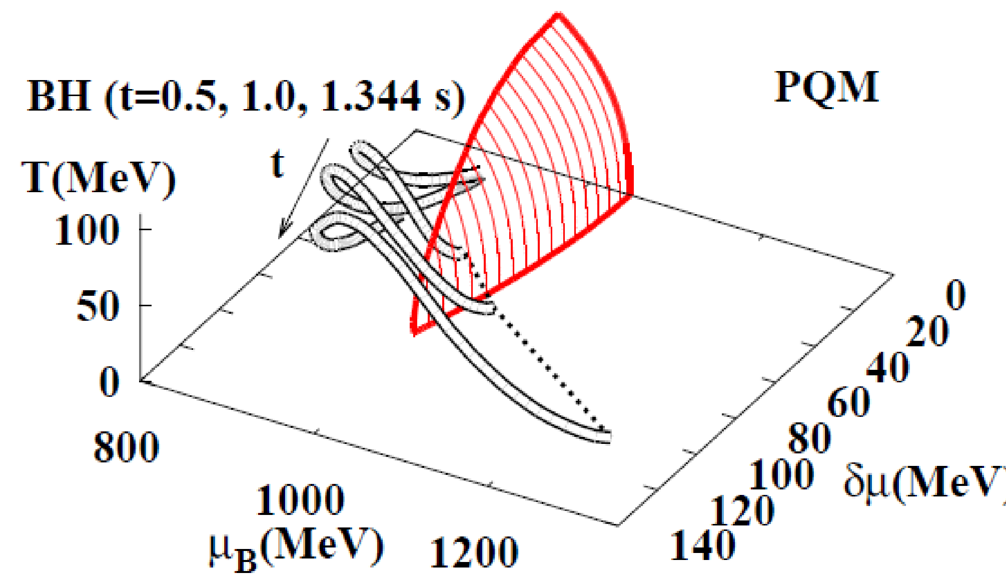
STAR (2014)

Phase diagram: 2D → 3D

- **Heavy-Ion Collisions: (T, μ) phase diagram**
- **Compact Star phenomena: Isospin chem. pot. is necessary → (T, μ, δμ) 3D phase diagram**



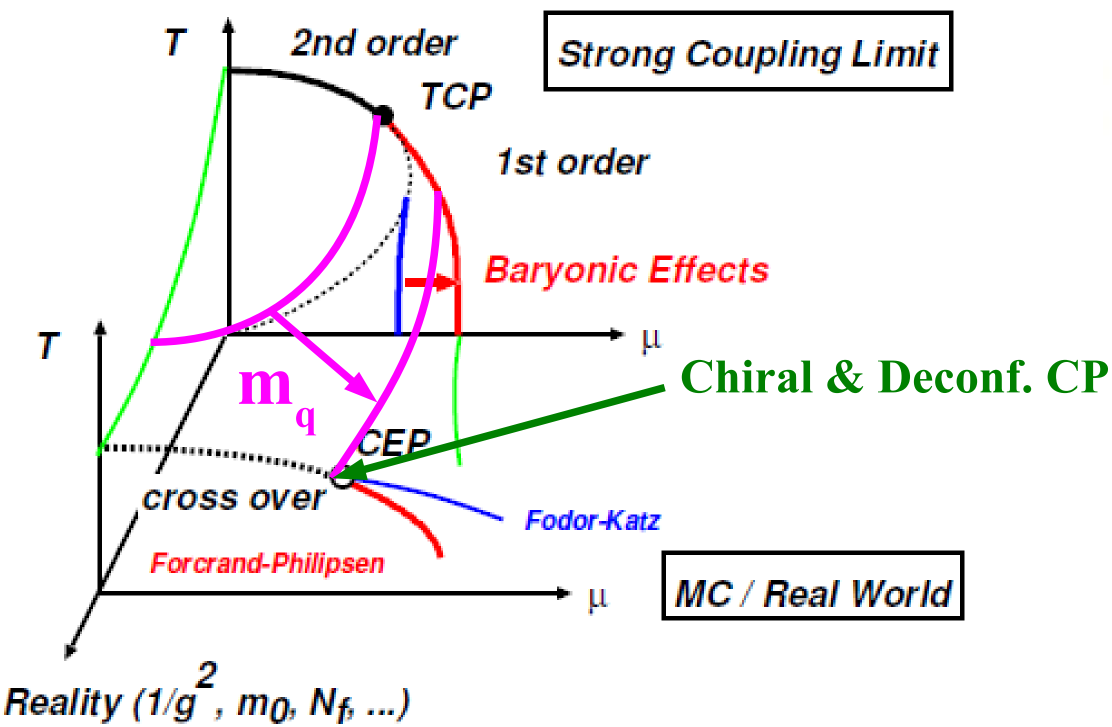
AO, Ueda, Nakano, Ruggieri, Sumiyoshi ('13)

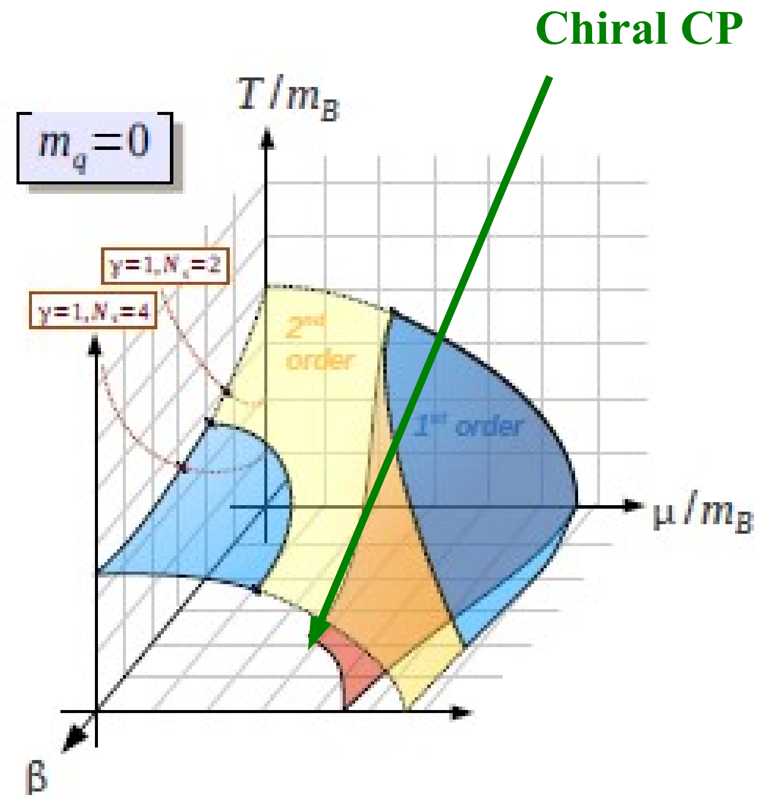


AO, Ueda, Nakano, Ruggieri, Sumiyoshi ('11)

Phase Diagram Evolution with $\beta=2N_c/g^2$

- **Extensive studies exist in the strong coupling limit (staggered, N_f=4), and the sign problem is under control.**
- **3D phase structure is related to the nature of CP.**





Kawamoto, Miura, AO, Ohnuma ('07) *de Forcrand, Langelage, Philipsen, Unger ('13)*

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- Introduction
  - QCD phase diagram under various conditions
- Strong coupling lattice QCD
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  - Finite coupling & Polyakov loop effects
    K. Mura, T. Z. Nakano, AO, N. Kawamoto, PRD80(2009), 074034;
    T. Z. Nakano, K. Miura, AO, PRD83(2011),016014.
    K. Miura, T.Z. Nakano, AO, N. Kawamoto (n preo.)
  - Fluctuations
    AO, T. Ichihara, T.Z. Nakano, PoS Lattice2012 (2012), 088;
    T. Ichihara, T. Z. Nakano, AO, PoS Lattice2013 (2013), 143;
    T. Ichihara, T. Z. Nakano, AO, arXiv:1401.4647.
    T. Ichihara, AO, LAT2014 (in prep.)
- (Sign problem in SC-LQCD)
- Summary

Strong Coupling Lattice QCD

Strong Coupling Lattice QCD

- **Effective action at strong coupling**

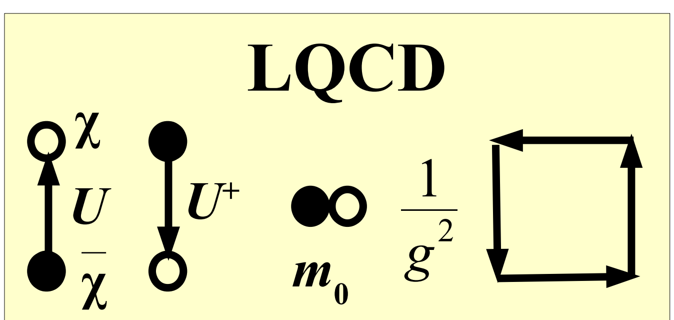
$$Z = \int \mathcal{D}[\chi, \bar{\chi}, U_0, U_j] \exp[-S_{\rm LQCD}(\chi, \bar{\chi}, U_0, U_j)]$$

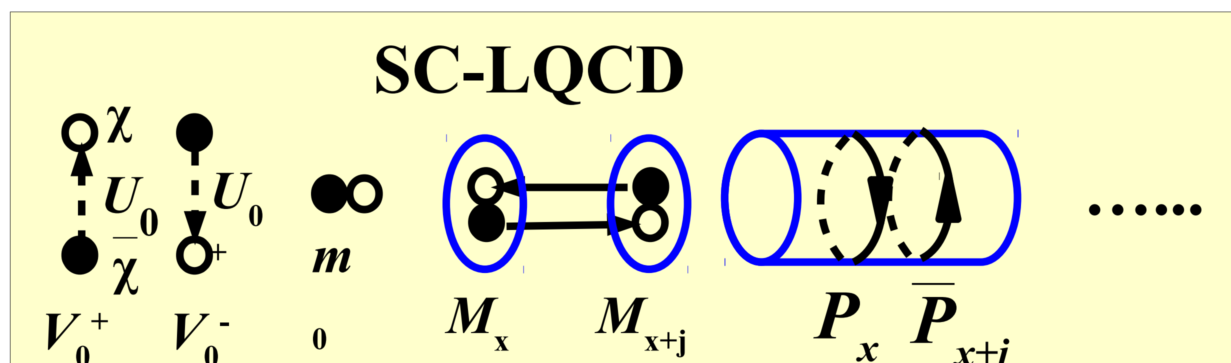
spatial link integral

$$= \int \mathcal{D}[\chi, \bar{\chi}, U_0] \exp[-S_{\rm eff}(\chi, \bar{\chi}, U_0)]$$

Damgaard,Kawamoto,Shigemoto('84), Ichinose('84), Faldt,Petersson('86), Nakano, Miura,AO ('09), Gocksch, Ogilvie ('85), Fukushima ('04), Nakano, Miua, AO ('11)

- **Integrate spatial links first.**
- **Given order in $1/g^2$ and, Leading order in 1/d (d=spatial dim.)**
- **Temporal Link + Nearest Neighbor Int. + Polyakov loop + ….**



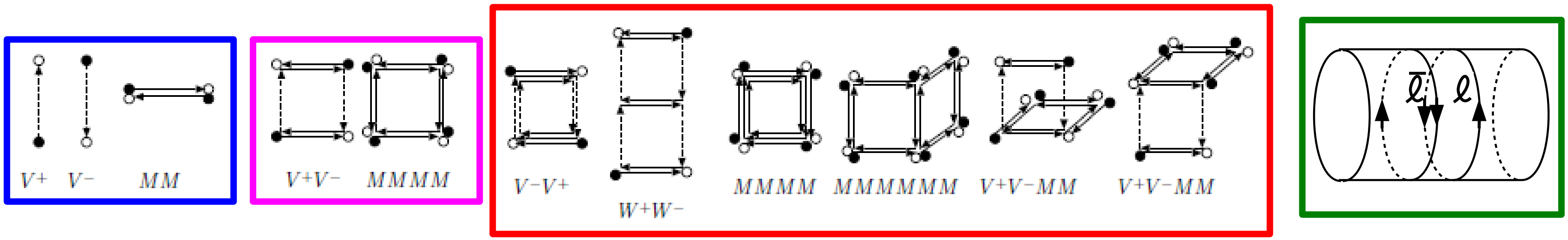


Finite Coupling Effects

- **Finite coupling corrections**

$$S_{\rm eff} = S_{\rm SCL} - \log\langle \exp(-S_G)\rangle = S_{\rm SCL} - \sum_{n=1} \frac{(-1)^n}{n!}\langle S_G^n \rangle_c$$

$\langle S_G^n \rangle_c$=Cumulant (connected diagram contr.) *c.f. R.Kubo('62)*



$$S_{\rm eff} = \frac{1}{2}\sum_x (V_x^+ - V_x^-) - \frac{b_\sigma}{2d}\sum_{x,j>0}[MM]_{j,x}$$

SCL (Kawamoto-Smit, '81)

$$+\frac{1}{2}\frac{\beta_\tau}{2d}\sum_{x,j>0}[V^+V^- + V^-V^+]_{j,x} - \frac{1}{2}\frac{\beta_s}{d(d-1)}\sum_{x,j>0,k>0,k\neq j}[MMMM]_{jk,x}$$

$$-\frac{\beta_{\tau\tau}}{2d}\sum_{x,j>0}[W^+W^- + W^-W^+]_{j,x} - \frac{\beta_{ss}}{4d(d-1)(d-2)}\sum_{\substack{x,j>0,|k|>0,|l|>0\\ |k|\neq j,|l|\neq j,|l|\neq|k|}}[MMMM]_{jk,x}[MM]_{j,x+\hat{l}}$$

NLO (Ichinose, '84; Faldt-Petersson, '86)

$$+\frac{\beta_{\tau s}}{8d(d-1)}\sum_{x,j>0,|k|\neq j}[V^+V^- + V^-V^+]_{j,x}\left([MM]_{j,x+\hat{k}} + [MM]_{j,x+\hat{k}+\hat{0}}\right)$$

NNLO (Nakano, Miura, AO, '09)

$$-\left(\frac{1}{g^2N_c}\right)^{N_\tau} N_c^2 \sum_{\mathbf{x},j>0}\left(\bar{P}_{\mathbf{x}}P_{\mathbf{x}+\hat{j}} + h.c.\right)$$

Polyakov loop (Gocksch, Ogilvie ('85), Fukushima ('04) Nakano, Miua, AO ('11))

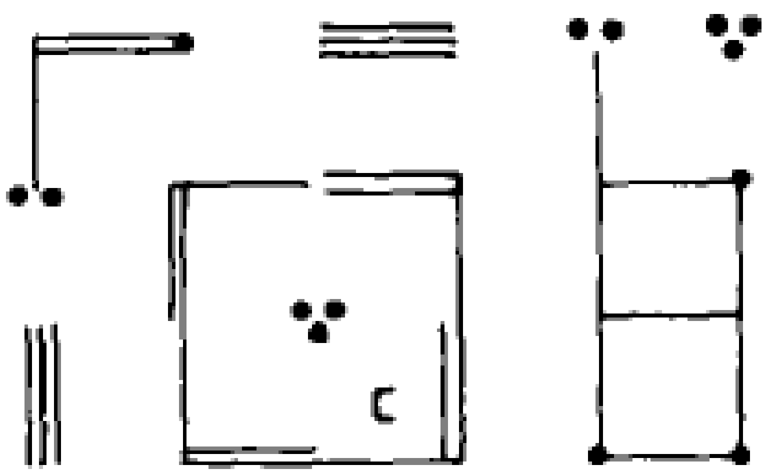
Monomer-Dimer-Polymer simulation

- **The partition fn. can be given as the sum of monomer-dimer-polymer (MDP) configuration weight in the strong coupling limit. The sign problem is mild.** *Karsch, Mutter ('89)*

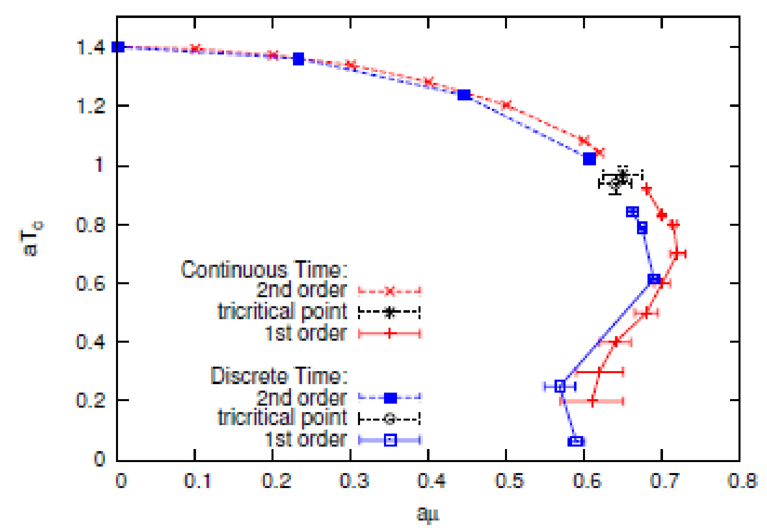
$$Z(2ma, \mu, r) = \sum_K w_K$$

$$w_K = (2ma)^{N_M} r^{2N_t} (1/3)^{N_1 N_2} \prod_x w(x) \prod_C w(C)$$

- **MDP with worm algorithm is applied to study the phase diagram** *de Forcrand, Fromm ('10), de Forcrand, Unger ('11)*



Karsch, Mutter ('89)

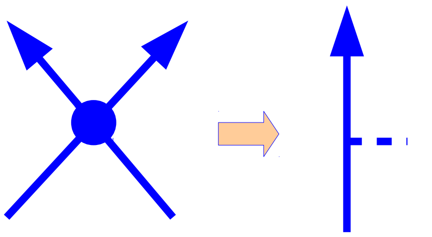


de Forcrand, Unger ('11)

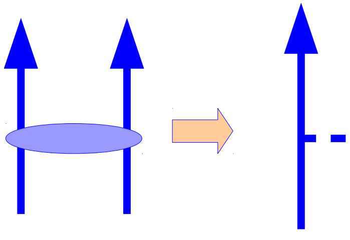
Mean Field & Auxiliary Field Monte-Carlo

- **One popular way to handle many-fermion int. = Bosonization (Hubbard-Stratonovich transf.)**

$$e^{MM} = \int d\sigma e^{-\sigma^2 - 2\sigma M} \quad (M = \bar{\chi}\chi)$$



- **Nearest neighbor Four-Fermi int. → σ and π (chiral partners) Extended HS transf.**



$$\exp[\alpha \sum_{x,j} M_x M_{x+\hat{j}}] = \int \mathcal{D}[\sigma, \pi] \exp\left[-S_{\rm int}\right]$$

$$S_{\rm int} = L^3 N_\tau \alpha \sum_{k, f(\mathbf{k})>0} f(\mathbf{k})(|\sigma_k|^2 + |\pi_k|^2) + \alpha \sum_{x,\pm j} M_x (\sigma + i\varepsilon\pi)_{x\pm\hat{j}}$$

$$(f(\mathbf{k}) = \sum_j \cos k_j \ , \quad \varepsilon = (-1)^{x_0+x_1+x_2+x_3})$$

- **Mean Field & Auxiliary Field MC (after χ and U_0 integral)**

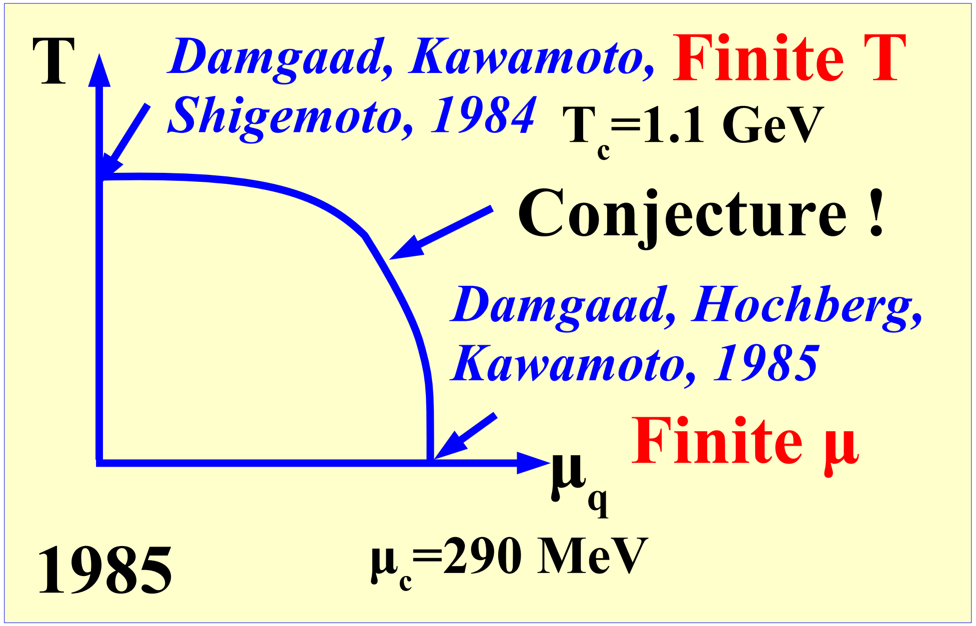
$$Z = \underline{\int \mathcal{D}[\sigma, \pi] \exp[-S_{\rm eff}(\sigma_k, \pi_k)]} \simeq \underline{\exp[-S_{\rm eff}(\sigma_k \delta_{k0}, \pi_k)]|_{\rm stat.}}$$

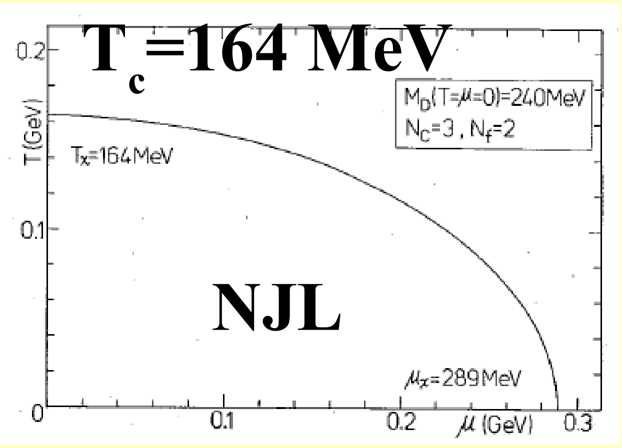
AFMC | **Mean Field approx.**

Phase Diagram in SC-LQCD

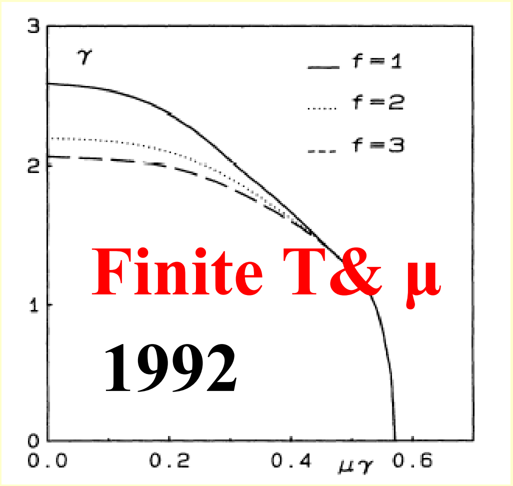
SC-LQCD phase diagram evolution in time...

- **Phase diagram in the strong coupling limit (mean field)**

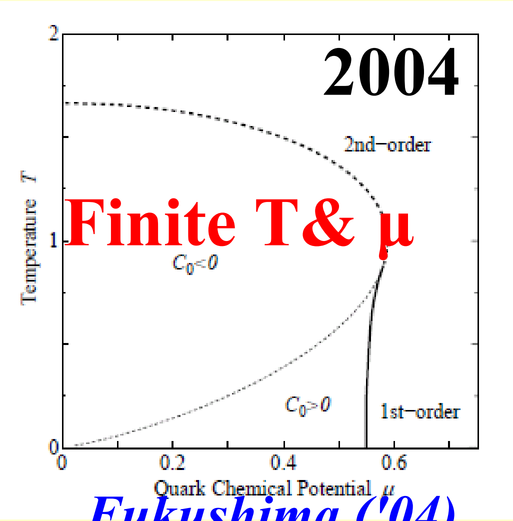




NJL: Hatsuda, Kunihiro ('85)
cf. Fukushima (Aug.5)



Bilic, Karsch, Redlich ('92)



Fukushima ('04)
Nishida ('04)

Tc is too high in SCL.
$R = \mu_c / T_c = (0.2\text{-}0.3)$
Mean field artifact ?
SCL artifact ?

