QCD・カオス・エントロピー生成

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in collaboration with
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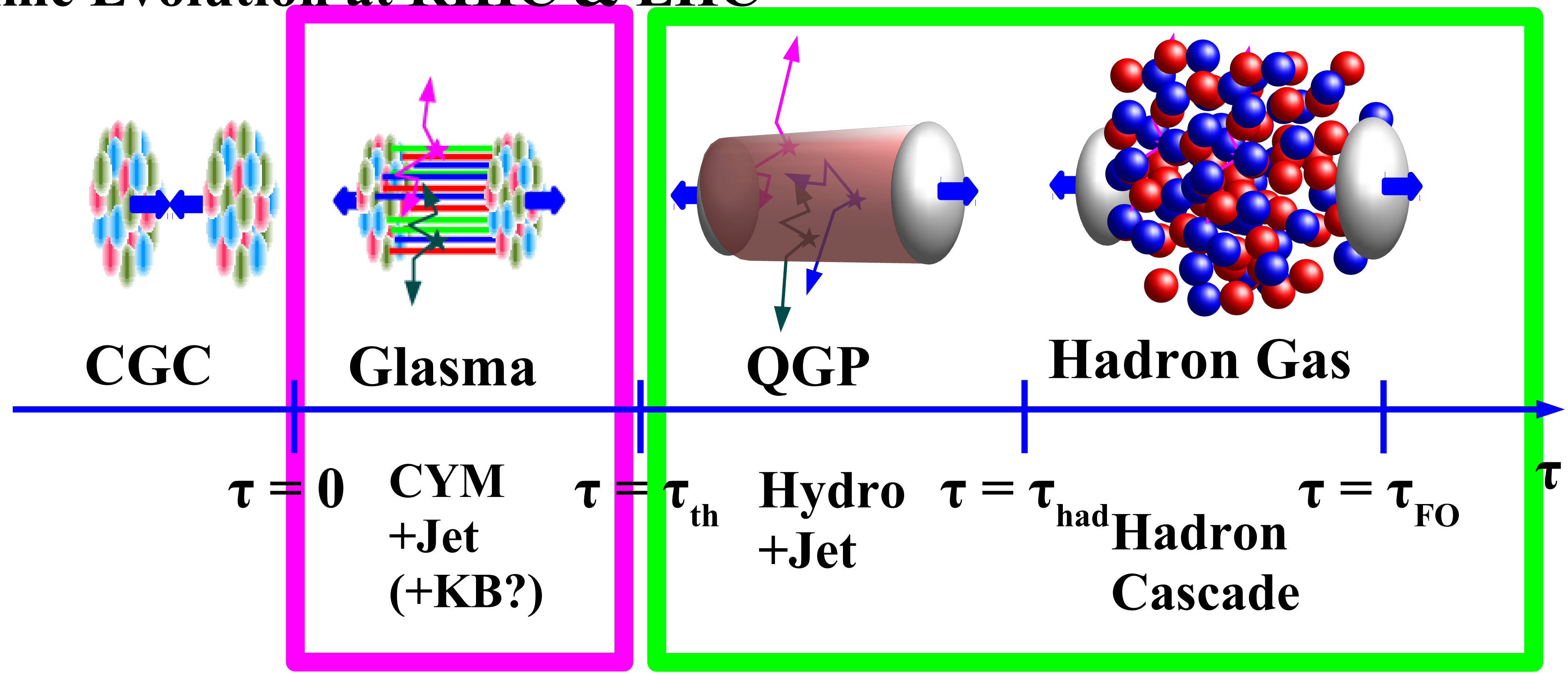
研究会「乱流 と QCD・重力」, Jan. 7, 2015, Osaka U.

- Introduction: Entropy production before QGP formation
- Entropy production in isolated quantum systems
- Entropy production in classical Yang-Mills fields
- Summary

Thermalization in High-Energy Heavy-Ion Collisions

- **Time Evolution at RHIC & LHC**



Theor. Challenges

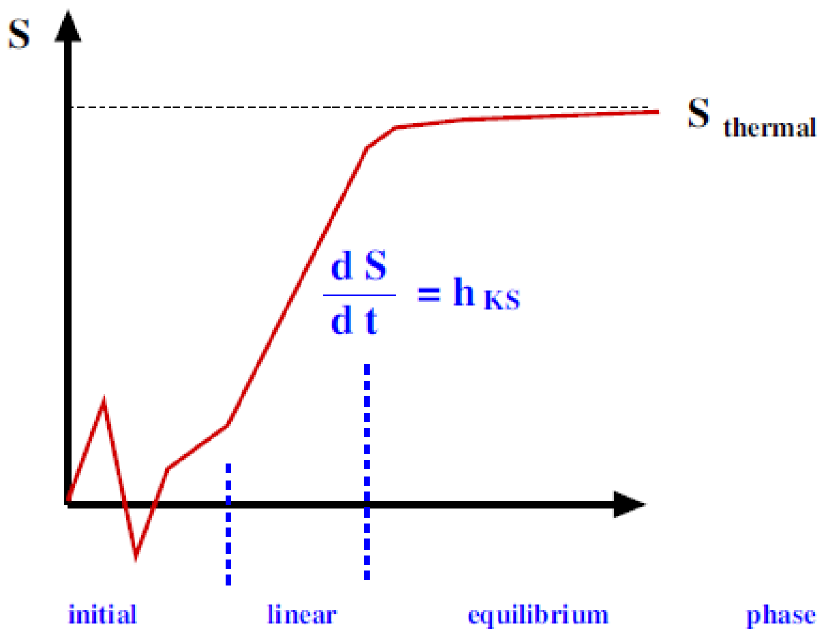
- **Thermalization under dynamical classical field**
- **Theoretically interesting and Phenomenologically important.**
  dN/ dη, init. cond. of hydro.