Finite Coupling Effects

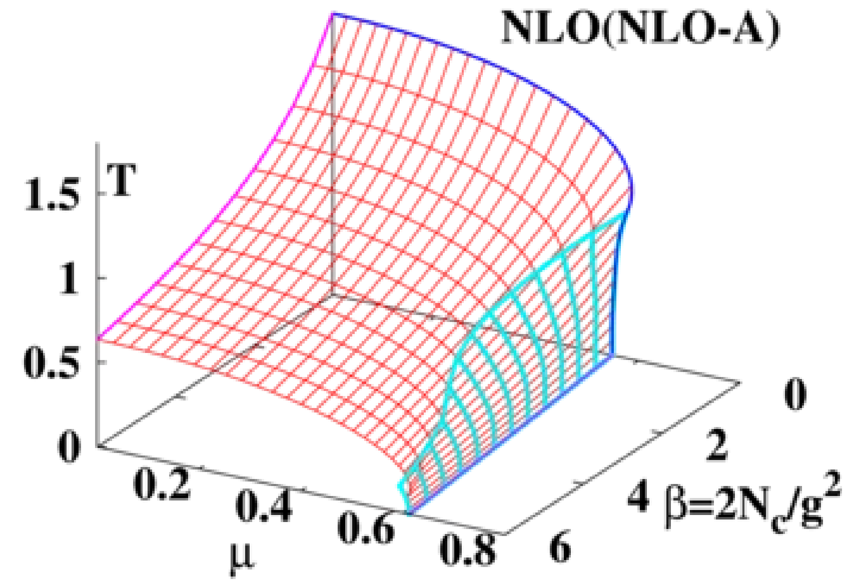
- **Shape of the phase diagram is compressed in T direction with β**

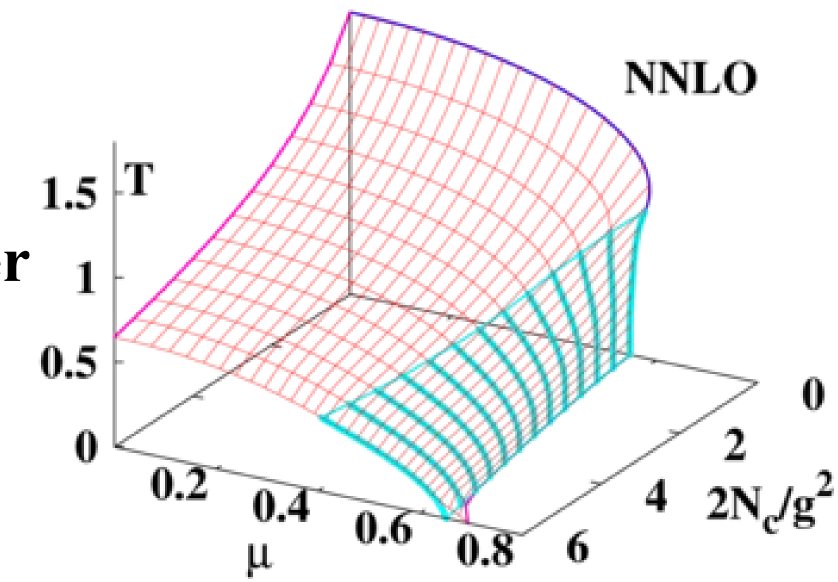
  → Improvements in $R = \mu_c / T_c$!

  - **MC (R > 1) → SCL (R = (0.3-0.45))**
    → NLO/NNLO (R ~ 1)
    → Real World (R~(2-4))

- **Critical Point**
  - **NLO: μ(CP) ~ Const.**
  - **NNLO: μ(CP) decreases with β consistent with the expected 1st order at μ=0 for N_f=4.**

    Kronfeld ('07), Pisarski, Wilczek ('84)
  - **μ(CP)/T(CP) ~ 1 ↔ MC (μ/T > 1)**
    Ejiri, ('08), Aoki et al.(WHOT,'08), Allton et al., ('03,'05)

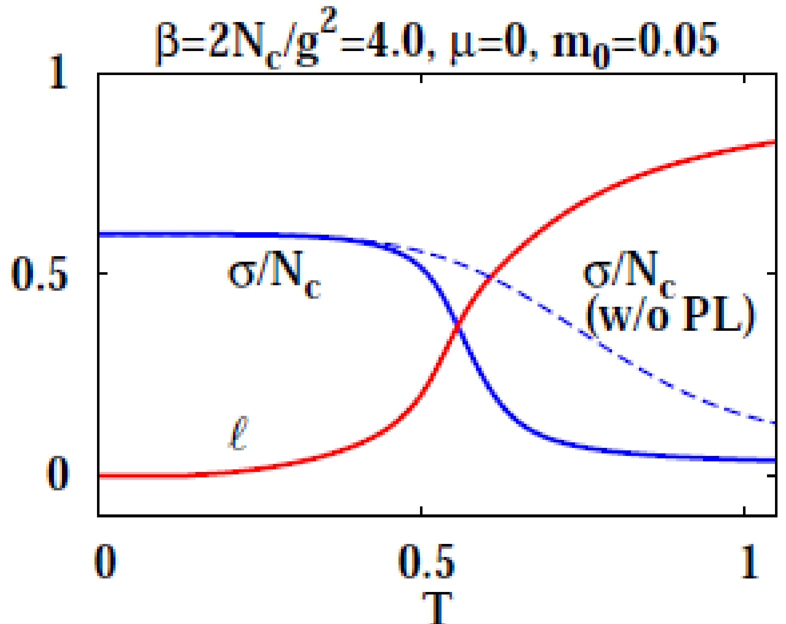


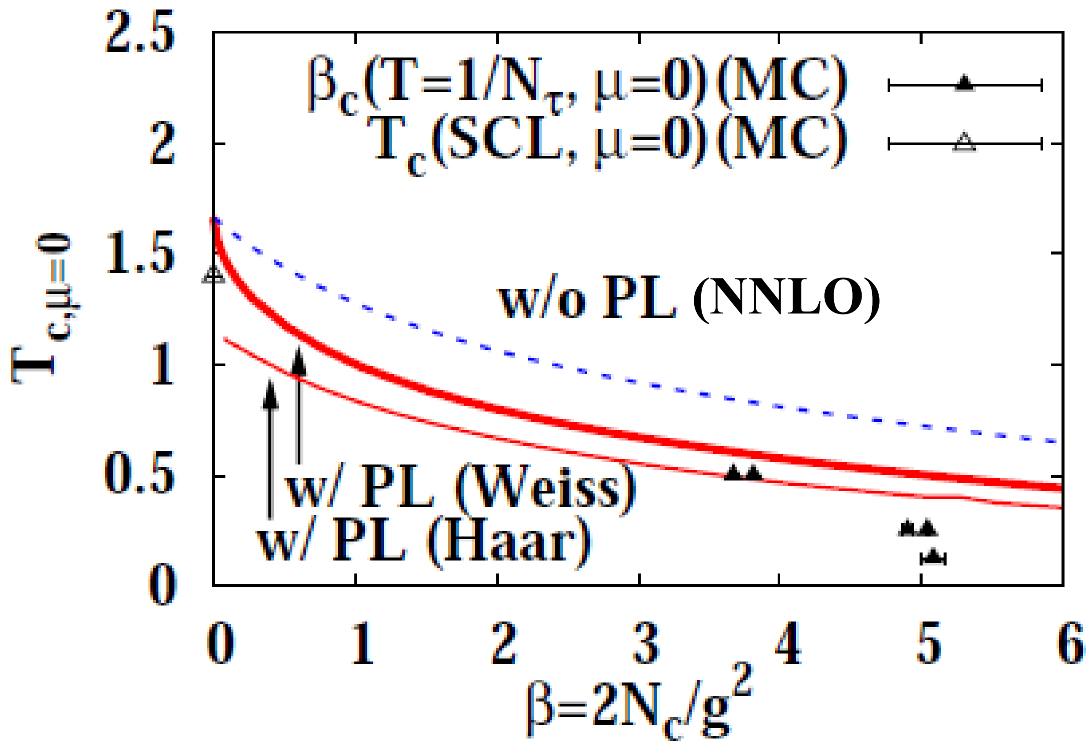


Miura, Nakano, AO, Kawamoto ('09)
Nakano, Miura, AO ('09)

Polyakov Loop Effects (μ=0)

- **P-SC-LQCD reproduces MC results of T_c(μ=0) ($\beta = 2N_c/g^2 \le 4$)**
  MC data: SCL (Karsch et al. (MDP), de Forcrand, Fromm (MDP)), N_τ =2 (de Forcrand, private), N_τ=4 (Gottlieb et al.('87), Fodor-Katz ('02)), N_τ =8 (Gavai et al.('90))
  - **Weiss MF method: Bosonization of Pol. loop action (includes Gaussian fluc. of PL)**
  - **Haar measure method: PL is replaced with c-number.**





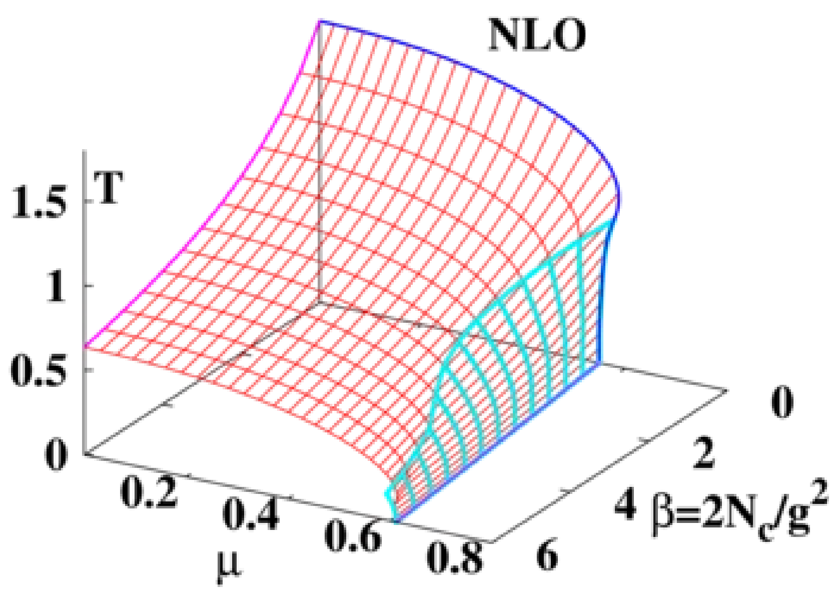
Lattice Unit

T. Z. Nakano, K. Miura, AO, PRD 83 (2011), 016014 [arXiv:1009.1518 [hep-lat]]

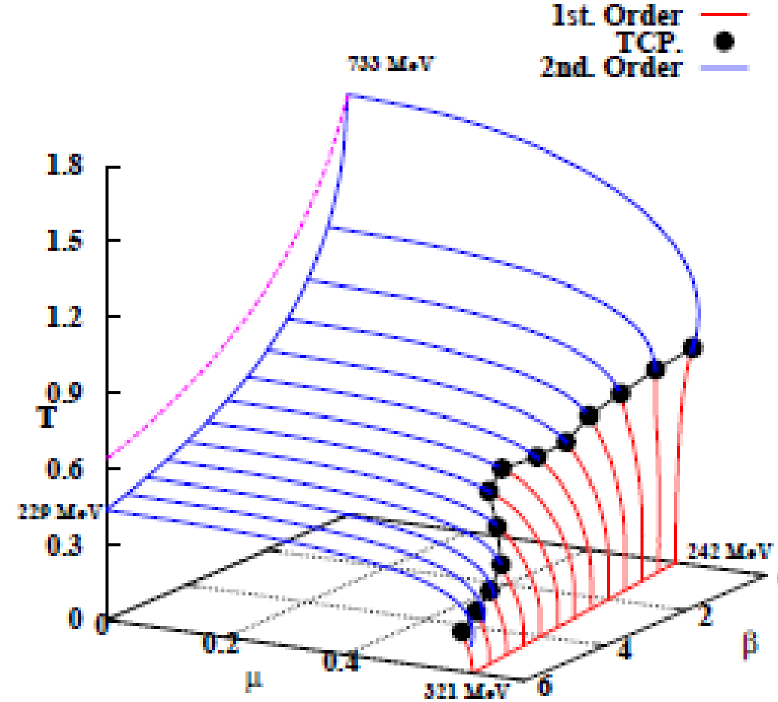
Polyakov Loop Effects

- **Polyakov loop effect is significant at μ=0.**
  - **Tc is reduced by 30 % at β=3, and MC results are roughly reproduced.**
- **Polyakov loop effect may be less significant at finite μ.**

Miura, Nakano, AO, Kawamoto (in prep.)



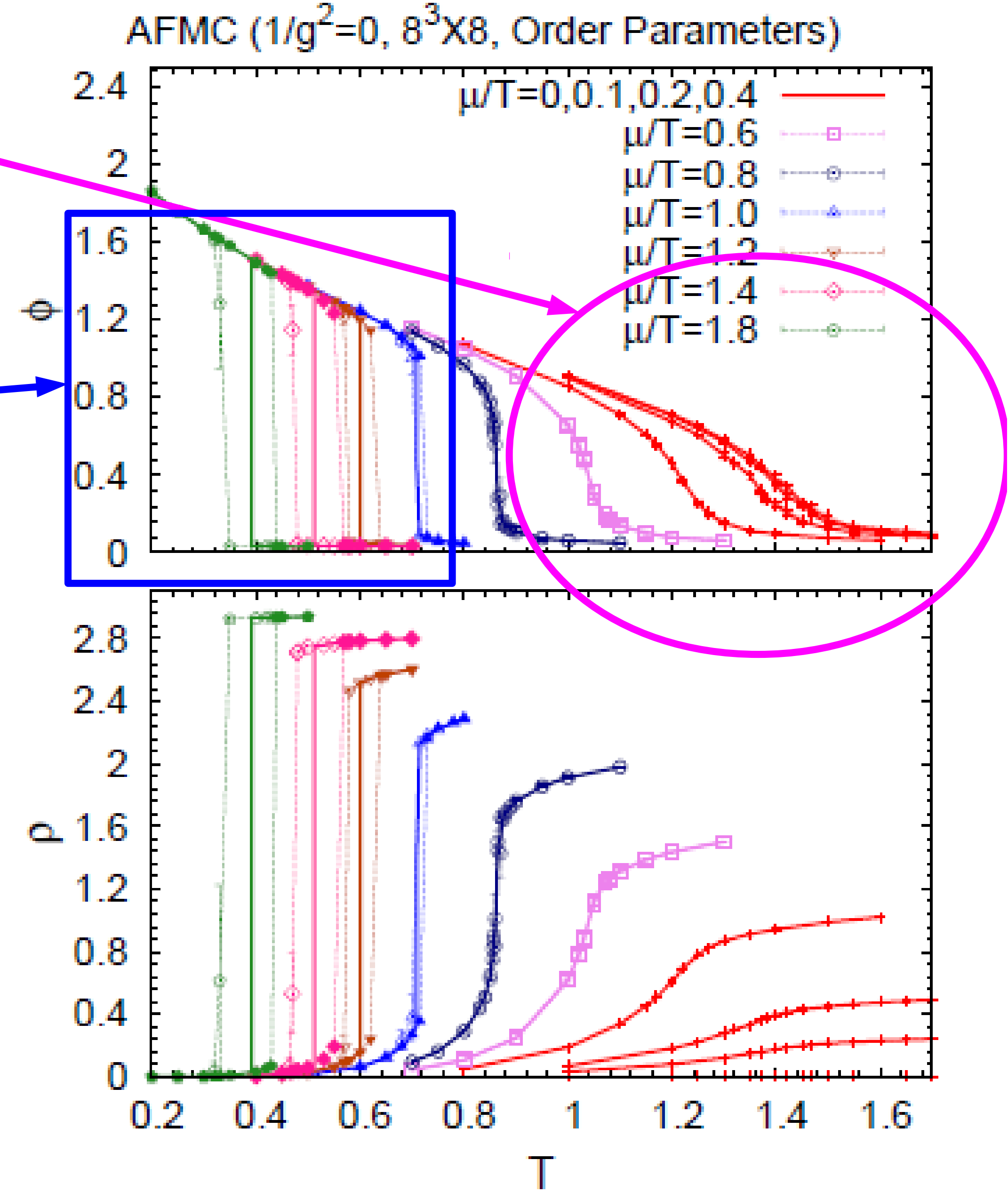
Miura, Nakano, AO, Kawamoto ('09)

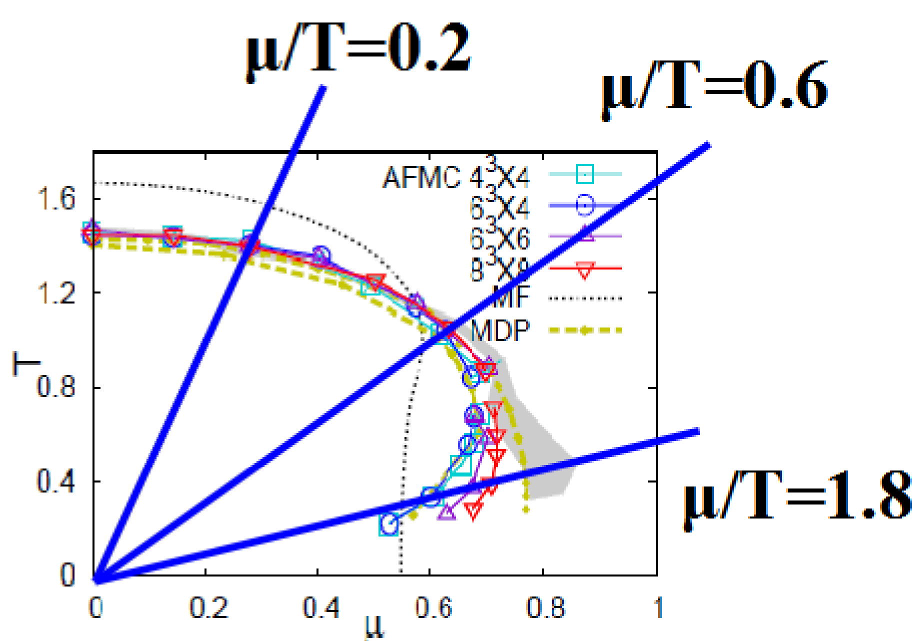


Miura, Nakano, AO, Kawamoto (in prep.)

Order Parameters in AFMC

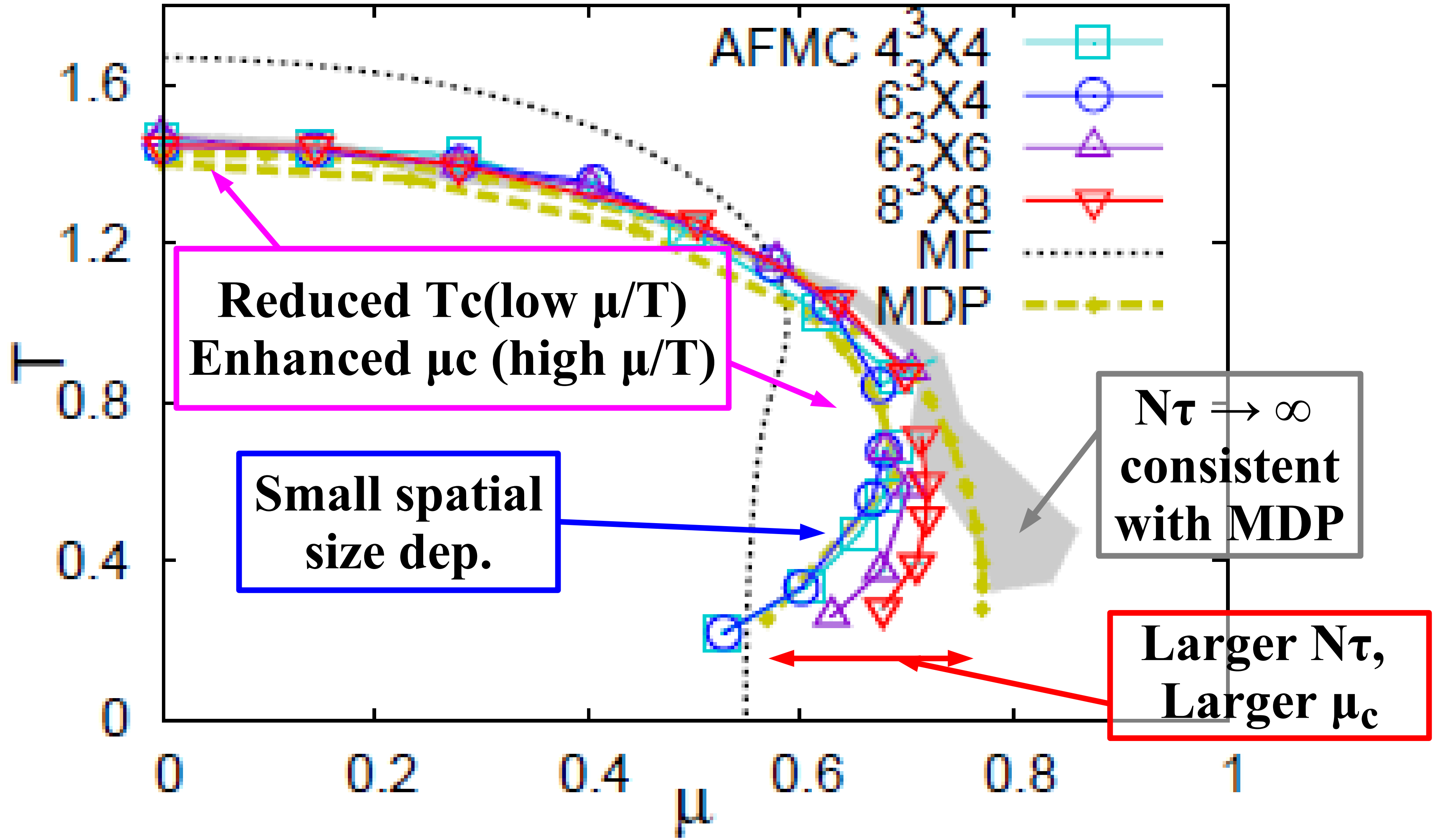
- **Low μ/T region**
  → 2nd order or crossover
  (would-be second)
- **High μ/T region**
  → sudden change
  & hysteresis
  (would-be first)





Ichihara, AO, Nakano ('14)

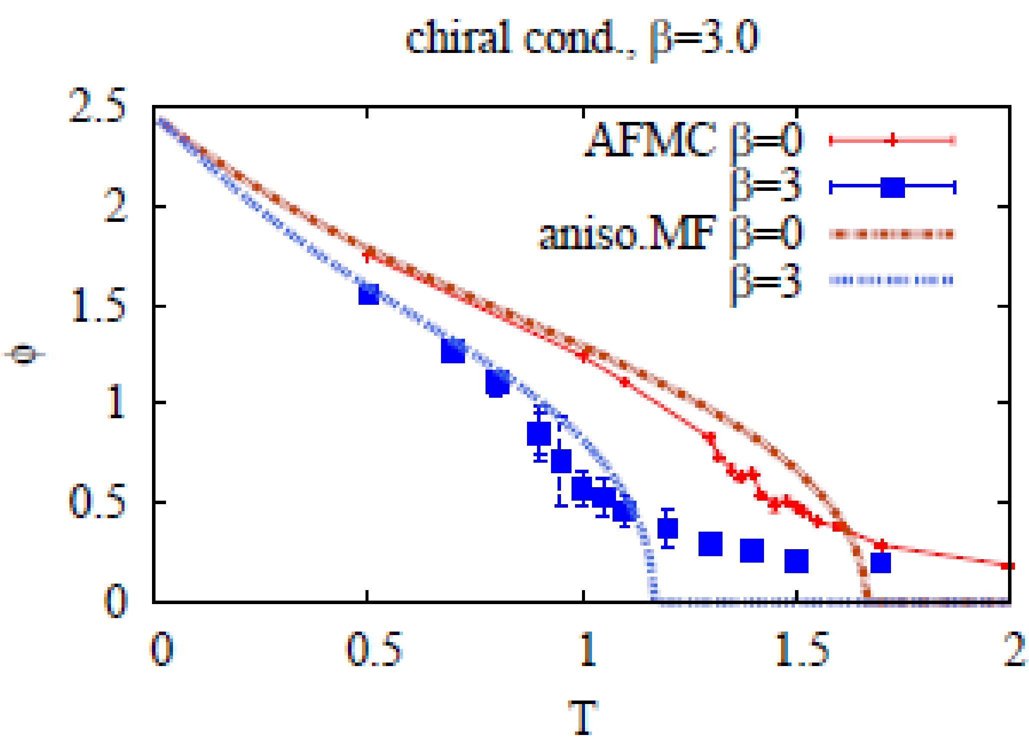
Fluctuation Effects



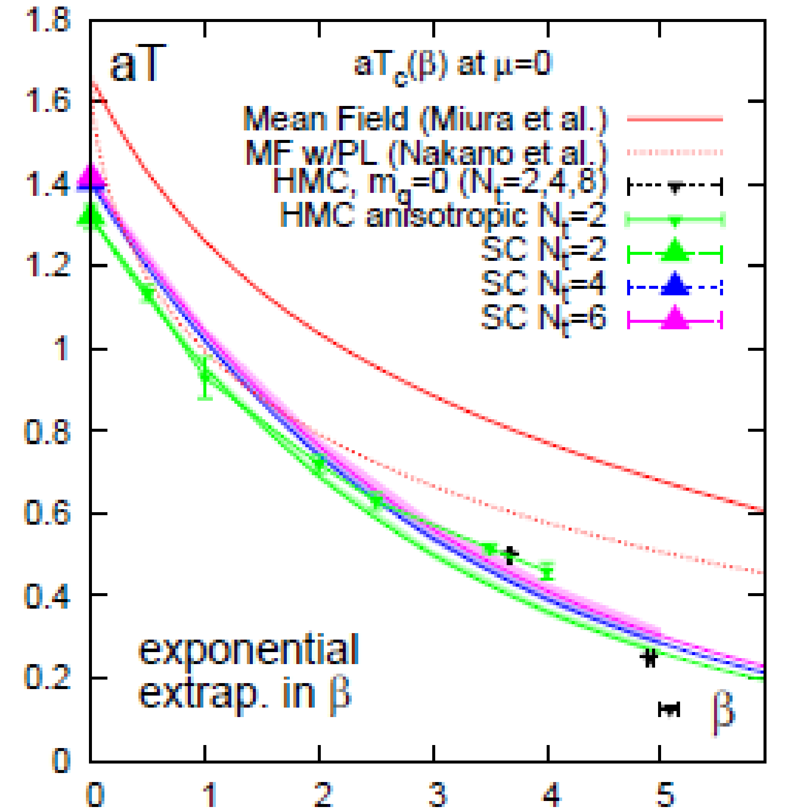
Ichihara, AO, Nakano ('14)
de Forcrand, Fromm ('10), de Forcrand, Unger ('11)

Fluctuation & Finite Coupling Effects

- **Fluctuation & Finite Coupling effects on Tc are in the same direction.**
  - **Fluctuation reduces Tc by (10-15) % in AFMC**



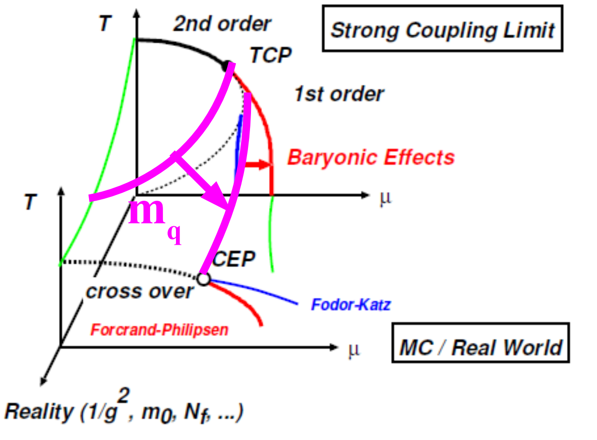
Ichihara, AO (LAT2014 proc., to be submitted)



de Forcrand, Langelage, Philipsen, Unger, 1406.4397v1

Summary

- Phase diagrams under various conditions (3D phase diagrams) are important to understand
  Compact star physics (isospin chem. pot., $\delta\,\mu$),
  Finite density lattice QCD (coupling, $\beta=2\,N_c/g^2$, imag. μ),
  Magnetic catalysis/inhibition, and more.
- Strong coupling lattice QCD is a powerful tool to understand finite density QCD at $\beta < (3\text{-}4)$ $(1/g^2 < (0.5\text{-}0.66))$.
  - At $\mu=0$, finite coupling, Polyakov loop and (chiral field) fluctuation reduces Tc by (30-40) %, (30-40) % and (10-15) % at $\beta \sim 3$.
    → consistent with Hybrid MC results.
- CP position is still sensitive to the truncations & method.
  - Do we see CEP or Lifshits pont in BES ?



Thank you

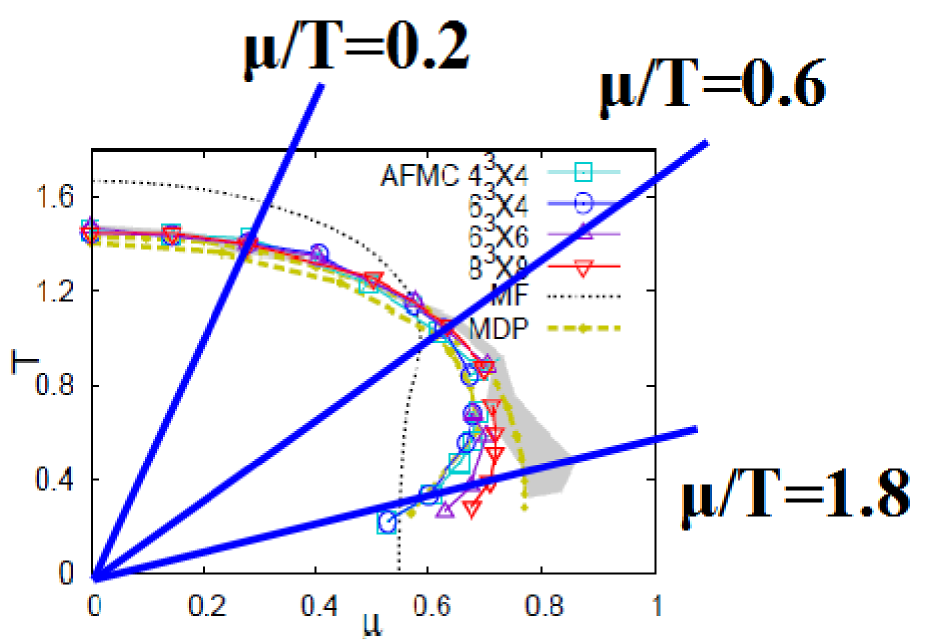
Sign problem in SC-LQCD

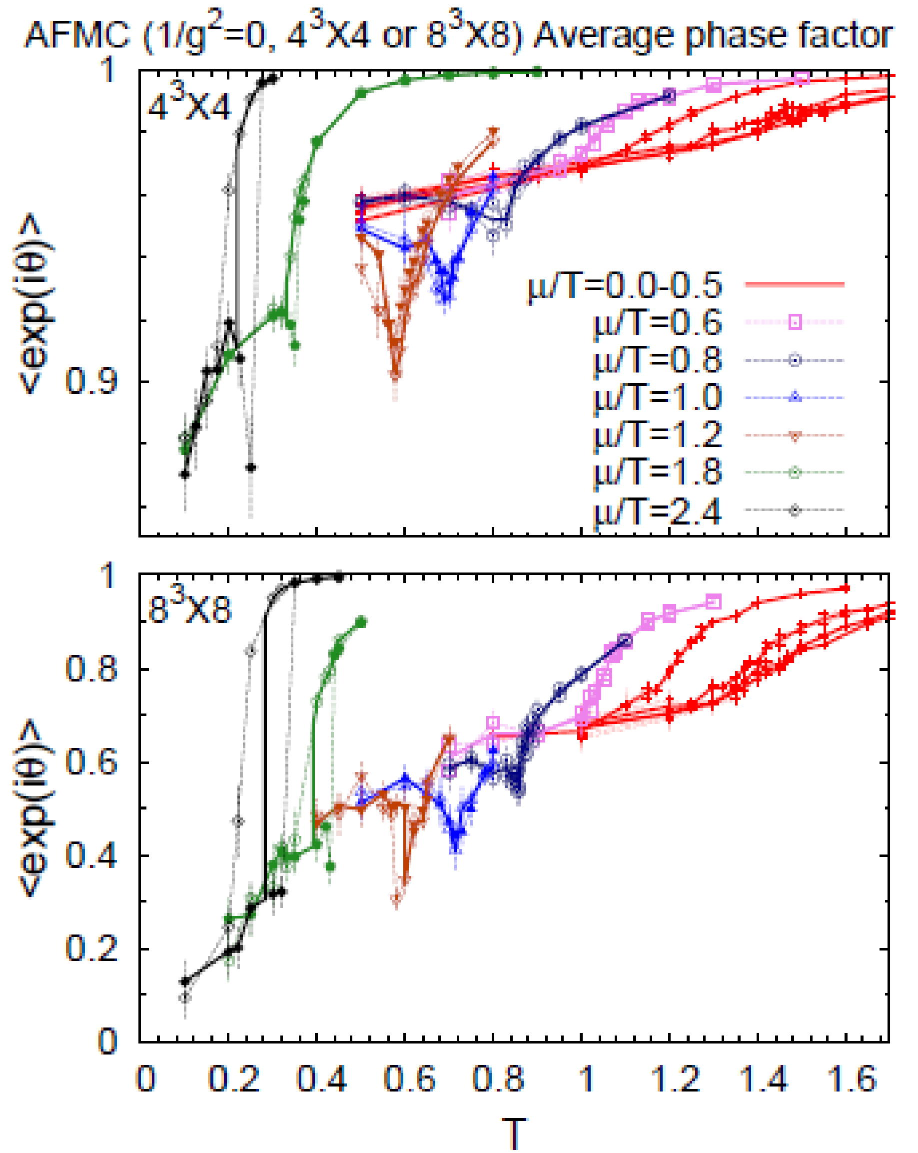
Average Phase Factor

- **Average phase factor = Weight cancellation**

$$\langle e^{i\theta} \rangle = Z_{\text{phase quenched}} \;/\; Z_{\text{full}}$$

- **AFMC results**
  - **$<e^{i\theta}>$ > 0.9 on 4^4 lattice**
  - **$<e^{i\theta}>$ > 0.1 on 8^4 lattice**





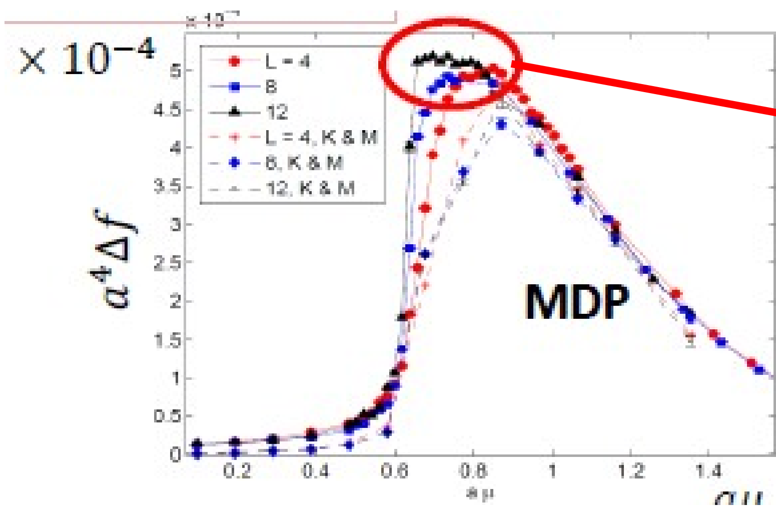
Ichihara, AO, Nakano ('14)

Discussion: Comparison with MDP

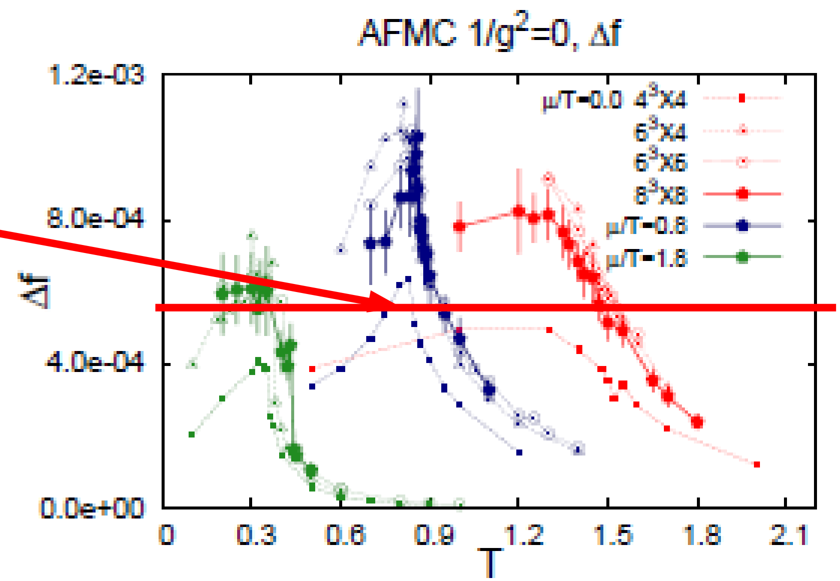
- **Free energy difference**

$$\langle \exp(i\theta) \rangle \equiv \exp(-\Omega \Delta f) \quad , \quad \Omega = \text{space-time volume}$$

- **MDP simulation on anisotropic lattice at finite T and μ**
  de Forcrand, Fromm ('10), de Forcrand, Unger ('11)
  - **Strong coupling limit.**
  - **Higher-order terms in 1/d expansion**
  - **No sign problem in the continuous time limit ($N\tau \rightarrow \infty$).**



de Forcrand, Unger ('11)



Ichihara, AO, Nakano ('14)

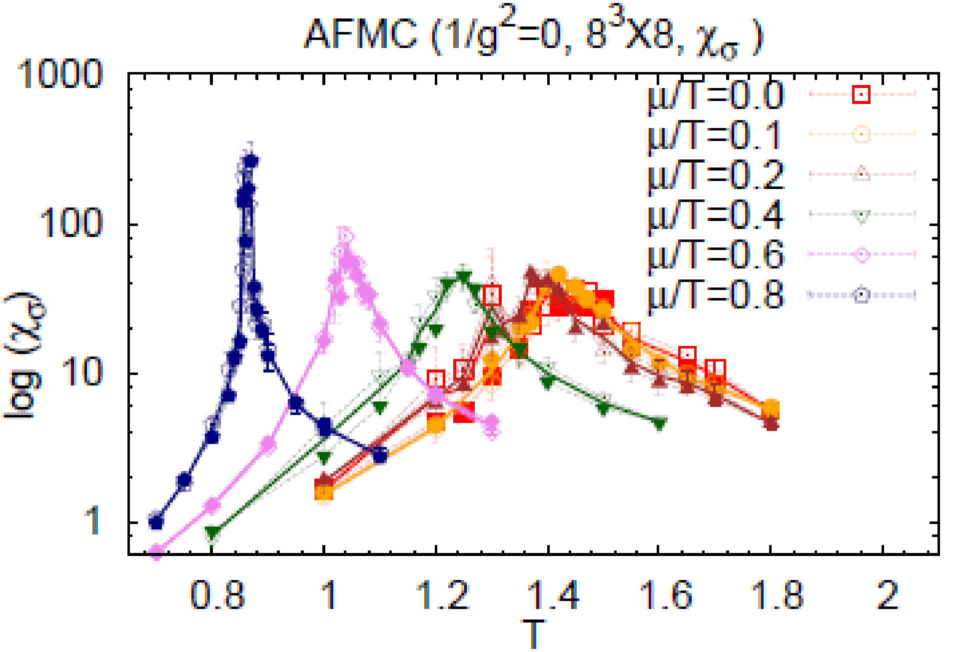
Ways to avoid the sign problem

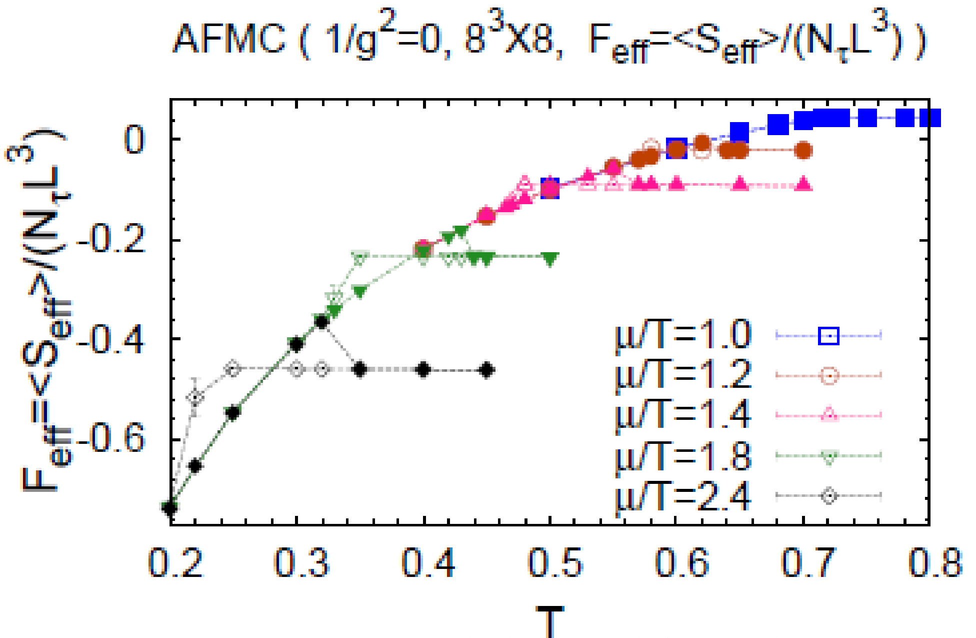
- **Complex Langevin simulation with Gauge cooling**
  Aartz, Bongiovanni, Seiler, Sexty, Stamatescu ('13)
- **Integral on Lefschetz Thimble**
  Fujii, Honda, Kato, Kikukawa, Komatsu, Sano ('13)
  Aurora Science Collab. ('12)

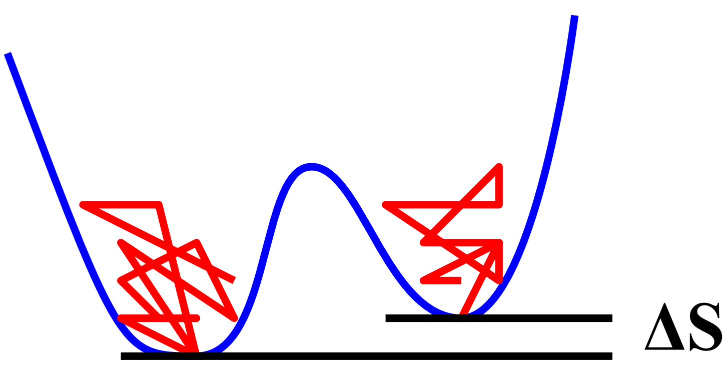
Phase boundary

- **Low μ/T region (would-be second) → Chiral susc. peak**
- **High μ/T region (would-be first) → Average eff. action from Wigner/NG init. cond.**

c.f. Exchange MC (Hukuyama)



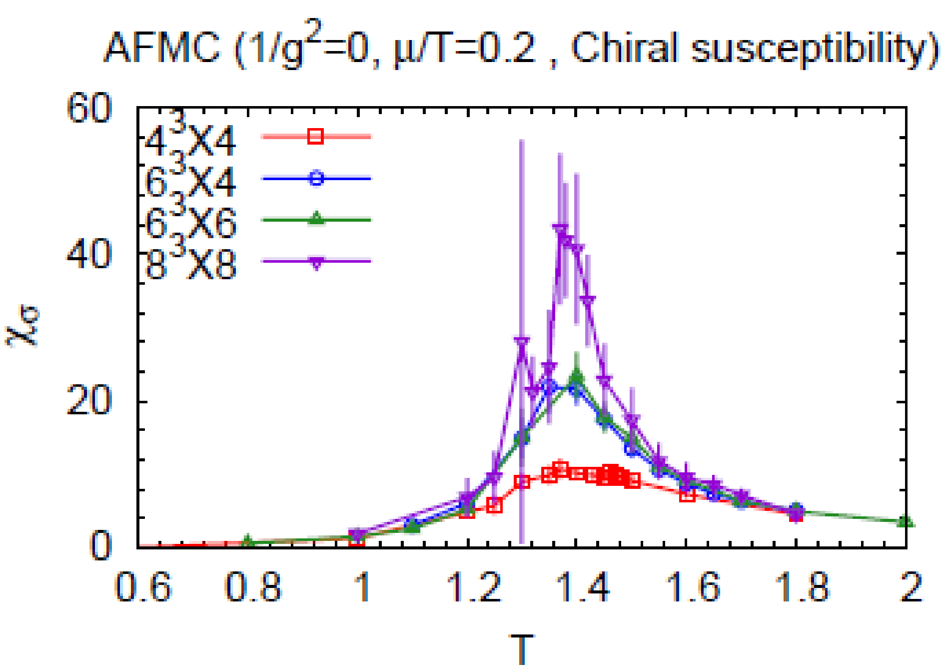


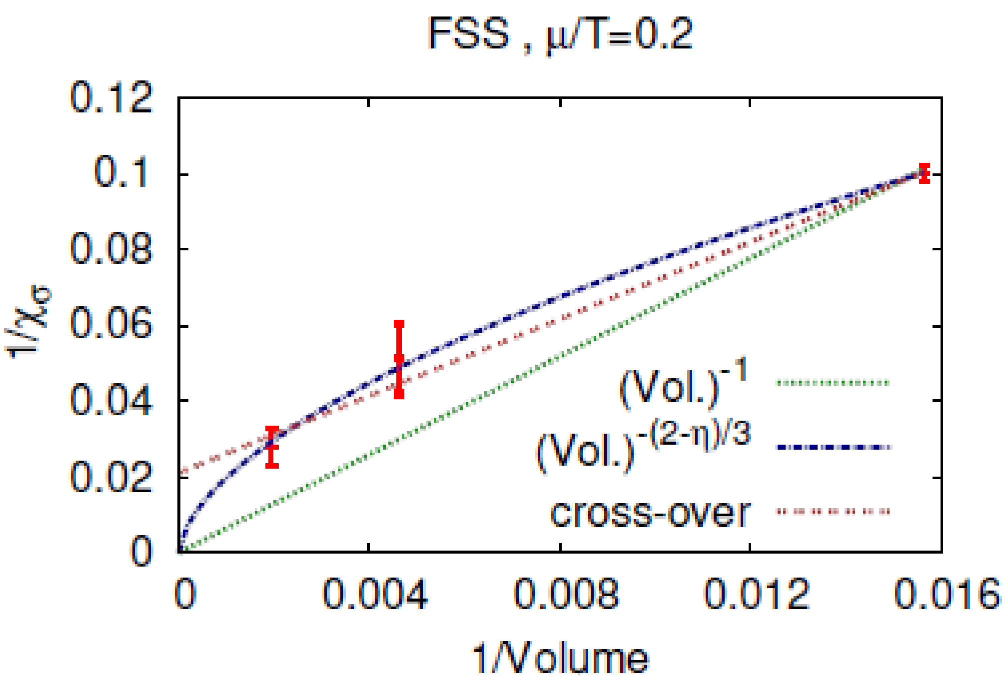


Ichihara, AO, Nakano ('14)

Finite Size Scaling of Chiral Susceptibility

- **Finite size scaling of χ_σ in the V (spatial vol.) → ∞ limit**
  - **Crossover: Finite**
  - **Second order: $\chi_\sigma \propto V^{(2-\eta)/3}$, η=0.0380(4) in 3d O(2) spin**
    Campostrini et al. ('01)
  - **First order: $\chi_\sigma \propto V$**
- **AFMC results : Not First order at low μ/T.**





Ichihara, AO, Nakano ('14)

Beyond the mean field approximation

- **Constant auxiliary field → Fluctuating auxiliary field**

$$S_{\text{eff}} = S_F^{(t)} + \sum_x m_x M_x + \frac{L^3}{4N_c} \sum_{\boldsymbol{k},\tau} f(\boldsymbol{k}) \left[|\sigma_{\boldsymbol{k},\tau}|^2 + |\pi_{\boldsymbol{k},\tau}|^2 \right]$$

$$m_x = m_0 + \frac{1}{4N_c} \sum_j \left((\sigma + i\varepsilon\pi)_{x+\hat{j}} + (\sigma + i\varepsilon\pi)_{x-\hat{j}} \right)$$

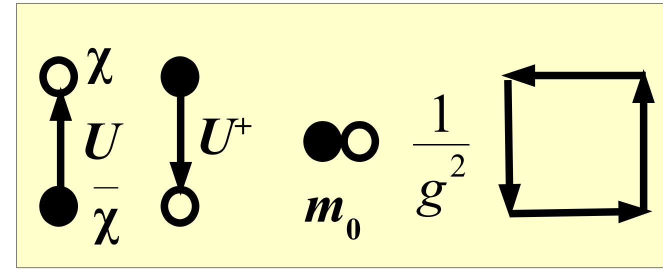
$$f(\boldsymbol{k}) = \sum_j \cos k_j \; < , \quad \varepsilon = (-1)^{x_0 + x_1 + x_2 + x_3}$$

- **Auxiliary Field Monte-Carlo (AFMC) integral**
  - **Another method: Monomer-Dimer-Polymer simulation**
    Mutter, Karsch ('89), de Forcrand, Fromm ('10), de Forcrand, Unger ('11)
- **Bosonization of "repulsive" mode: Extended HS transf.**
  → Introducing " *i* " leads to the complex Fermion determinant.
  Miura, Nakano, AO (09), Miura, Nakano, AO, Kawamoto (09)

Strong Coupling Effective Action

- **Lattice QCD action at strong coupling**

anisotropy



$$S_{\rm LQCD}=\frac{\gamma}{2}\sum_x\left[V^+(x)-V^-(x)\right]+m_0\sum_x M_x$$

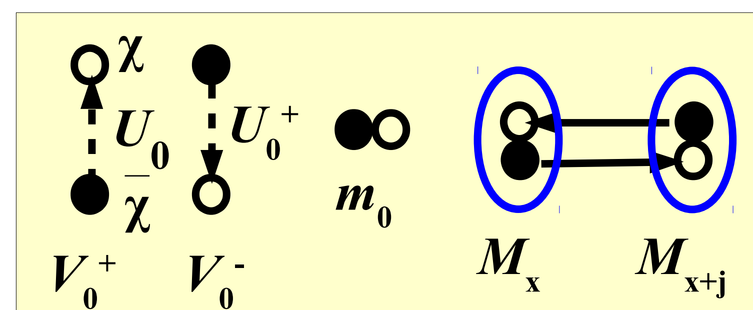
$$+\frac{1}{2}\sum_{x,j}\eta_j(x)\left[\bar\chi_x U_{j,x}\chi_{x+\hat j}-\bar\chi_{x+\hat j}U^+_{j,x}\chi_x\right]+S_G$$

$$V^+(x)=e^{\mu/\gamma^2}\bar\chi_x U_{x,0}\chi_{x+\hat 0},\quad V^-(x)=e^{-\mu/\gamma^2}\bar\chi_{x+\hat 0}U^+_{x,0}\chi_x,\quad M_x=\bar\chi_x\chi_x$$

- **Strong Coupling Limit (LO in $1/g^2$ and $1/d$)**

$$S_{\rm eff}=\frac{\gamma}{2}\sum_x\left[V^+(x)-V^-(x)\right]-\frac{1}{4N_c}\sum_{x,j}M_x M_{x+\hat j}+m_0\sum_x M_x$$

Damgaard, Kawamoto, Shigemoto ('84)



  - **Integrate spatial links first.**
  - **Leading order in $1/g^2$ and $1/d$**
  - **Temporal Link + Nearest Neighbor Int.**

(d=spatial dim.)

Auxiliary Field Effective Action

- **Fermion det. + U0 integral can be done analytically.**
  → Auxiliary field effective action

$$S_{\rm eff}^{\rm AF} = \sum_{\boldsymbol{k},\tau,f(\boldsymbol{k})>0} \frac{L^3 f(\boldsymbol{k})}{4N_c}\left[|\sigma_{\boldsymbol{k},\tau}|^2 + |\pi_{\boldsymbol{k},\tau}|^2\right]$$

$$-\sum_{\boldsymbol{x}} \log\left[X_N(\boldsymbol{x})^3 - 2X_N(\boldsymbol{x}) + 2\cosh(3\mu/T)\right]$$

$$X_N(\boldsymbol{x}) = X_N[m(\boldsymbol{x},\tau)] \qquad (\text{known func.})$$

$$m_x = m_0 + \frac{1}{4N_c}\sum_j \left((\sigma + i\varepsilon\pi)_{x+\hat{j}} + (\sigma + i\varepsilon\pi)_{x-\hat{j}}\right)$$

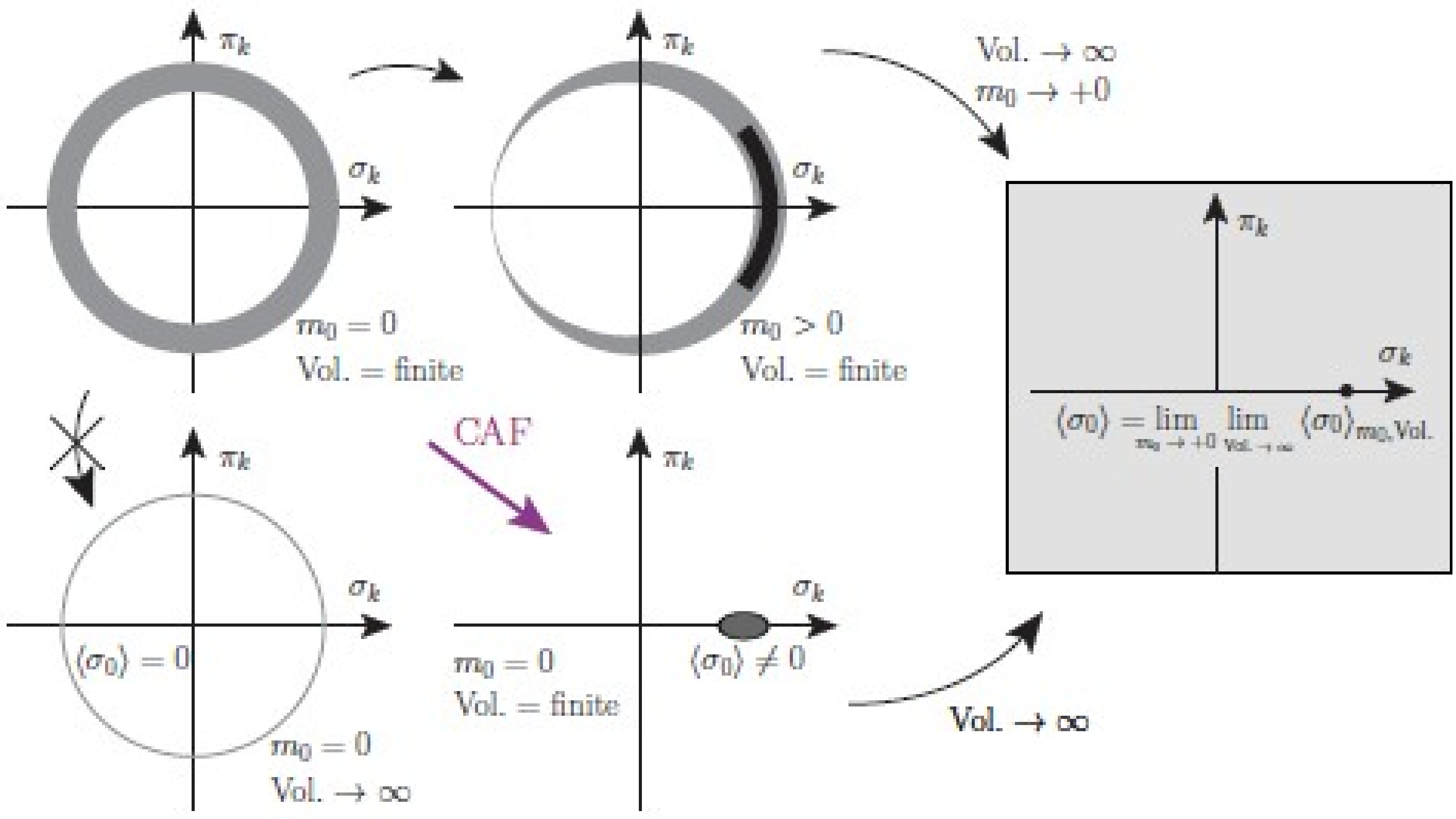
- **X_N = Known function of m(x, τ)** *Faldt, Petersson ('86)*

  For constant m, X_N= 2 cosh (N_τ arcsinh (m/γ))

- **Imag. part from X_N → Relatively smaller at large μ/T**
- **Imag. part from low momentum AF cancels due to iε factor.**

Chiral Angle Fixing

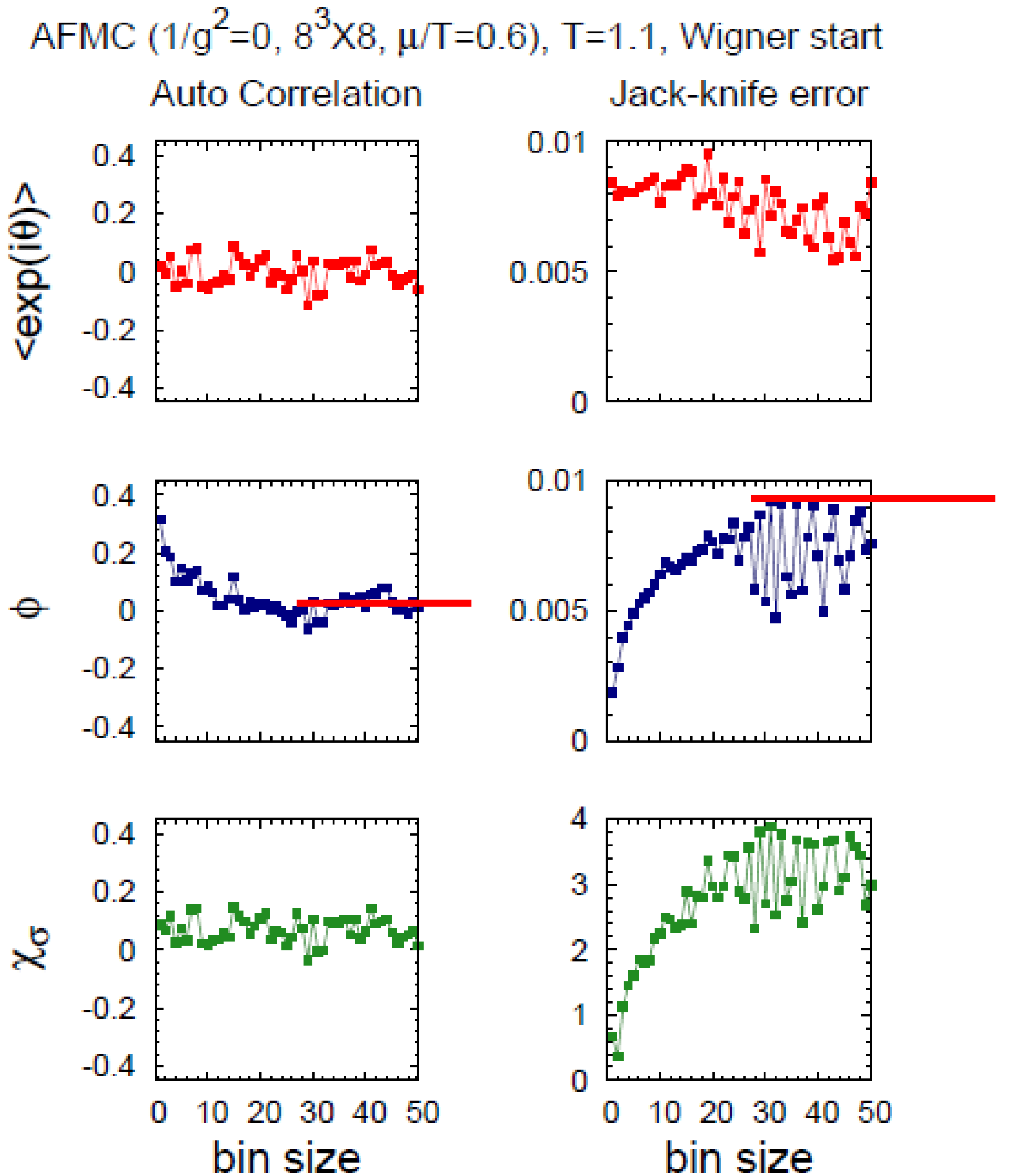
How can we simulate correct thermodynamic chiral limit using finite volume simulations ?



Ichihara, AO, Nakano ('14)

c.f. rms spin is adopted in spin systems *Kurt, Dieter ('10)*

Error estimate by Jack-knife method

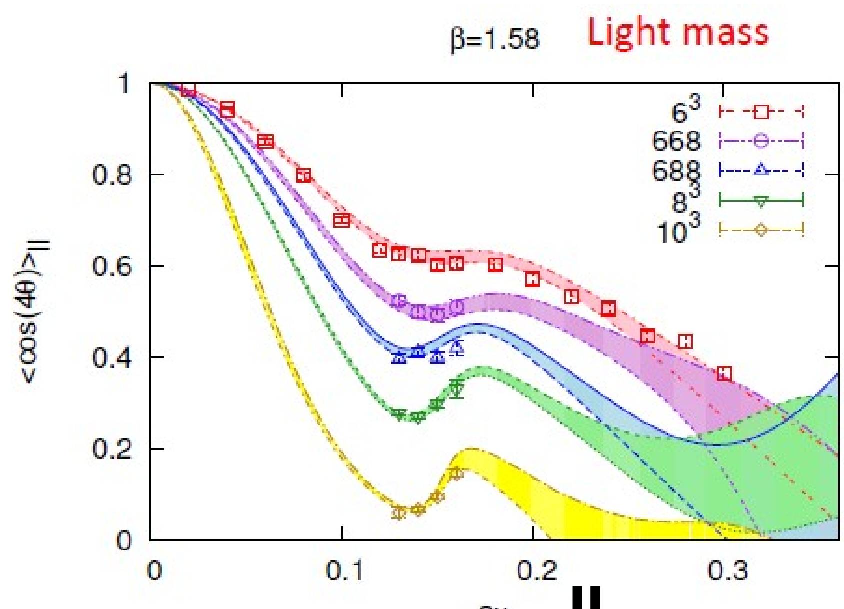


Error
= Jack-knife error after autocorrelation disappears

Ichihara, AO, Nakano ('14)

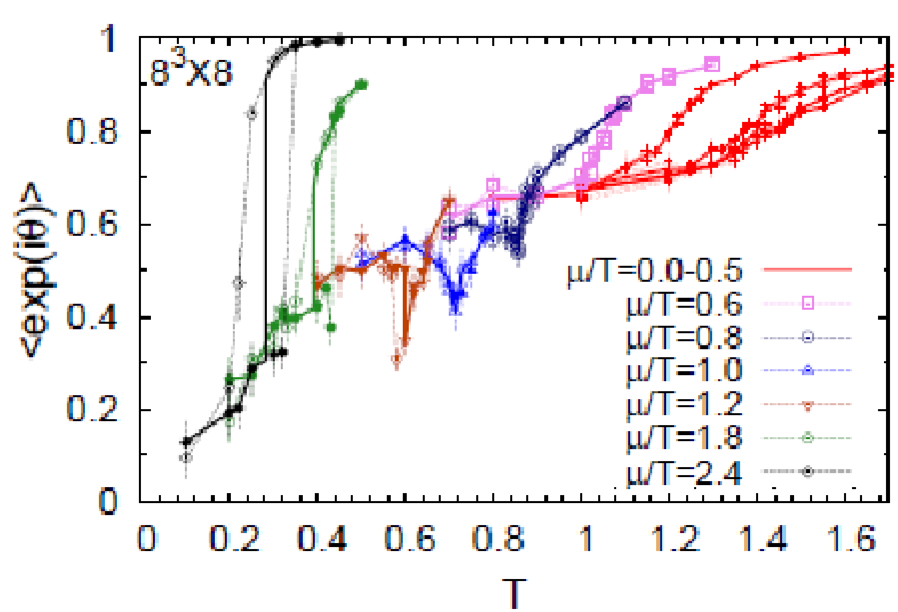
Comparison with Direct Simulation at finite coupling

- **Lattice MC simulation at finite μ and finite β with Nf=4**
  Takeda et al. ('13)
  - **Ave. Phase Factor ~ 0.3 at aμ ~ 0.15 (8^3 x 4, $a\mu_c = am_\pi/2 \sim 0.7$)**
- **AFMC**
  - **Ave. Phase Factor ~ 0.6 around the transition (8^4, SCL)**



Takeda, Jin, Kuramashi, Y.Nakamura, Ukawa, Lattice 2013

$a\mu_c = am_\pi/2 \sim 0.7$



Ichihara, AO, Nakano ('14)

- **Summary of formulation (MDP, AFMC)**

Can we suppress the sign problem ?

Sign problem

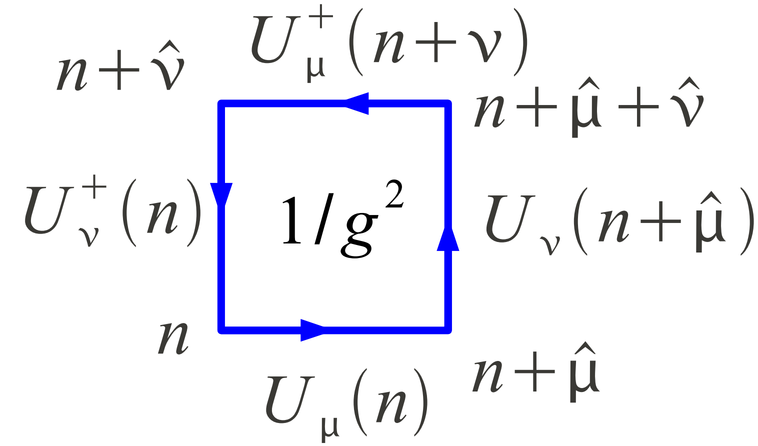
- **Sign problem**

Thank you

Lattice QCD action

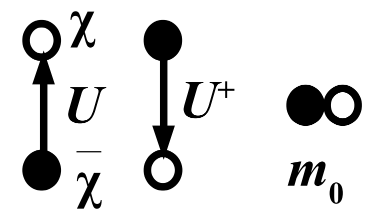
- **Gluon field → Link variables** $U_\mu(x) \simeq \exp(i g A_\mu)$
- **Gluon action → Plaquette action**

$$S_G = \frac{2N_c}{g^2} \sum_{plaq.} \left[1 - \frac{1}{N_c} \mathrm{Re}\,\mathrm{tr}\, U_{\mu\nu}(n) \right]$$



  - **Loop → surface integral of "rotation" $F_{\mu\nu}$ in the U(1) case.**

- **Quark action (staggered fermion)**



$$S_F = \frac{1}{2} \sum_x \left[\bar{\chi}_x e^{\mu} U_{0,x} \chi_{x+\hat{0}} - \bar{\chi}_{x+\hat{0}} e^{-\mu} U^+_{0,x} \chi_x \right]$$

$$+ \frac{1}{2} \sum_{x,j} \eta_j(x) \left[\bar{\chi}_x U_{x,j} \chi_{x+\hat{j}} - \bar{\chi}_{x+\hat{j}} U^+_{x,j} \chi_x \right] + \sum_x m_0 M_x$$

$$= S_F^{(t)} + S_F^{(s)} + \sum_x m_0 M_x$$

- **One-link integral**

$$\int dU\, U_{ab} U^{+}_{cd} = \frac{1}{N_c} \delta_{ad} \delta_{bc}$$

- **Wilson loop in pure Yang-Mills theory**

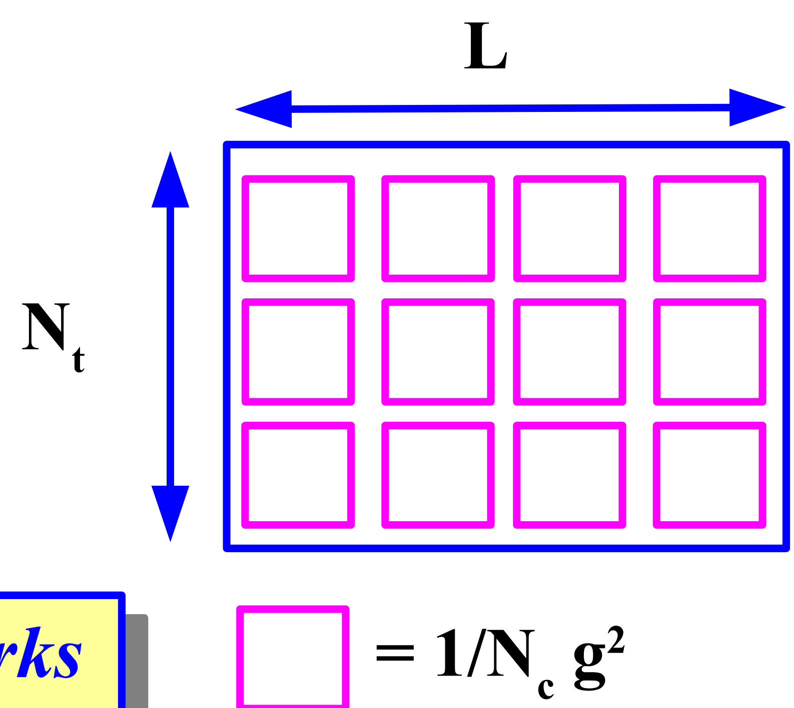
$$\langle W(C = L \times N_\tau) \rangle = \frac{1}{Z} \int DU\, W(C) \exp\left[\frac{1}{g^2} \sum_P \mathrm{tr}\left(U_P + U_P^{+}\right) \right]$$

$$= \exp\left(-V(L) N_\tau\right)$$

in the strong coupling limit

$$\langle W(C) \rangle = N \left(\frac{1}{g^2 N} \right)^{L N_\tau}$$

$$\rightarrow V(L) = L \log\left(g^2 N\right)$$



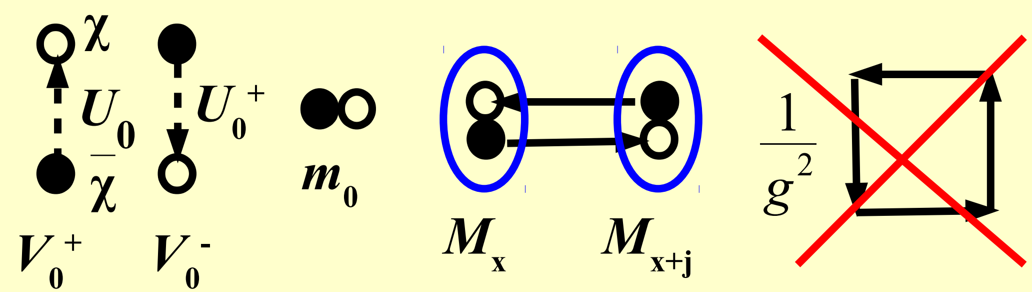
Linear potential between heavy-quarks → Confinement (Wilson, 1974)

Link integral → Effective action

- **Effective action in the strong coupling limit (SCL)**
  - **Ignore plaquette action ($1/g^2$)**
    → We can integrate each link independently !
  - **Integrate out *spatial* link variables of min. quark number diagrams (1/d expansion)**

$$S_{\text{eff}} = S_F^{(t)} - \frac{1}{4N_c} \sum_{x,j} M_x M_{x+\hat{j}} + m_0 \sum_x M_x \quad \left(M_x = \bar{\chi}_x \chi_x \right)$$

Damgaard, Kawamoto, Shigemoto ('84)

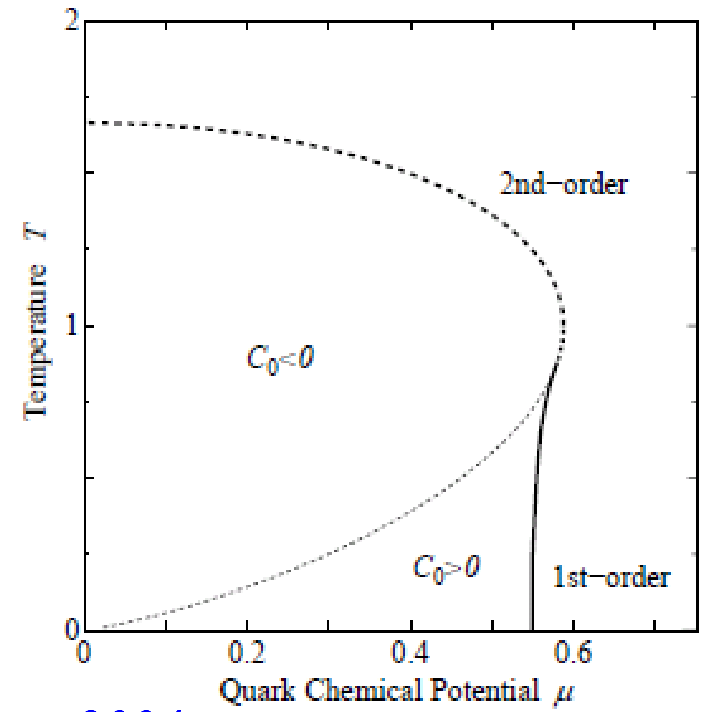


$$\int dU \, U_{ab} U^+_{cd} = \delta_{ad} \delta_{bc} / N_c$$

Lattice QCD in SCL
→ Fermion action
with nearest neighbor
four Fermi interaction

Phase diagram in SC-LQCD (mean field)

- **"Standard" simple procedure in Fermion many-body problem**
  - **Bosonize interaction term (Hubbard-Stratonovich transformation)**
  - **Mean field approximation (constant auxiliary field)**
  - **Fermion & temporal link integral**
    Damgaard, Kawamoto, Shigemoto ('84); Faldt, Petersson ('86); Bilic, Karsch, Redlich ('92); Fukushima ('04); Nishida ('04)



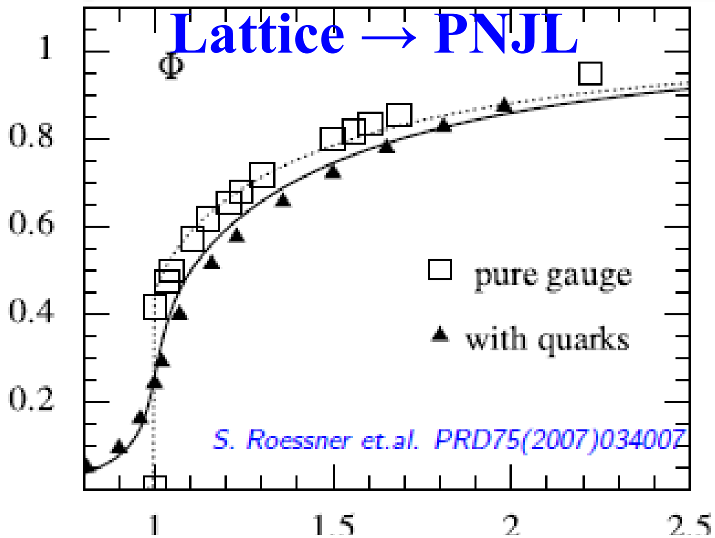
Fukushima, 2004

Thank you

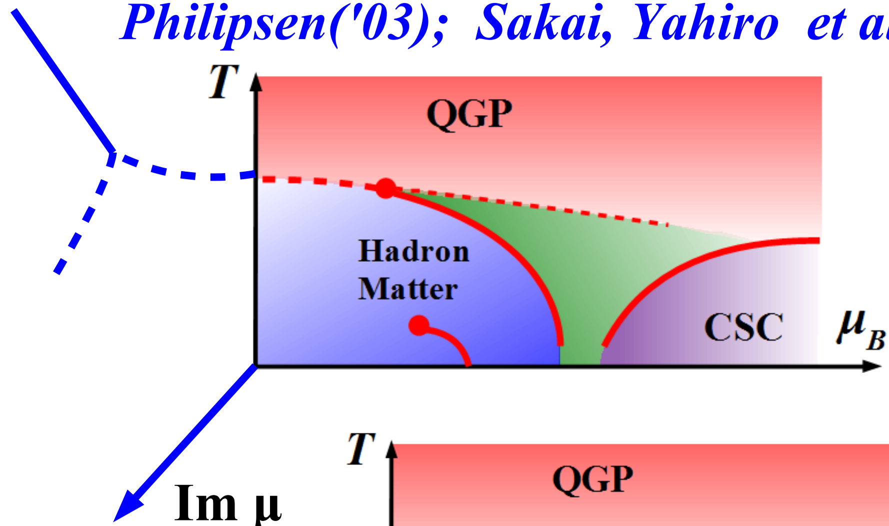
Phase Diagram under Various Conditions

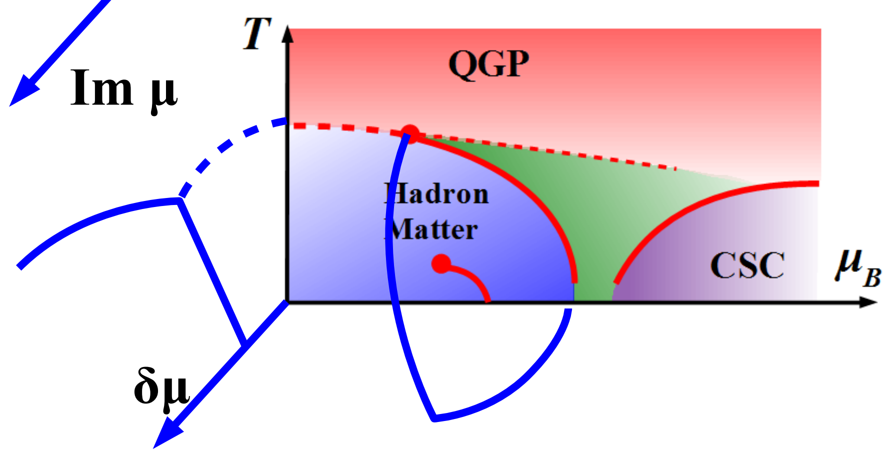
- **Vacuum hadron properties + finite T (μ=0) lattice data**

  + 3D phase diagram !

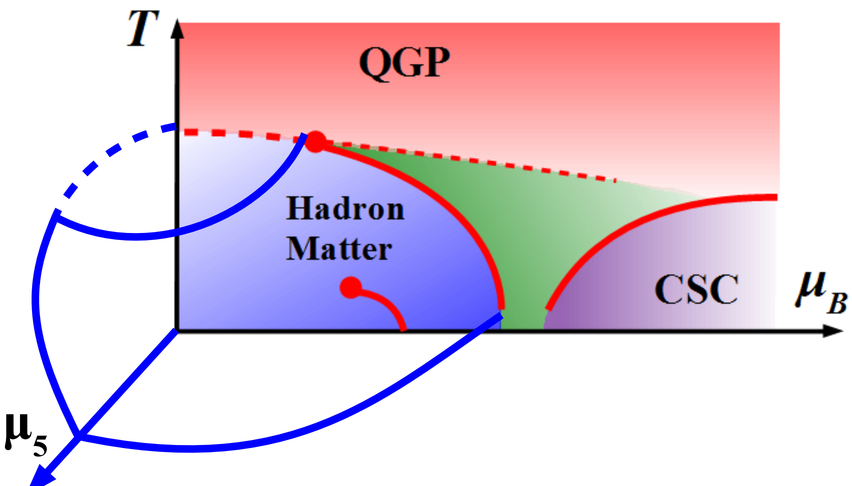


D'Elia, Lombardo('03); de Forcrand, Philipsen('03); Sakai, Yahiro et al. ('10),





Kogut, Sinclair ('04); Sakai et al.('10); AO, Ueda, Nakano, Ruggieri, Sumiyoshi ('11)



Fukushima et al.('08); Fukushima,Ruggieri,Gatto('10), Ruggieri ('10);Yamamoto('11); Nakano,AO (in prep.)

No sign prob. at μ_B=0

How can we investigate QCD phase diagram ?

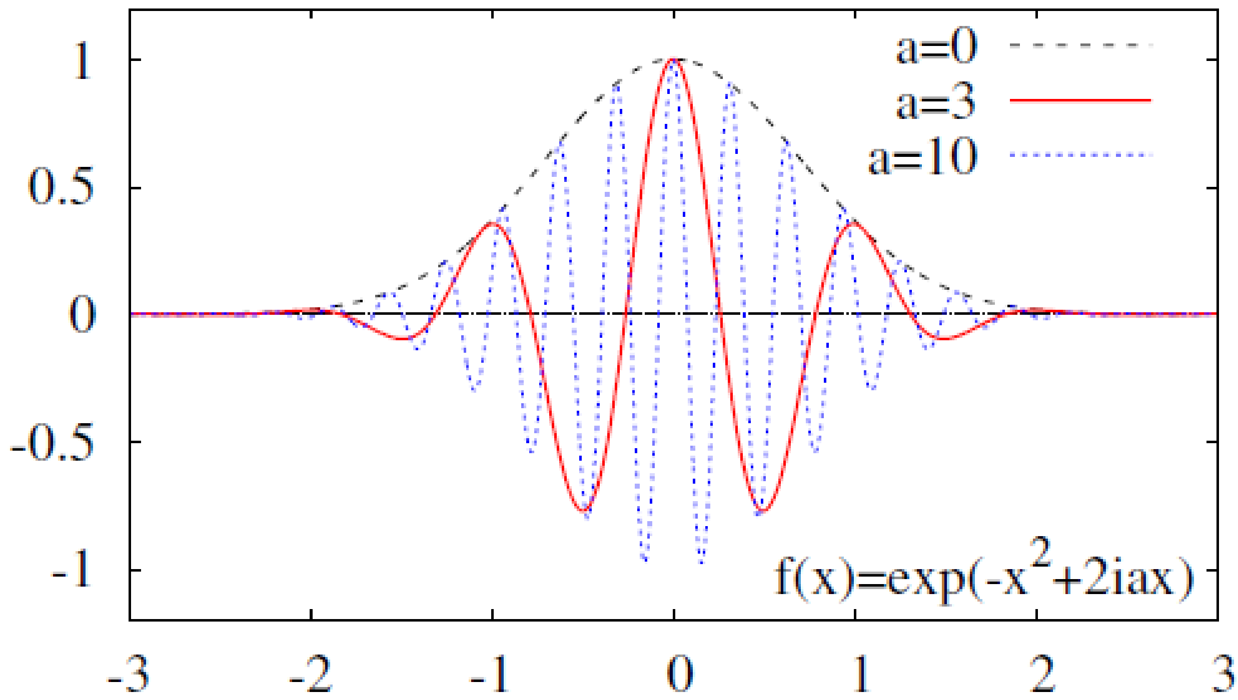
- **Non-pert. & ab initio approach**
  = Monte-Carlo simulation of lattice QCD
  but lattice QCD at finite μ has the sign problem.

Sign Problem

- **Monte-Carlo integral of oscillating function**

$$Z=\int dx \exp(-x^2+2iax)=\sqrt{\pi}\exp(-a^2)$$

$$\langle O\rangle=\frac{1}{Z}\int dx\, O(x)\, e^{-x^2+2iax}$$

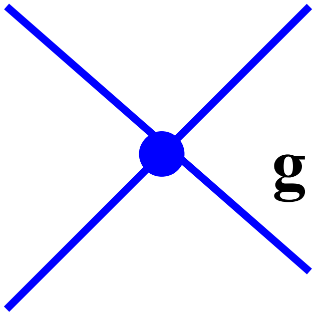


Easy problem for human is not necessarily easy for computers.

Sign Problem (cont.)

- **Generic problem in quantum many-body problems**
  - **Example: Euclid action of interacting Fermions**

$$S=\sum_{x,y}\bar{\psi}_x D_{x,y}\psi_y+g\sum_x(\bar{\psi}\psi)_x(\bar{\psi}\psi)_x$$



  - **Bosonization and MC integral (g>0 → repulsive)**

$$\exp(-g M_x M_x)=\int d\sigma_x \exp(-g\sigma_x^2-2igσ_x M_x)\quad (M_x=(\bar{\psi}\psi)_x)$$

$$Z=\int D[\psi,\bar{\psi},\sigma]\exp\left[-\bar{\psi}(D+2ig\sigma)\psi-g\sum_x\sigma_x^2\right]$$

$$=\int D[\sigma]\ \underline{\mathrm{Det}(D+2ig\sigma)}\exp\left[-g\sum_x\sigma_x^2\right]$$

complex Fermion det.
→ complex stat. weight
→ sign problem

Sign problem in lattice QCD

- **Fermion determinant (= stat. weight of MC integral) becomes complex at finite μ in LQCD.**
  - **γ_5 Hermiticity**

$$Z=\int D[U,q,\bar{q}]\ \exp(-\bar{q}D(\mu,U)q-S_G(U))$$
$$=\int D[U]\ \mathrm{Det}(D(\mu,U))\ \exp(-S_G(U))$$

$$\gamma_5 D(\mu,U)\gamma_5=[D(-\mu^*,U^+)]^+$$
$$\rightarrow \mathrm{Det}(D(\mu,U))=[\mathrm{Det}(D(-\mu^*,U^+))]^*$$

  - **Fermion det. (Det D) is real for zero μ (and pure imag. μ)**
  - **Fermion det. is complex for finite real μ.**
  - **Phase quenched weight = Weight at isospin chem. pot.**

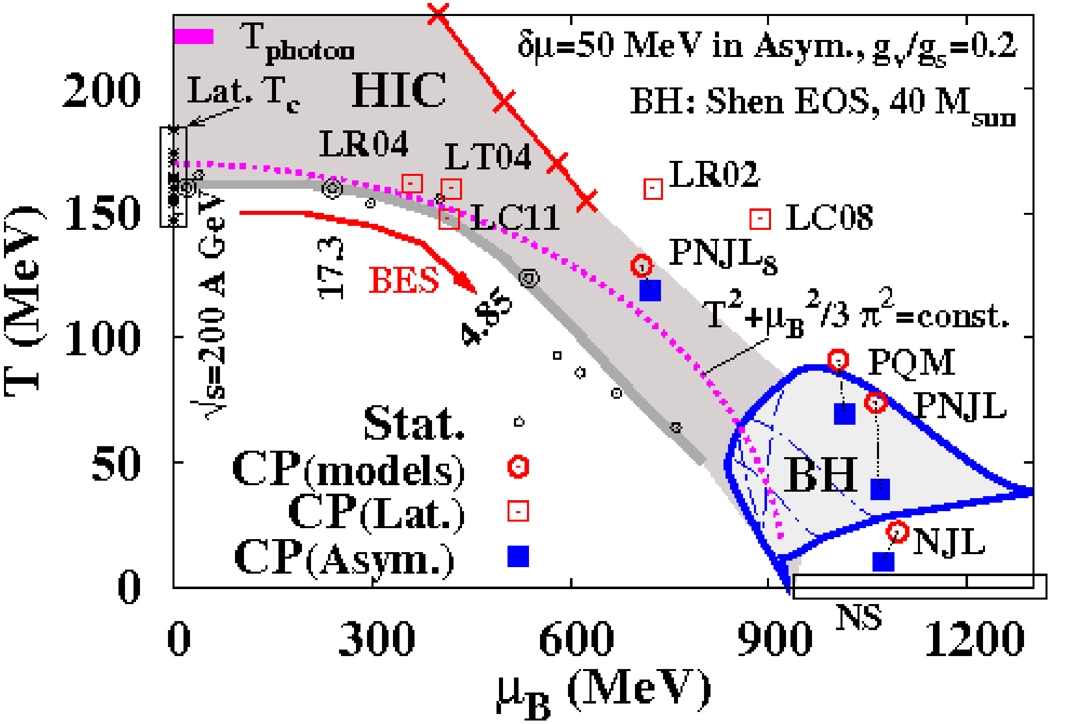
$$Z_{\text{phase quench}}(T,\mu_u=\mu_d=\mu)=Z_{\text{full}}(T,\mu_u=-\mu_d=\mu)$$

How can we investigate QCD phase diagram ?

- **Non-pert. & ab initio approach**
  = Monte-Carlo simulation of lattice QCD
  but lattice QCD at finite μ has the sign problem.

- **Effective model and/or Approximations are necessary.**

  - **Effective models:**
    NJL, PNJL, PQM, ...
    Model dependence is large.
  - **Approximation / Truncation**
    Taylor expansion,
    Imag. μ , Canonical,
    Re-weighting,
    Strong coupling LQCD
  - **Alternative method**
    Fugacity expansion,
    Histogram method,
    Complex Langevin



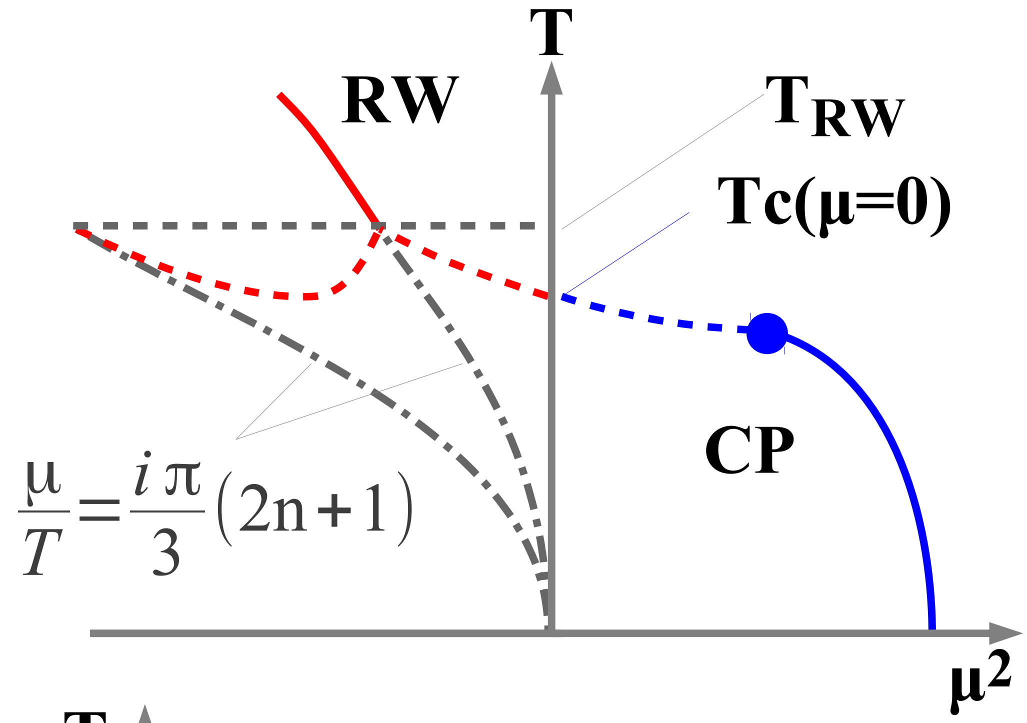
Lattice QCD at fnite μ

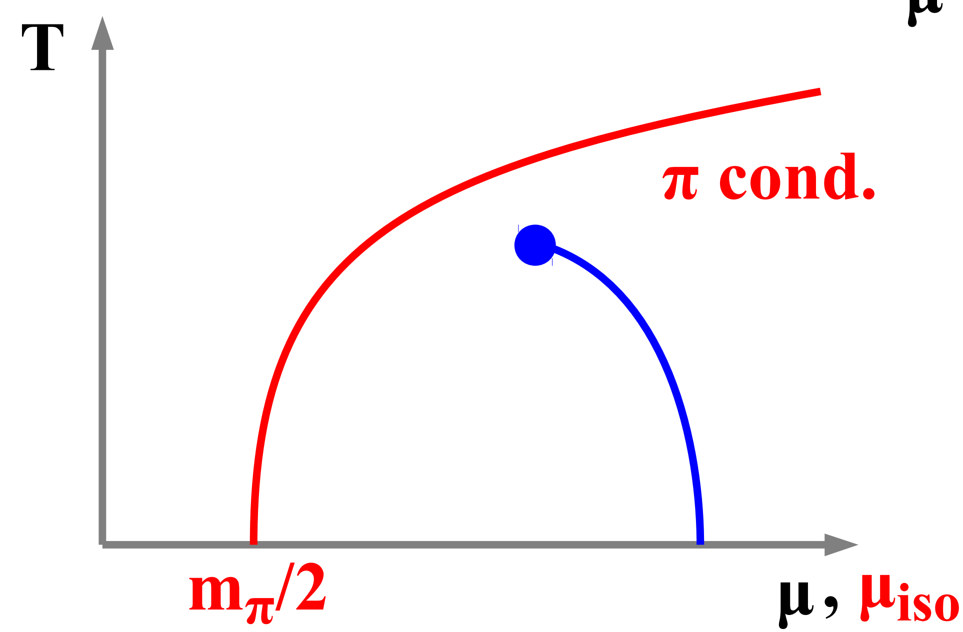
- **Various method work at small μ (μ/T < 1).**
- **Large μ**
  - **Roberge-Weiss transition**
    → Conv. μ/T < π/3 at T>T_{RW}
  - **No go theorem**
    Splittorff ('06), Han, Stephanov ('08), Hanada, Yamamoto ('11), Hidaka, Yamamoto ('11)

    Phase quenched sim.
    ~ Isospin chem. pot.

$$Z_{\text{phase quench}}(T, \mu_u = \mu_d = \mu) = Z_{\text{full}}(T, \mu_u = -\mu_d = \mu)$$

→ CP in π cond. phase (Silver Blaze)

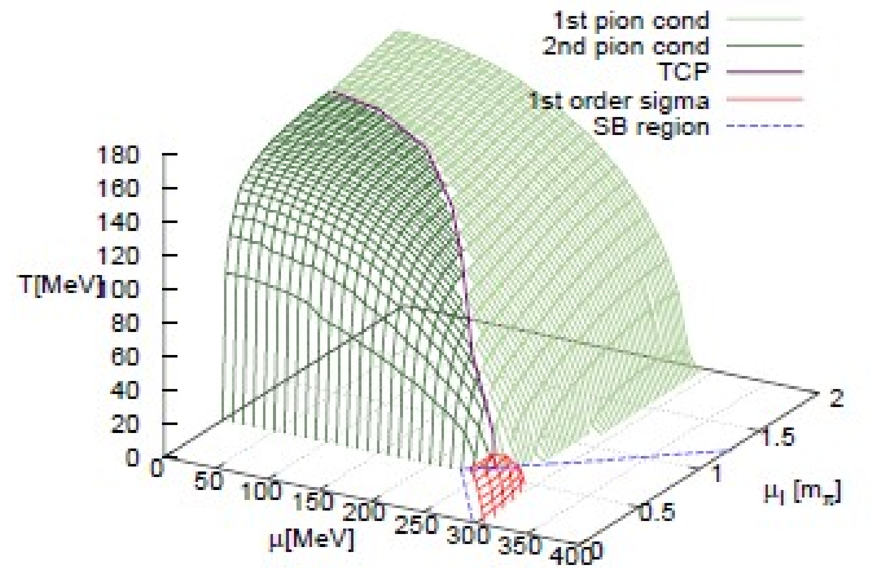


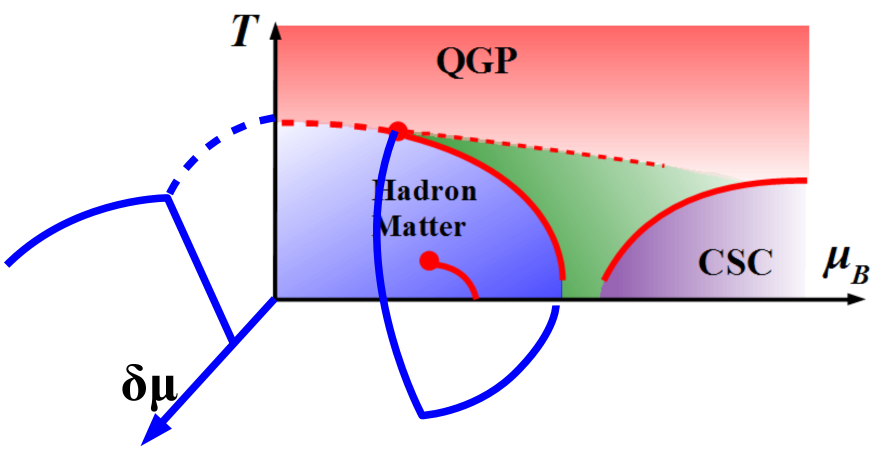


Phase diagram in isospin chemical potential space

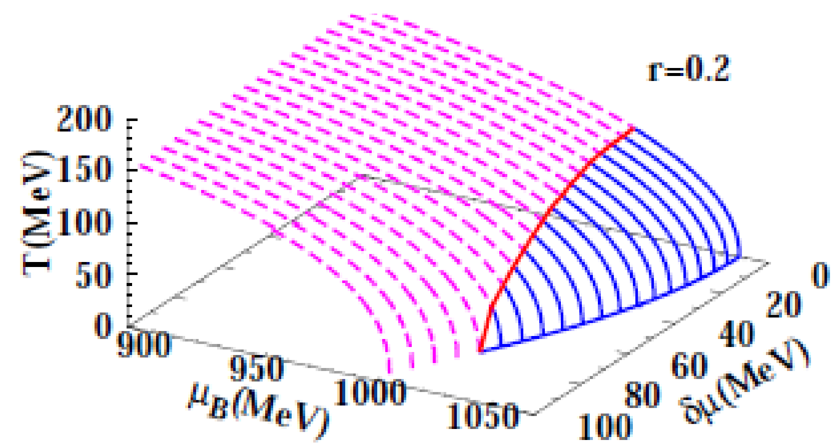
- Important in Compact Astrophys. phen.
- At vanishing quark chem. pot., there is no sign problem.
- Interesting 3D phase structure.

FRG: Kamikado, Strodthoff, von Smekal, Wambach ('13)





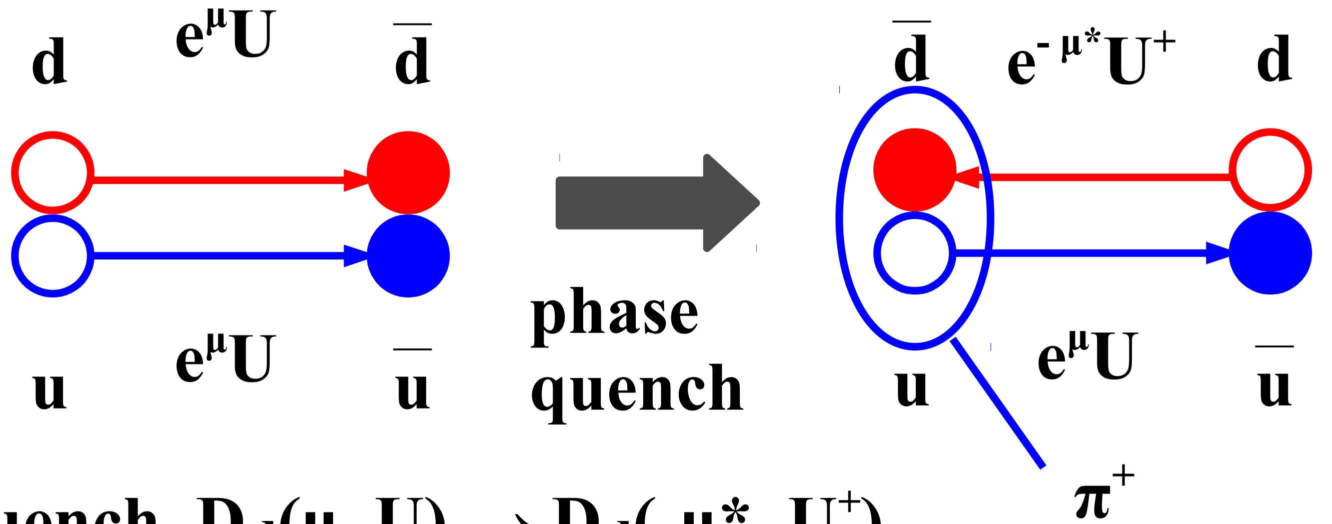
Kogut, Sinclair ('04); Sakai et al.('10); AO, Ueda, Nakano, Ruggieri, Sumiyoshi ('11)



PQM: Ueda, Nakano, AO, Ruggieri, Sumiyoshi ('13)

Silver Blaze

- "Watson, the dog did not bark at night. This is the evidence that he is the criminal who stole Silver Blaze."
- In physics,
  "If $\delta\mu > m_\pi/2$ at low T and you do not have pion condensation, that theory should be wrong."



- Phase quench $D_d(\mu, U) \to D_d(-\mu^*, U^+)$
  → We can compose pions from original di-quark configuration.
- To do: Directly sample with complex S (CLE), Integrate U first (SC-LQCD), and some other method....

How can we investigate QCD phase diagram ?

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    Strong coupling LQCD
  - **Alternative method**
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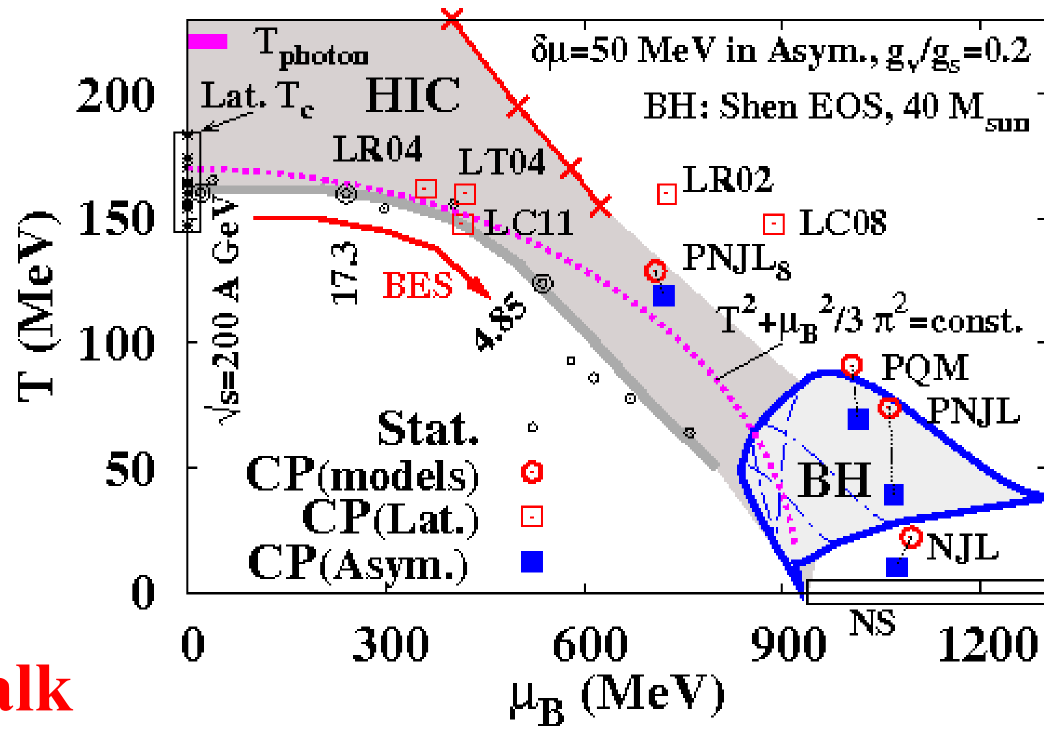
    Histogram method,
    Complex Langevin

This talk

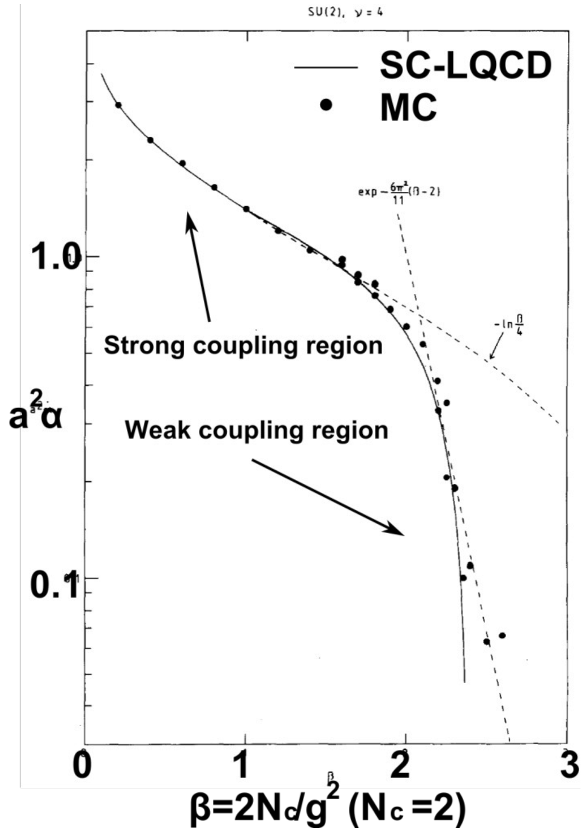
Nakamura, Nagata
Ejiri
Stamatescu



Strong coupling lattice QCD

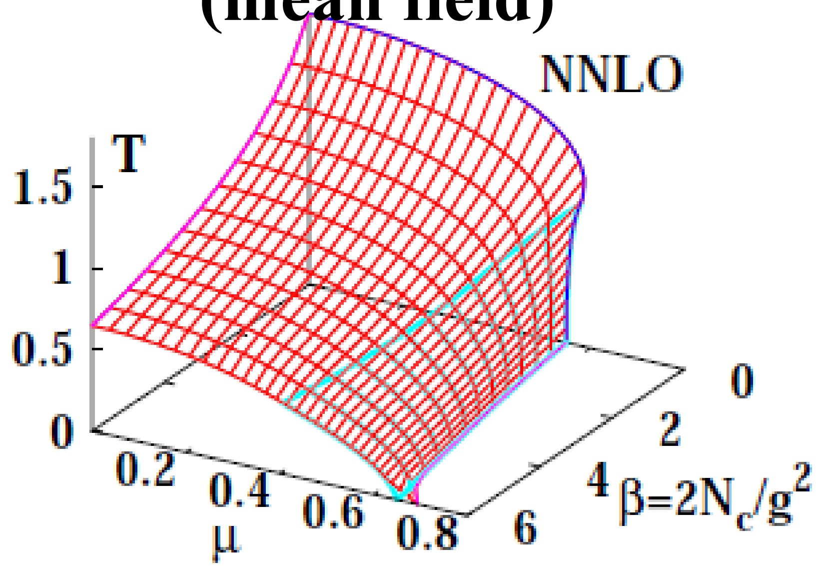
Strong Coupling Lattice QCD

Pure YM



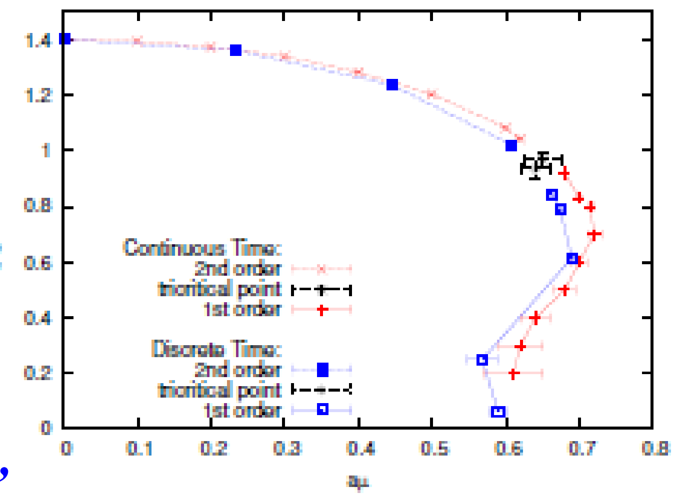
Wilson ('74), Creutz ('80), Munster ('80, '81), Lottini, Philipsen, Langelage's ('11)

Phase diagram (mean field)



Kawamoto ('80), Kawamoto, Smit ('81), Damagaard, Hochberg, Kawamoto ('85), Bilic, Karsch, Redlich ('92), Fukushima ('03); Nishida ('03), Kawamoto, Miura, AO, Ohnuma ('07). Miura, Nakano, AO, Kawamoto ('09) Nakano, Miura, AO ('10)

Fluctuations



Mutter, Karsch ('89), de Forcrand, Fromm ('10), de Forcrand, Unger ('11), AO, Ichihara, Nakano ('12), Ichihara, Nakano, AO ('13)

SC-LQCD: Setups & Disclaimer

- **Effective action in SCL ($1/g^0$), NLO ($1/g^2$), NNLO ($1/g^4$) terms and Polyakov loop.**

  NLO: Faldt-Petersson ('86), Bilic-Karsch-Redlich ('92)
  Conversion radius > 6 in pure YM ? Osterwalder-Seiler ('78)

- **One species of unrooted staggered fermion (N_f=4 @ cont.)**

  Moderate N_f deps. of phase boundary: BKR92, Nishida('04), D'Elia-Lombardo ('03)

- **Leading order in 1/d expansion (d=3=space dim.)**
  → Min. # of quarks for a given plaquette configurations, no spatial B hopping term.

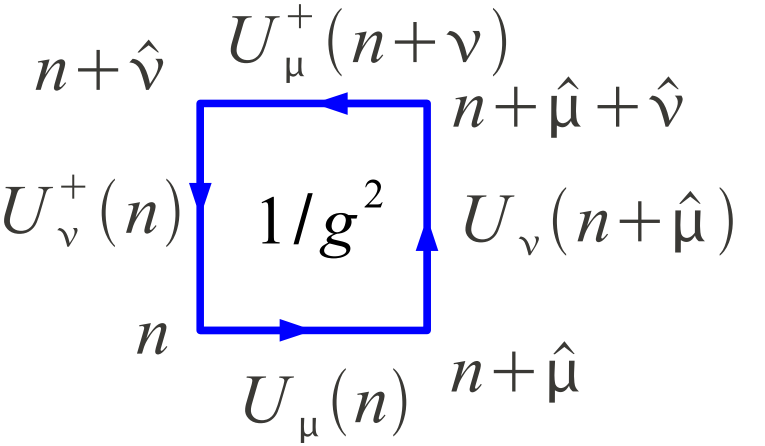
- **Different from "strong couling" in "large N_c"**

> ***Still far from "Realistic", but SC-LQCD would tell us useful qualitative features of the phase diagram and EOS.***

Lattice QCD action

- **Gluon field → Link variables** $U_\mu(x) \simeq \exp(i g A_\mu)$
- **Gluon action → Plaquette action**

$$S_G = \frac{2N_c}{g^2} \sum_{plaq.} \left[1 - \frac{1}{N_c} \mathrm{Re\,tr}\, U_{\mu\nu}(n) \right]$$



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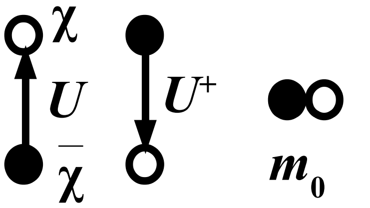
- **Quark action (staggered fermion)**



$$S_F = \frac{1}{2} \sum_x \left[\bar{\chi}_x e^{\mu} U_{0,x} \chi_{x+\hat{0}} - \bar{\chi}_{x+\hat{0}} e^{-\mu} U^+_{0,x} \chi_x \right]$$

$$+ \frac{1}{2} \sum_{x,j} \eta_j(x) \left[\bar{\chi}_x U_{x,j} \chi_{x+\hat{j}} - \bar{\chi}_{x+\hat{j}} U^+_{x,j} \chi_x \right] + \sum_x m_0 M_x$$

$$= S_F^{(t)} + S_F^{(s)} + \sum_x m_0 M_x$$

Link integral → Area Law

- **One-link integral**

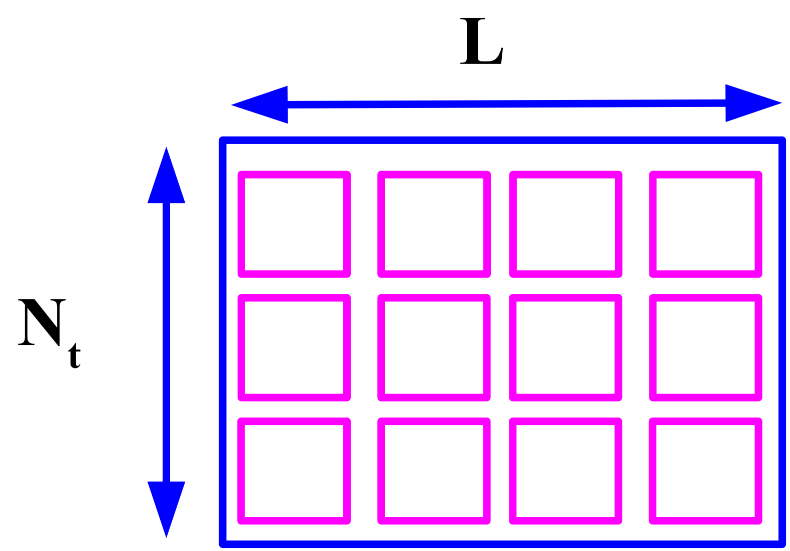
$$\int dU\, U_{ab} U^{+}_{cd} = \frac{1}{N_c} \delta_{ad} \delta_{bc}$$

- **Wilson loop in pure Yang-Mills theory**

$$\langle W(C = L \times N_\tau) \rangle = \frac{1}{Z} \int DU\, W(C) \exp\left[\frac{1}{g^2} \sum_P \mathrm{tr}\left(U_P + U_P^{+}\right) \right]$$
$$= \exp\left(-V(L) N_\tau\right)$$

in the strong coupling limit

$$\langle W(C) \rangle = N \left(\frac{1}{g^2 N} \right)^{L N_\tau}$$
$$\rightarrow V(L) = L \log\left(g^2 N\right)$$



= $1/N_c\, g^2$

Linear potential between heavy-quarks → Confinement (Wilson, 1974)

Link integral → Effective action

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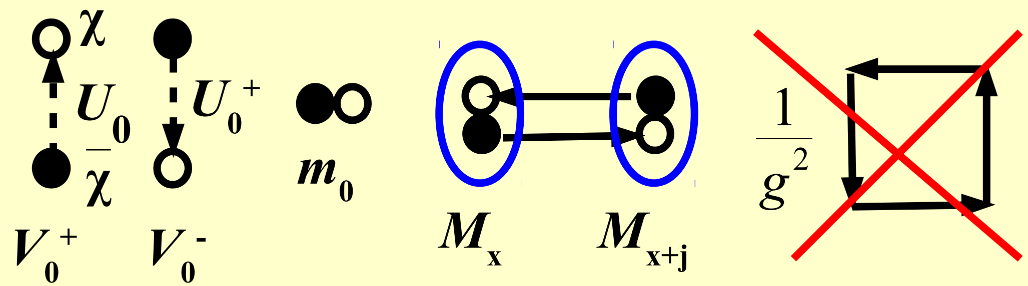
  - **Ignore plaquette action ($1/g^2$)**
    → We can integrate each link independently !
  - **Integrate out *spatial* link variables of min. quark number diagrams (1/d expansion)**

$$S_{\text{eff}} = S_F^{(t)} - \frac{1}{4N_c}\sum_{x,j} M_x M_{x+\hat{j}} + m_0 \sum_x M_x \quad \left(M_x = \bar{\chi}_x \chi_x\right)$$

Damgaard, Kawamoto, Shigemoto ('84)

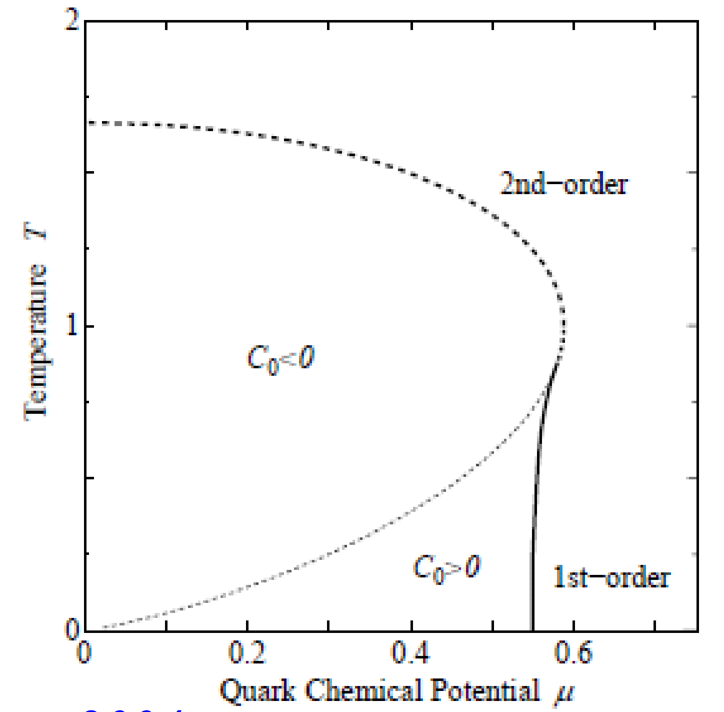


$$\int dU\, U_{ab} U^+_{cd} = \delta_{ad}\delta_{bc}/N_c$$

Lattice QCD in SCL
→ Fermion action
with nearest neighbor
four Fermi interaction

Phase diagram in SC-LQCD (mean field)

- "Standard" simple procedure in Fermion many-body problem
  - Bosonize interaction term (Hubbard-Stratonovich transformation)
  - Mean field approximation (constant auxiliary field)
  - Fermion & temporal link integral
    Damgaard, Kawamoto, Shigemoto ('84); Faldt, Petersson ('86); Bilic, Karsch, Redlich ('92); Fukushima ('04); Nishida ('04)



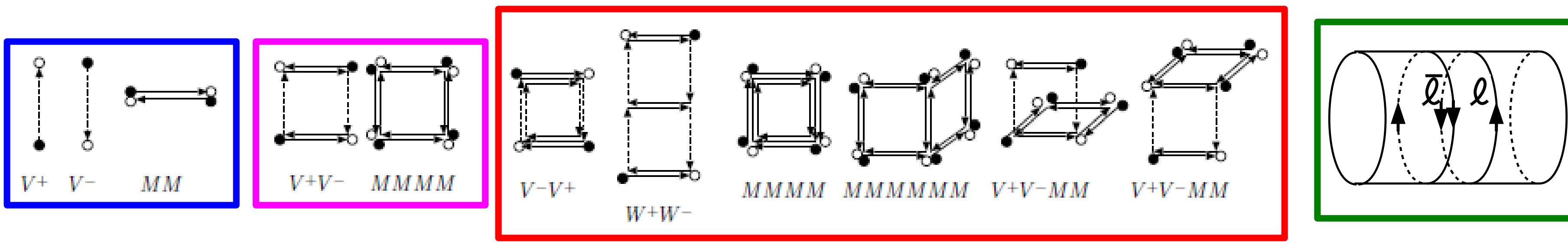
Fukushima, 2004

Finite Coupling Effects

- **Effective Action with finite coupling corrections**
  Integral of exp(-S_G) over spatial links with exp(-S_F) weight → S_{eff}

$$S_{\text{eff}} = S_{\text{SCL}} - \log\langle \exp(-S_G)\rangle = S_{\text{SCL}} - \sum_{n=1} \frac{(-1)^n}{n!}\langle S_G^n\rangle_c$$

$\langle S_G^n \rangle_c$=Cumulant (connected diagram contr.) *c.f. R.Kubo('62)*



$$S_{\text{eff}} = \frac{1}{2}\sum_x (V_x^+ - V_x^-) - \frac{b_\sigma}{2d}\sum_{x,j>0}[MM]_{j,x}$$

SCL (Kawamoto-Smit, '81)

$$+\frac{1}{2}\frac{\beta_\tau}{2d}\sum_{x,j>0}[V^+V^- + V^-V^+]_{j,x} - \frac{1}{2}\frac{\beta_s}{d(d-1)}\sum_{x,j>0,k>0,k\neq j}[MMMM]_{jk,x}$$

NLO (Faldt-Petersson, '86)

$$-\frac{\beta_{\tau\tau}}{2d}\sum_{x,j>0}[W^+W^- + W^-W^+]_{j,x} - \frac{\beta_{ss}}{4d(d-1)(d-2)}\sum_{\substack{x,j>0,|k|>0,|l|>0\\ |k|\neq j,|l|\neq j,|l|\neq|k|}}[MMMM]_{jk,x}[MM]_{j,x+\hat{l}}$$

$$+\frac{\beta_{\tau s}}{8d(d-1)}\sum_{x,j>0,|k|\neq j}[V^+V^- + V^-V^+]_{j,x}\left([MM]_{j,x+\hat{k}} + [MM]_{j,x+\hat{k}+\hat{0}}\right)$$

NNLO (Nakano, Miura, AO, '09)

$$-\left(\frac{1}{g^2N_c}\right)^{N_\tau} N_c^2 \sum_{\mathbf{x},j>0}\left(\bar{P}_{\mathbf{x}}P_{\mathbf{x}+\hat{j}} + h.c.\right)$$

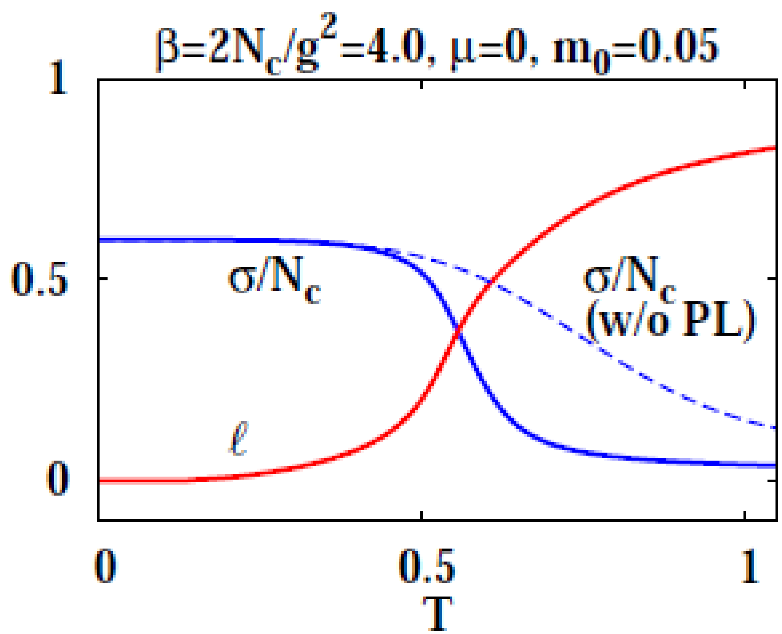
Polyakov loop (Gocksch, Ogilvie ('85), Fukushima ('04)
Nakano, Miua, AO ('11))

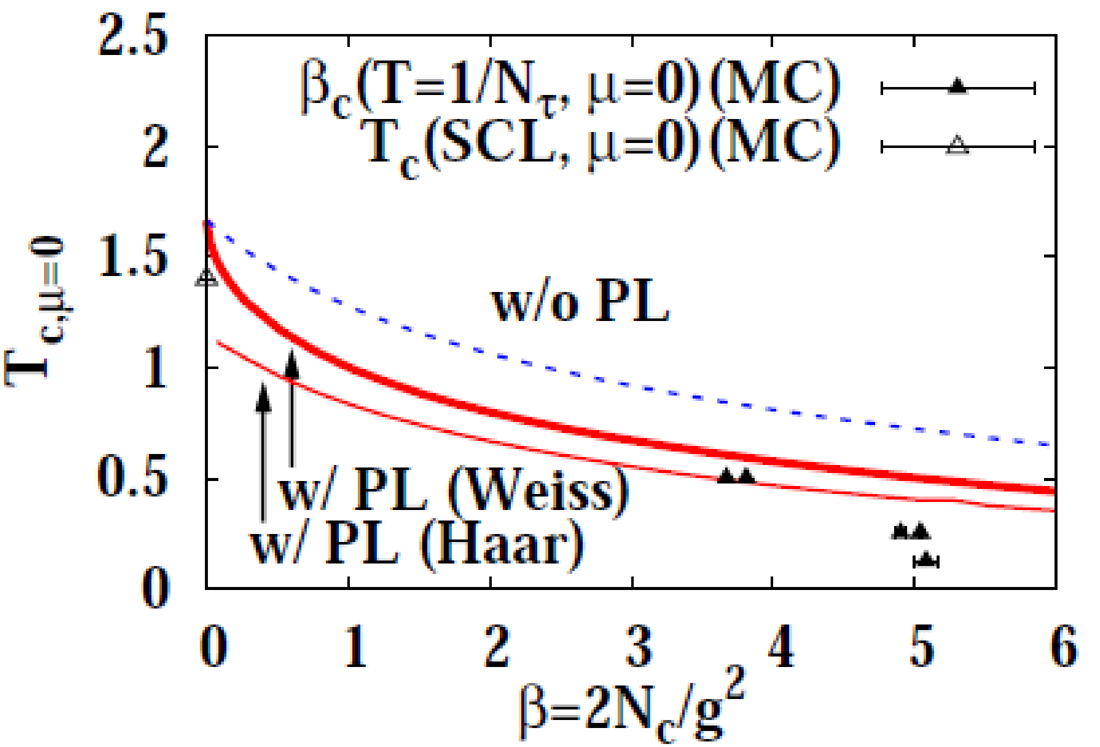
SC-LQCD with Polyakov Loop Effects at μ=0

T. Z. Nakano, K. Miura, AO, PRD 83 (2011), 016014 [arXiv:1009.1518 [hep-lat]]

- **P-SC-LQCD reproduces MC results of $T_c(\mu=0)$ ($\beta= 2N_c/g^2 \leq 4$)**

MC data: SCL (Karsch et al. (MDP), de Forcrand, Fromm (MDP)), N_τ =2 (de Forcrand, private), N_τ=4 (Gottlieb et al.('87), Fodor-Katz ('02)), N_τ =8 (Gavai et al.('90))





Lattice Unit

Beyond the mean field approximation

- **Constant auxiliary field → Fluctuating auxiliary field**

$$S_{\text{eff}} = S_F^{(t)} + \sum_x m_x M_x + \frac{L^3}{4N_c} \sum_{\boldsymbol{k},\tau} f(\boldsymbol{k}) \left[|\sigma_{\boldsymbol{k},\tau}|^2 + |\pi_{\boldsymbol{k},\tau}|^2 \right]$$

$$m_x = m_0 + \frac{1}{4N_c} \sum_j \left((\sigma + i\varepsilon\pi)_{x+\hat{j}} + (\sigma + i\varepsilon\pi)_{x-\hat{j}} \right)$$

$$f(\boldsymbol{k}) = \sum_j \cos k_j \; < , \quad \varepsilon = (-1)^{x_0 + x_1 + x_2 + x_3}$$

- **Auxiliary Field Monte-Carlo (AFMC) integral**
  - **Another method: Monomer-Dimer-Polymer simulation**
    Mutter, Karsch ('89), de Forcrand, Fromm ('10), de Forcrand, Unger ('11)
- **Bosonization of "repulsive" mode: Extended HS transf.**
  → Introducing " *i* " leads to the complex Fermion determinant.
  Miura, Nakano, AO (09), Miura, Nakano, AO, Kawamoto (09)

Origin of the sign problem in AFMC

- **Extended Hubbard-Stratonovich transformation**

Miura, Nakano, AO ('09), Miura, Nakano, AO, Kawamoto ('09)

$$e^{\alpha AB} = \int d\varphi\, d\phi\, e^{-\alpha\left[(\varphi+(A+B)/2)^2+(\phi+i(A-B)/2)^2-AB\right]}$$

$$= \int d\varphi\, d\phi\, e^{-\alpha\left[\varphi^2+\phi^2+\varphi(A+B)+i\phi(A-B)\right]}$$

Complex

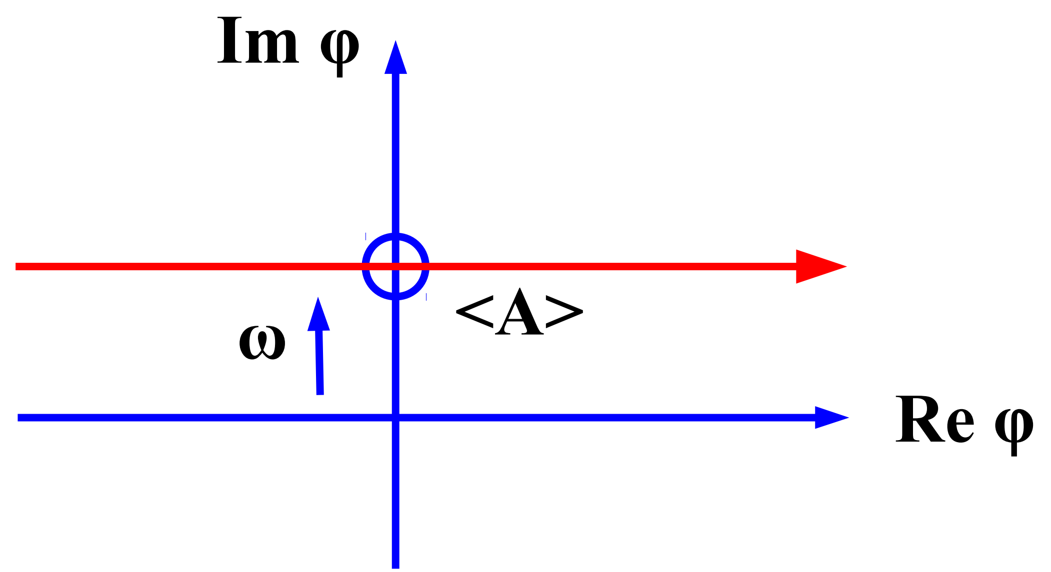
We need "i" to bosonize product of different kind.
→ Fermion determinant becomes complex.

- **Bosonization in AFMC in the strong coupling limit**

$$\exp\left\{\alpha f(\mathrm{k})\left[M_{-\mathbf{k},\tau}M_{\mathbf{k},\tau} - M_{-\bar{\mathbf{k}},\tau}M_{\bar{\mathbf{k}},\tau}\right]\right\}$$
$$= \int d\sigma_{\mathbf{k},\tau}\, d\sigma^*_{\mathbf{k},\tau}\, d\pi_{\mathbf{k},\tau}\, d\pi^*_{\mathbf{k},\tau} \exp\left\{-\alpha f(\mathrm{k})\left[|\sigma_{\mathbf{k},\tau}|^2 + |\pi_{\mathbf{k},\tau}|^2\right.\right.$$
$$\left.\left. +\sigma^*_{\mathbf{k},\tau}M_{\mathbf{k},\tau} + M_{-\mathbf{k},\tau}\sigma_{\mathbf{k},\tau} - i\pi^*_{\mathbf{k},\tau}M_{\bar{\mathbf{k}},\tau} - iM_{-\bar{\mathbf{k}},\tau}\pi_{\mathbf{k},\tau}\right]\right\}$$

- **Mean field treatment of repulsive interaction**

$$
\begin{aligned}
e^{-\alpha A^2} &= \int d\phi \exp\left(-\alpha\left[\phi^2 - 2i\phi A\right]\right) \\
&= \int d\phi \exp\left(-\alpha\left[(\phi + i\omega)^2 - 2i(\phi + i\omega) A\right]\right) \\
&= \int d\phi \exp\left(-\alpha\left[\phi^2 + 2i\phi(\omega - A) - \omega^2 + 2\omega A\right]\right) \\
&\simeq \exp\left(\alpha\left[\omega^2 - 2\omega A\right]\right) \quad (\phi = i\omega,\quad \omega = \langle A \rangle)
\end{aligned}
$$



Auxiliary Field Effective Action

- **Fermion det. + U0 integral can be done analytically.**
  → Auxiliary field effective action

$$S_{\rm eff}^{\rm AF} = \sum_{\boldsymbol{k},\tau,f(\boldsymbol{k})>0} \frac{L^3 f(\boldsymbol{k})}{4N_c}\left[|\sigma_{\boldsymbol{k},\tau}|^2 + |\pi_{\boldsymbol{k},\tau}|^2\right] - \sum_{\boldsymbol{x}} \log\left[X_N(\boldsymbol{x})^3 - 2X_N(\boldsymbol{x}) + 2\cosh(3\mu/T)\right]$$

$$X_N(\boldsymbol{x}) = X_N[m(\boldsymbol{x},\tau)] \qquad (\text{known func.})$$

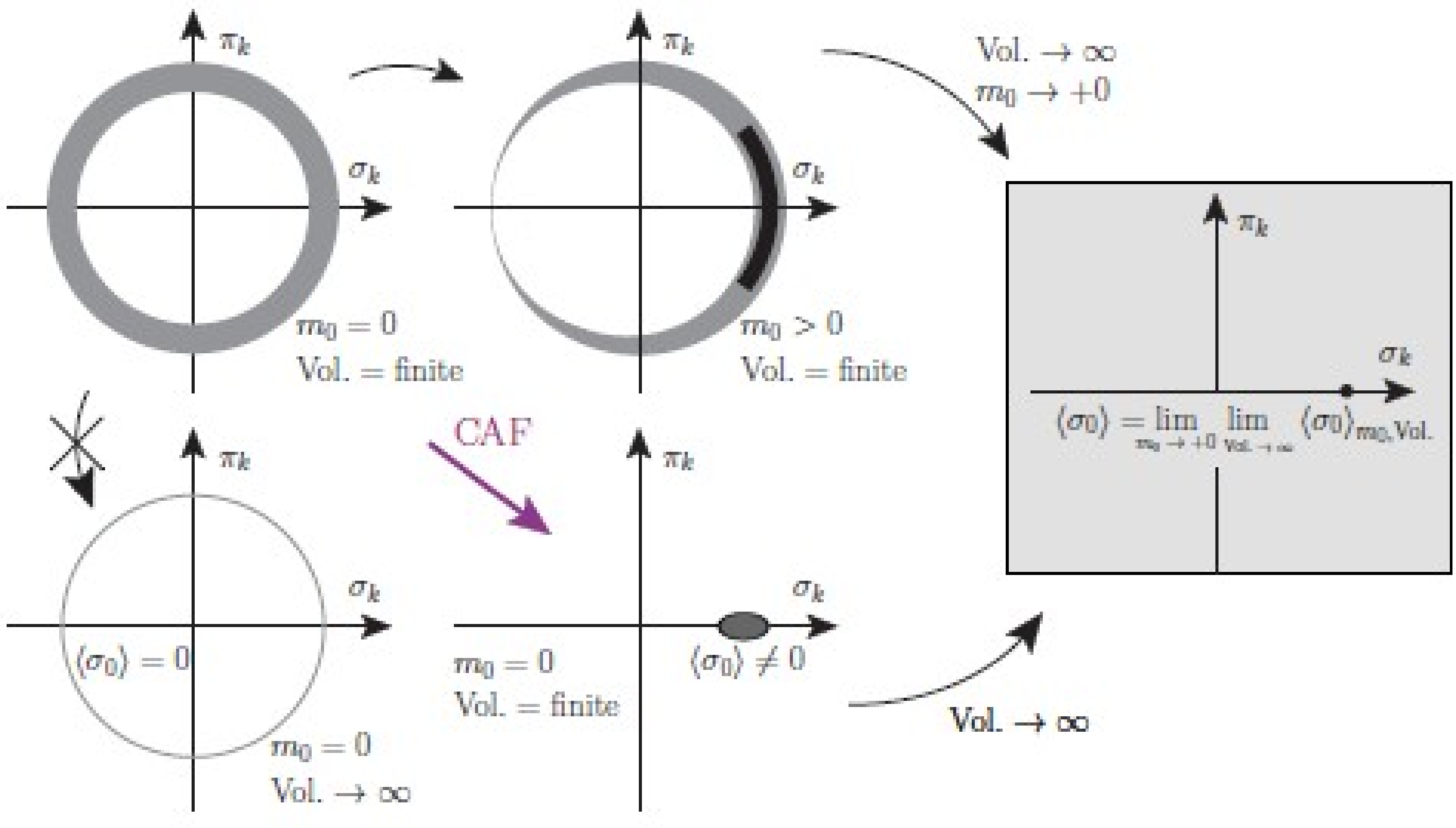
$$m_x = m_0 + \frac{1}{4N_c}\sum_j \left((\sigma + i\varepsilon\pi)_{x+\hat{j}} + (\sigma + i\varepsilon\pi)_{x-\hat{j}}\right)$$

- **X_N = Known function of m(x, τ)** *Faldt, Petersson ('86)*

  For constant m, $X_N = 2\cosh(N_\tau \,\mathrm{arcsinh}\,(m/\gamma))$
- **Imag. part from X_N → Relatively smaller at large μ/T**
- **Imag. part from low momentum AF cancels due to iε factor.**

Chiral Angle Fixing

How can we simulate correct thermodynamic chiral limit using finite volume simulations ?

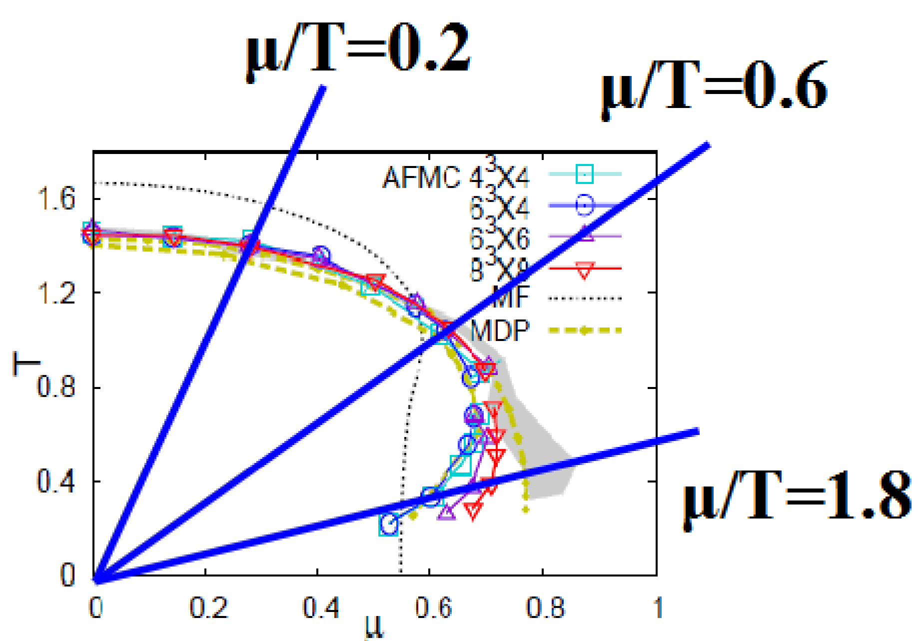


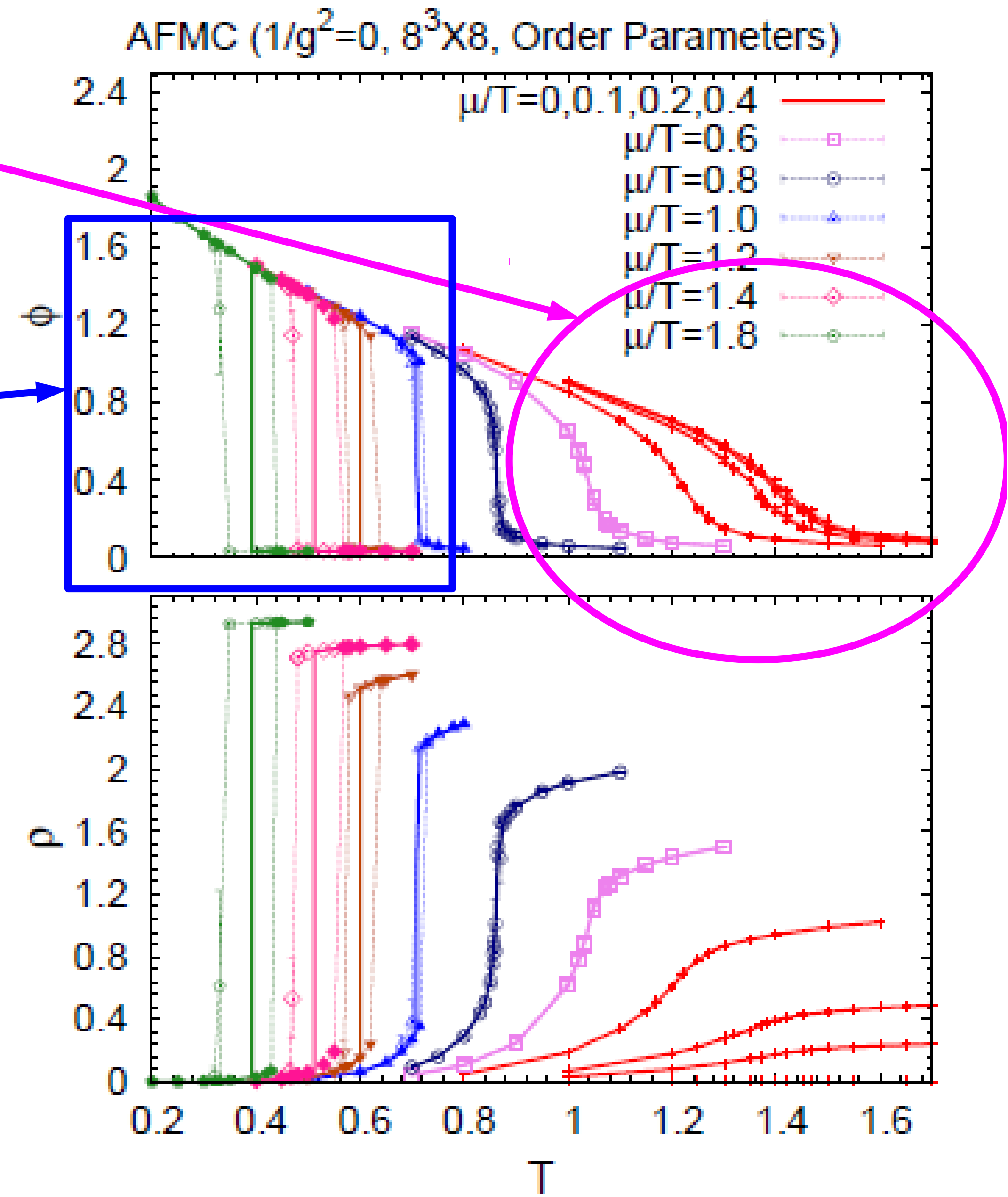
Ichihara, AO, Nakano ('14)

c.f. rms spin is adopted in spin systems *Kurt, Dieter ('10)*

Order Parameters

- **Low μ/T region**
  → 2nd order or crossover
  (would-be second)
- **High μ/T region**
  → sudden change
  & hysteresis
  (would-be first)





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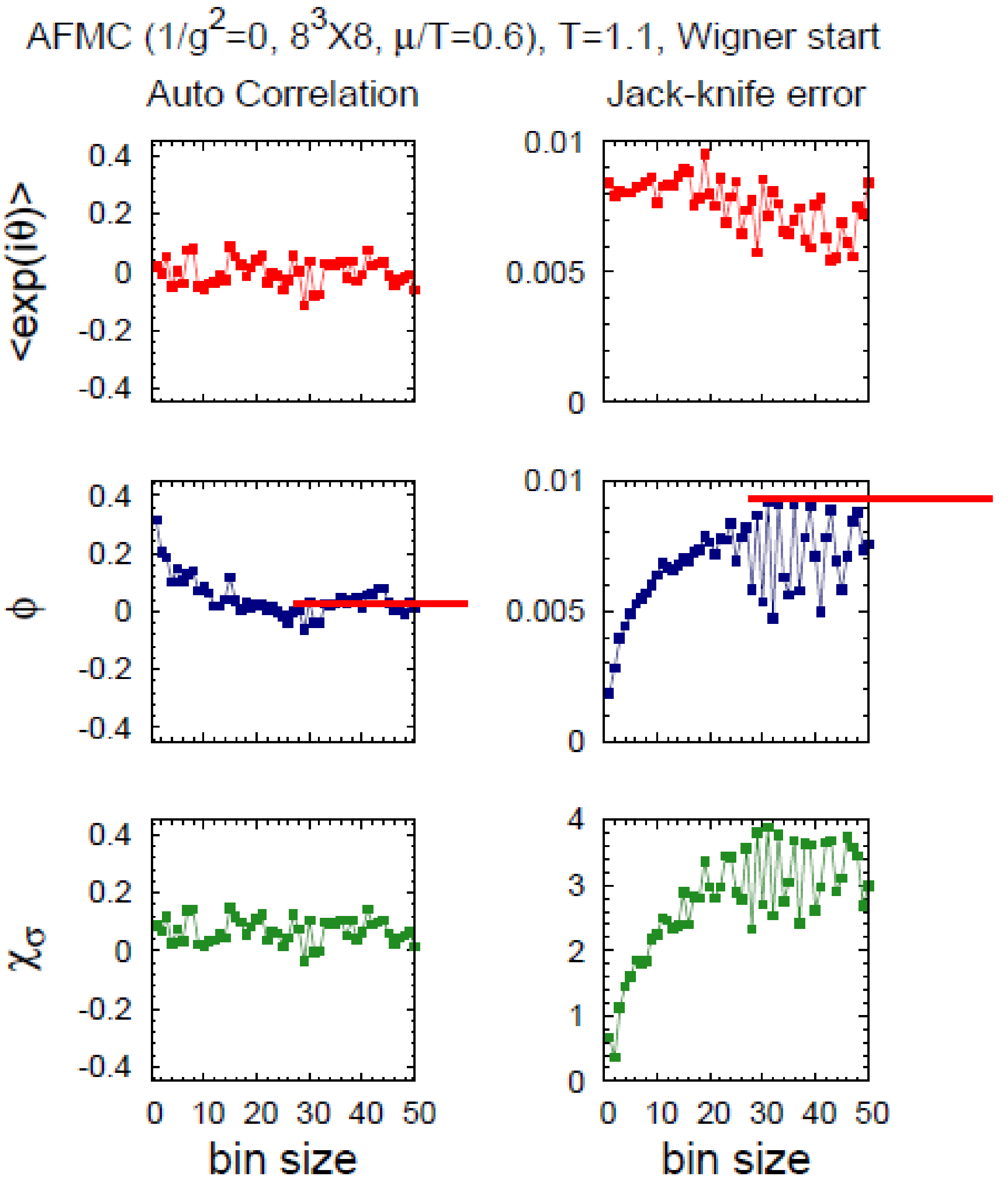
Error estimate by Jack-knife method



Error
= Jack-knife error after autocorrelation disappears

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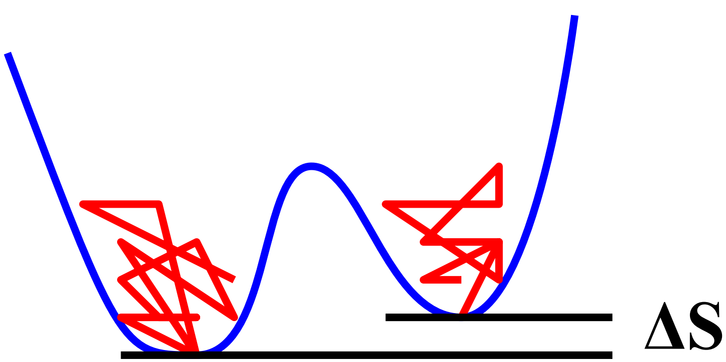
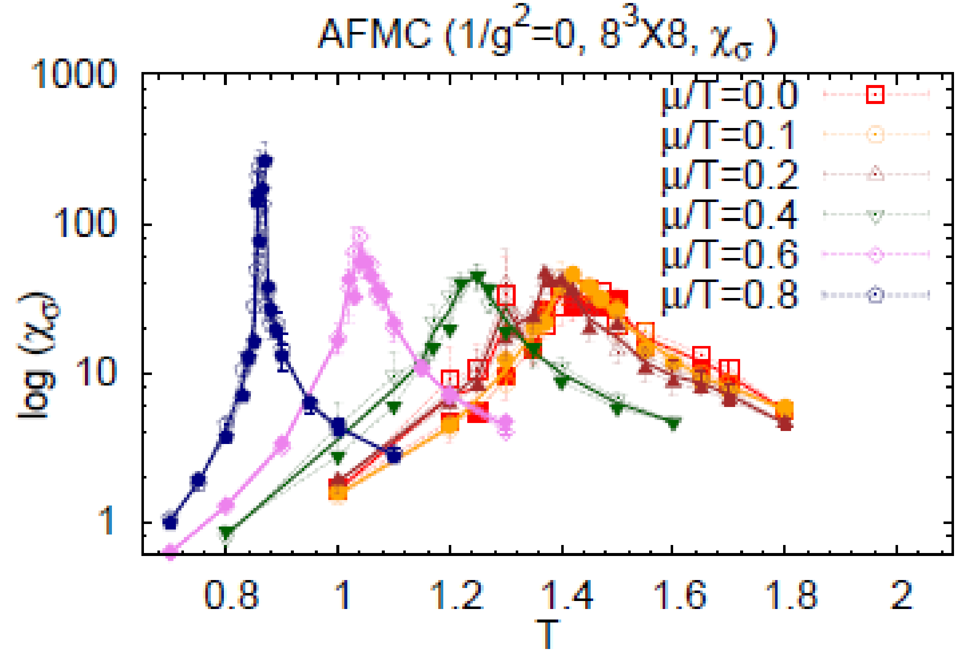
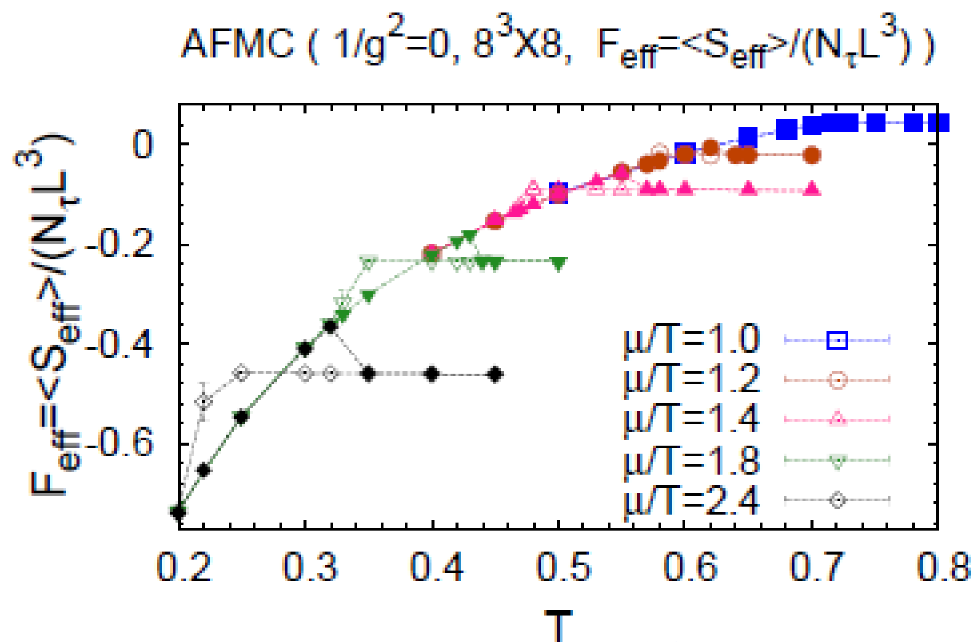
Phase boundary

- **Low μ/T region (would-be second) → Chiral susc. peak**
- **High μ/T region (would-be first) → Average eff. action from Wigner/NG init. cond.**

  c.f. Exchange MC (Hukuyama)







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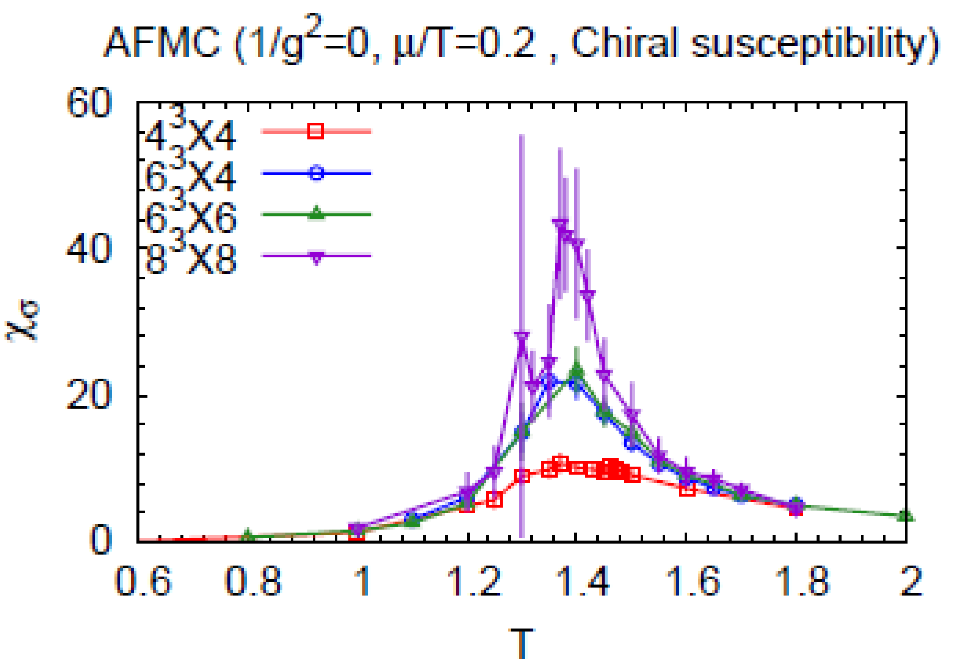
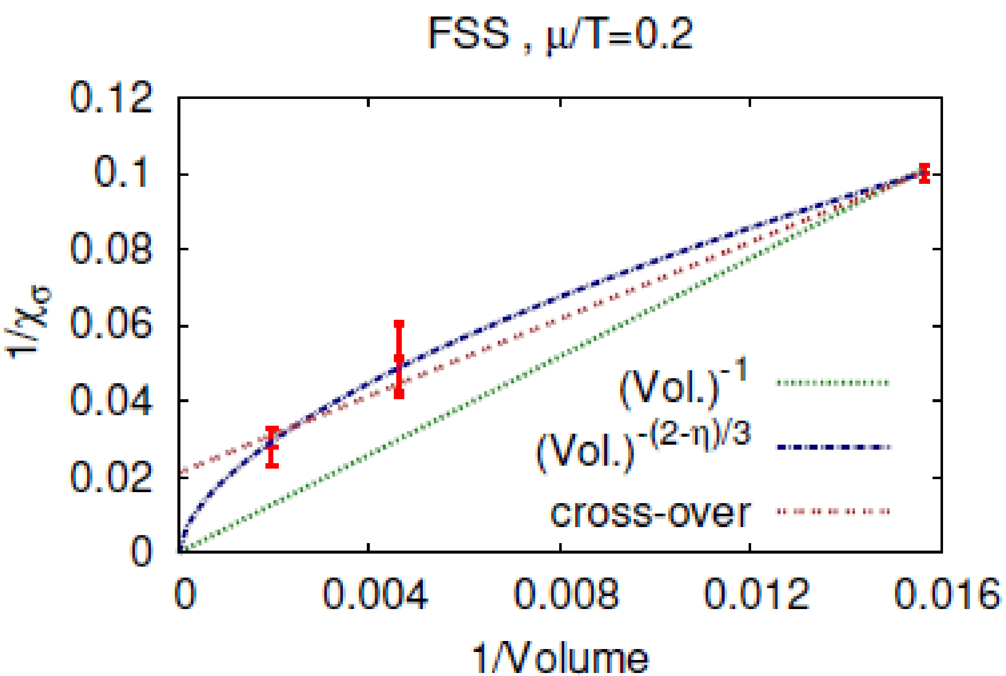
Finite Size Scaling of Chiral Susceptibility

- **Finite size scaling of χ_σ in the V (spatial vol.) → ∞ limit**
  - **Crossover: Finite**
  - **Second order: $\chi_\sigma \propto V^{(2-\eta)/3}$, η=0.0380(4) in 3d O(2) spin**
    Campostrini et al. ('01)
  - **First order: $\chi_\sigma \propto V$**
- **AFMC results : Not First order at low μ/T.**





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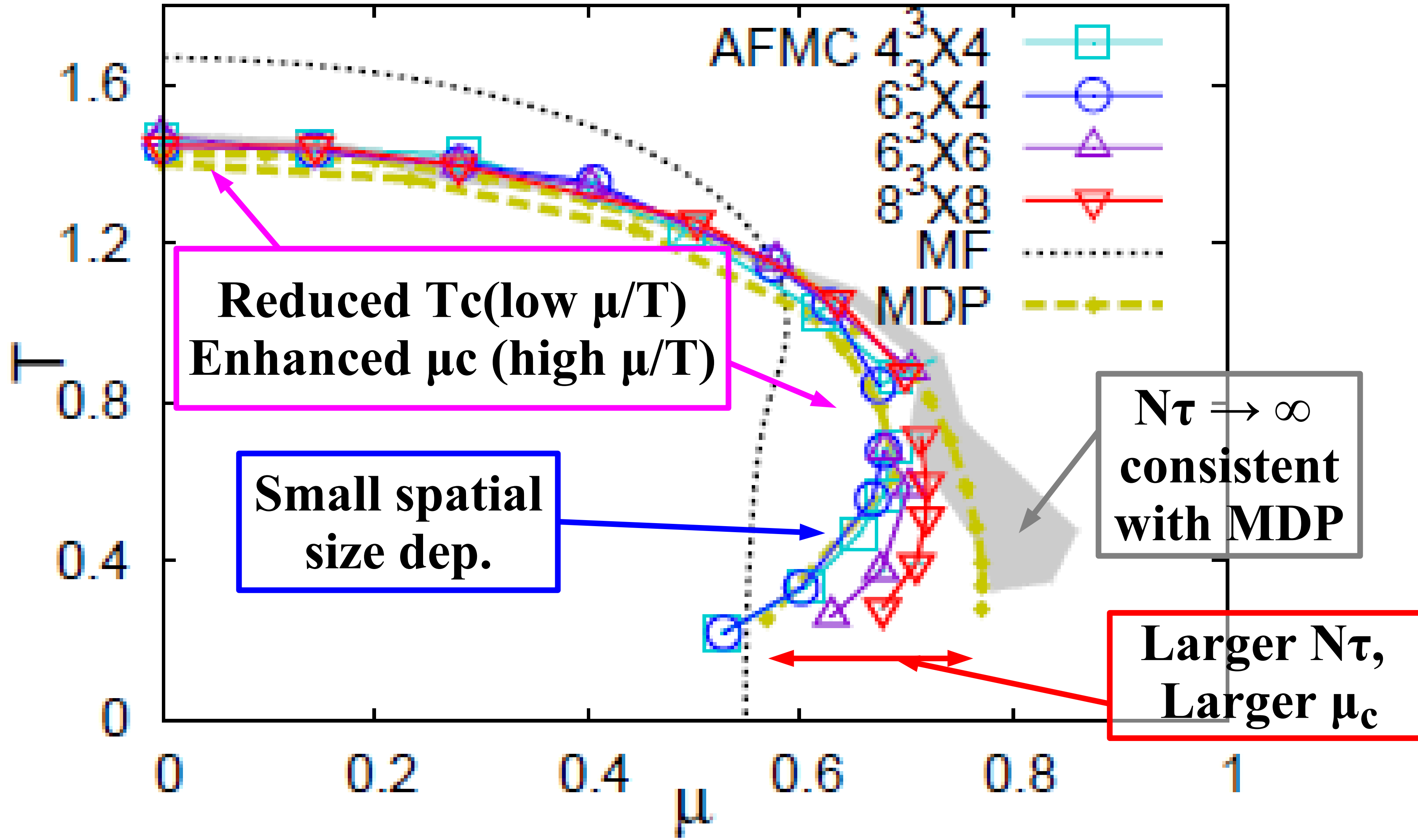
Phase diagram



Ichihara, AO, Nakano ('14)

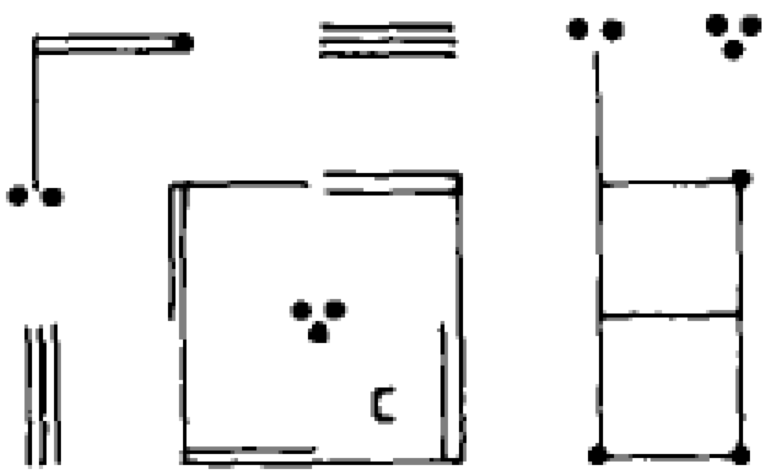
Monomer-Dimer-Polymer simulation

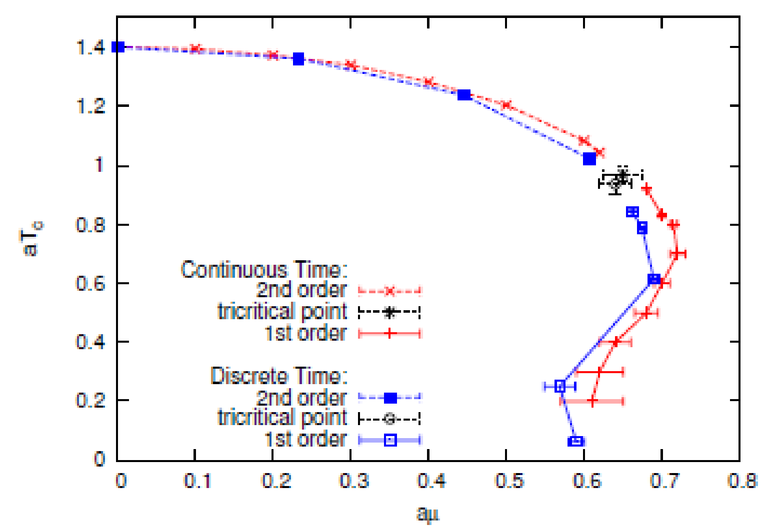
- **The partition function of LQCD can be given as the sum of monomer-dimer-polymer (MDP) configuration weight. The sign problem is mild.** *Karsch, Mutter ('89)*

$$Z(2ma, \mu, r) = \sum_K w_K$$

$$w_K = (2ma)^{N_M} r^{2N_t} (1/3)^{N_1 N_2} \prod_x w(x) \prod_C w(C)$$

- **MDP with worm algorithm is applied to study the phase diagram**
  de Forcrand, Fromm ('10), de Forcrand, Unger ('11)





Karsch, Mutter ('89) *de Forcrand, Unger ('11)*

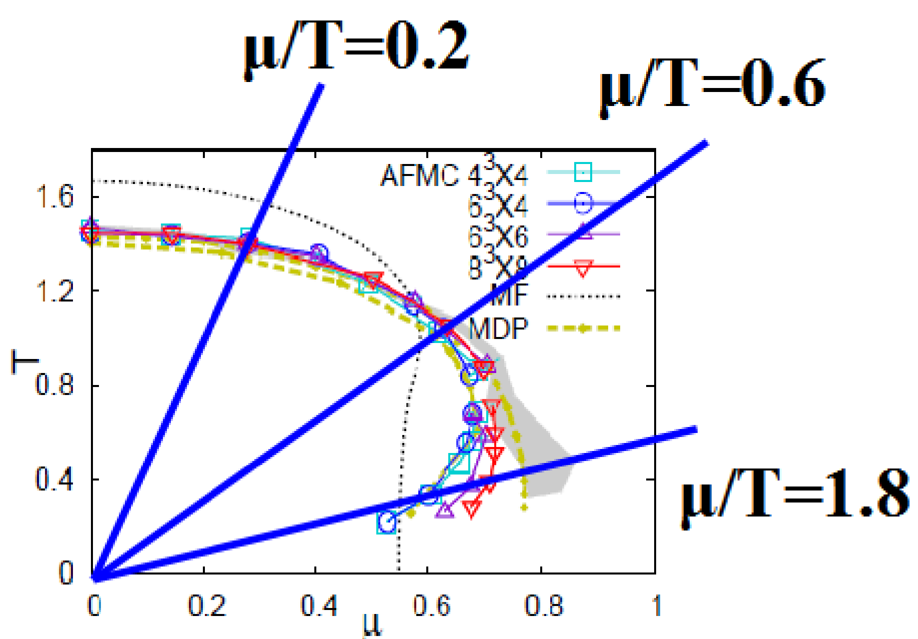
Sign problem in SC-LQCD

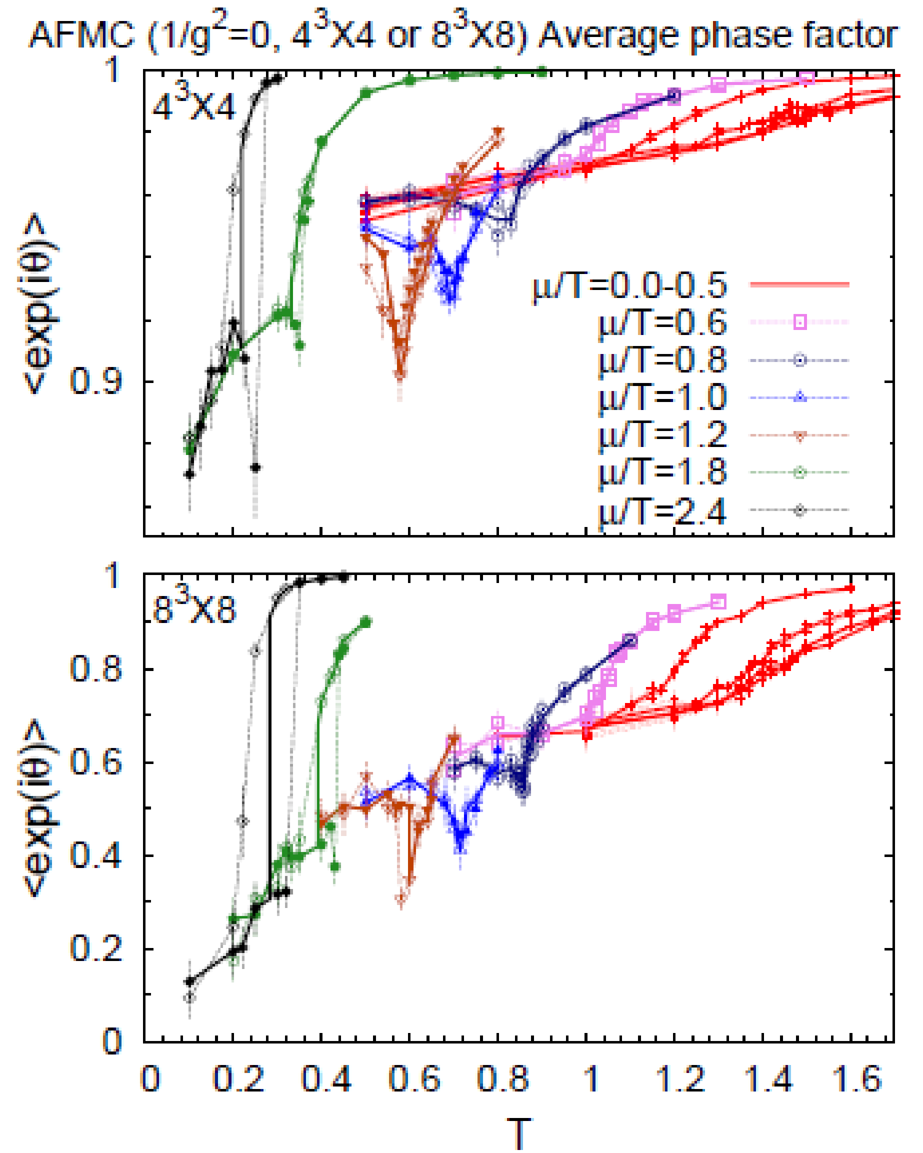
Average Phase Factor

- **Average phase factor = Weight cancellation**

$$\langle e^{i\theta}\rangle = Z_{\text{phase quenched}} \;/\; Z_{\text{full}}$$

- **AFMC results**
  - $\langle e^{i\theta}\rangle > 0.9$ on 4^4 lattice
  - $\langle e^{i\theta}\rangle > 0.1$ on 8^4 lattice

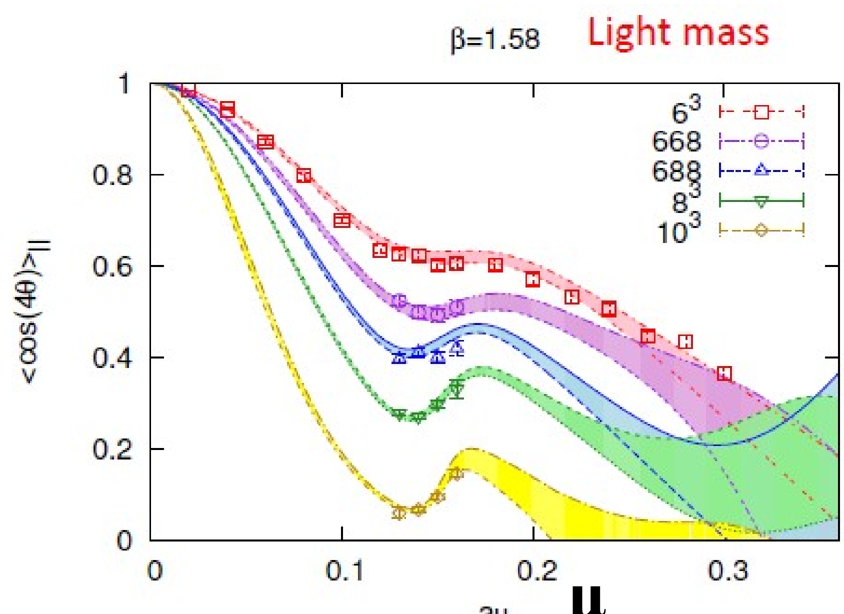


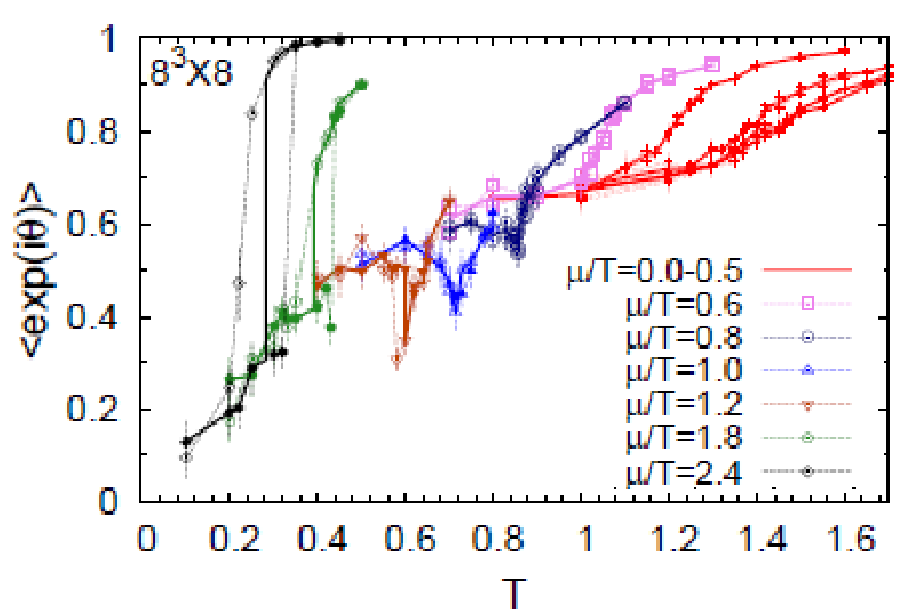


Ichihara, AO, Nakano ('14)

Comparison with Direct Simulation at finite coupling

- **Lattice MC simulation at finite μ and finite β with Nf=4**
  Takeda et al. ('13)
  - **Ave. Phase Factor ~ 0.3 at aμ ~ 0.15 (8^3 x 4, $a\mu_c = am_\pi/2 \sim 0.7$)**
- **AFMC**
  - **Ave. Phase Factor ~ 0.6 around the transition (8^4, SCL)**





Takeda, Jin, Kuramashi, Y.Nakamura, Ukawa, Lattice 2013

$a\mu_c = am_\pi/2 \sim 0.7$

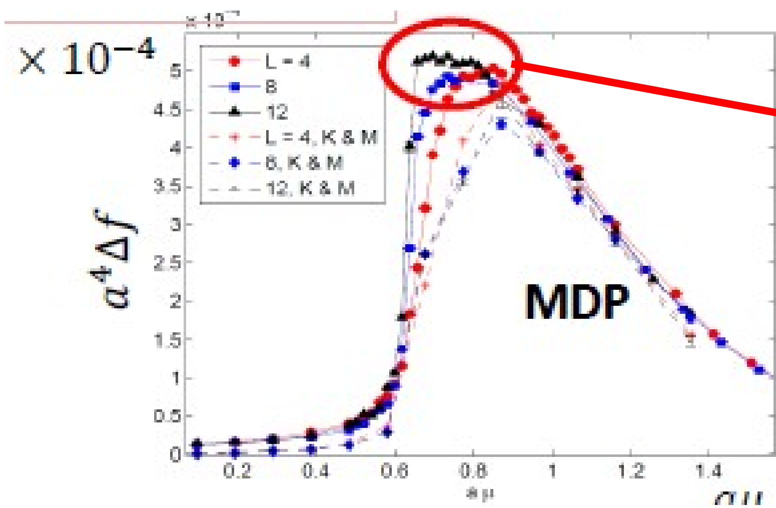
Ichihara, AO, Nakano ('14)

Discussion: Comparison with MDP

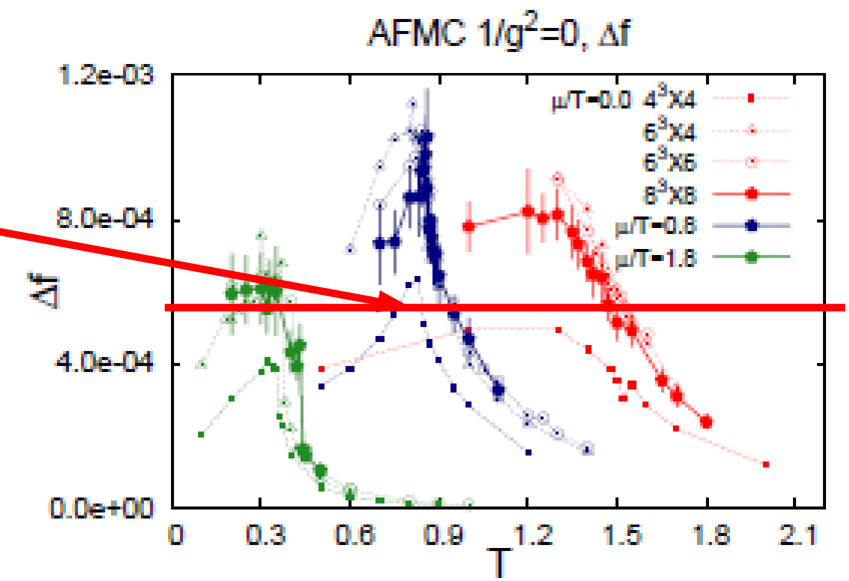
- **Free energy difference**

$$\langle \exp(i\theta) \rangle \equiv \exp(-\Omega \Delta f) \quad , \quad \Omega = \text{space-time volume}$$

- **MDP simulation on anisotropic lattice at finite T and μ**
  de Forcrand, Fromm ('10), de Forcrand, Unger ('11)
  - **Strong coupling limit.**
  - **Higher-order terms in 1/d expansion**
  - **No sign problem in the continuous time limit (Nτ → ∞).**



de Forcrand, Unger ('11)



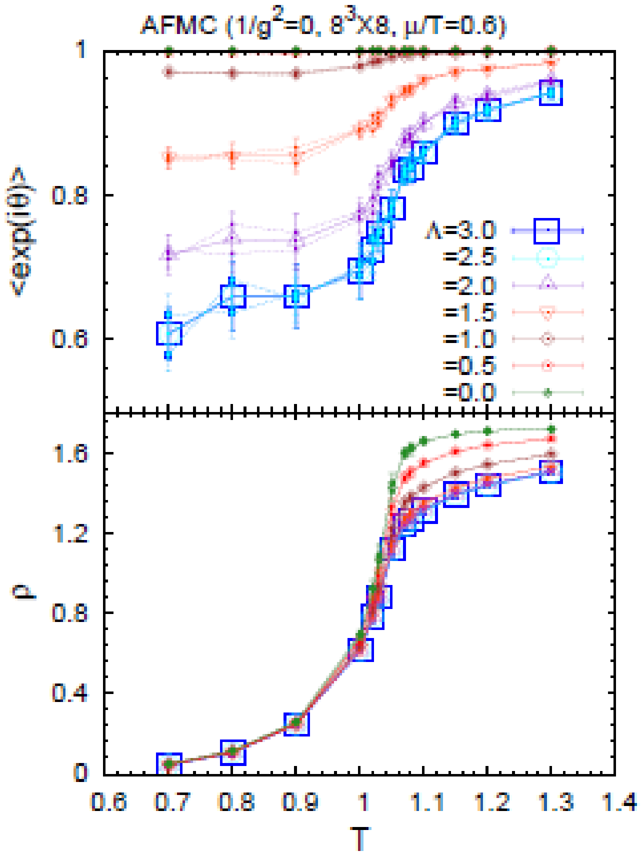
Ichihara, AO, Nakano ('14)

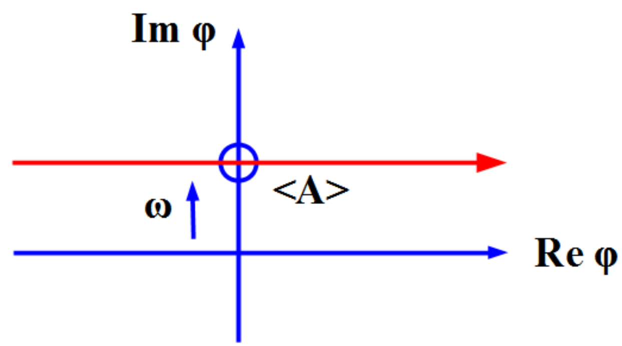
Summary

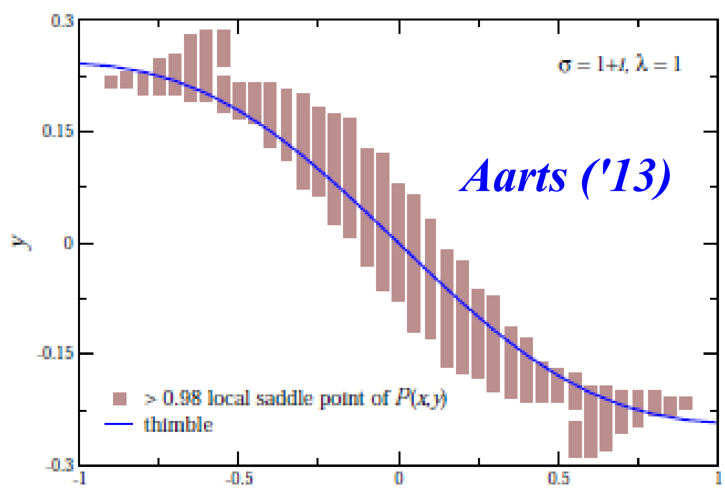
- **Strong coupling lattice QCD is a promising tool in finite density lattice QCD.**
  - **Strong coupling limit + finite coupling correction + Polyakov loop → MC results of Tc is roughly reproduced.**
  - **Fluctuation effects can be included in auxiliary field Monte-Carlo**
  - **Sign problem could be partially solved in the strong coupling limit. Two independent methods show the same phase boundary, and the spatial size dependence is small. (Monomer-dimer-polymer simulation, AFMC)**
- **Challenge**
  - **Finite coupling + Fluctuations** *Unger et al. ('13)*
    Different type of Fermion
    Minimally doubled fermion, Misumi, Kimura, AO ('12)
    Higher order terms in 1/d expansion,
    ….

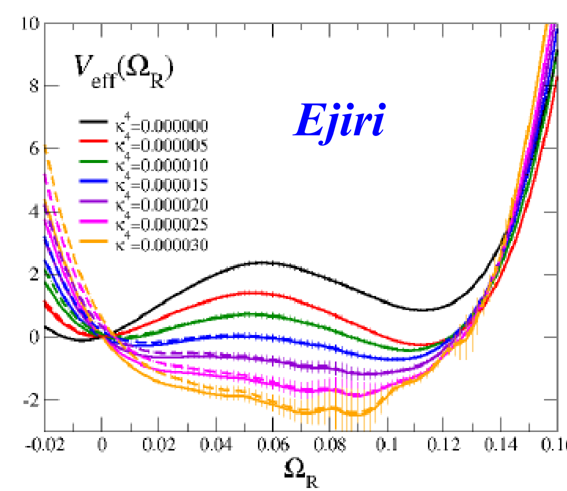
Real Challenge: How to live with the sign problem

- **Idea 1: Cutoff or Gauss integral of high momentum modes**
- **Idea 2: Change the integral path**
- **Idea 3: Combination of Fugacity exp. or Histogram method**









Thank you