Phen. Challenges

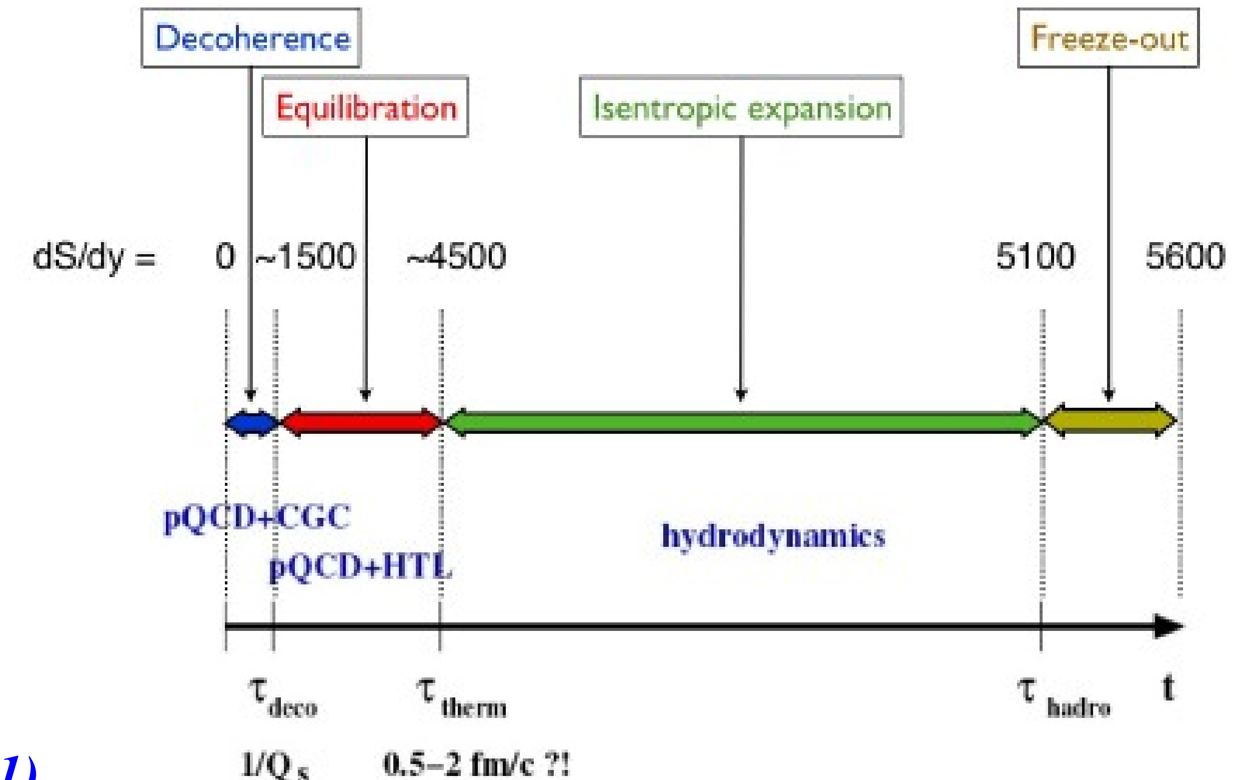
flow, jet, hard probes
→ hydro., transport coef., E-loss, hadron prop., phase diagram, ...

Entropy Production in Glasma

- **Huge entropy must be produced before QGP formation !**
  - **Thermalization time ~ (0.5-2.0) fm/c**
  - **Instability ? Rapid glasma decay ? Entropy of classical field ?**



B.Muller and A. Schaefer,
Int. J. Mod. Phys. E20, 2235 (2011)



R.J. Fries et al, arXiv 0906.5293

We discuss the CYM entropy and its production rate with emphasis on the chaoticity

Contents

- Introduction
- Entropy production in quantum mechanics
  - Chaoticity, Lyapunov exponent, and Kolmogorov-Sinaii entropy
  - Coarse graining and Husimi-Wehrl entropy
  - KS and HW entropy in quantum mechanics
    T. Kunihiro, B. Müller, A. Ohnishi, A. Schäfer, PTP 121 ('09), 555.
    H. Tsukiji, H. Iida, T. Kunihiro, A. Ohnishi, T. T. Takahashi, in prep.
- Entropy production in QCD (classical Yang-Mills field)
  - Wigner and Husimi functionals
  - KS, HW, and decoherence entropy of CYM
    T. Kunihiro, B. Müller, AO, A. Schäfer, T.T. Takahashi, A. Yamamoto, PRD82('10),114015.
    H.Iida, T.Kunihiro, B.Müller, AO, A.Schäfer, T.T.Takahashi, PRD88('13),094006.
    H.Iida, T.Kunihiro, AO, T. T. Takahashi, arXiv:1410.7309 [hep-ph].
    S. Tsutsui, H. Iida, T. Kunihiro, AO, arXiv:1411.3809.
    H. Tsukiji, H. Iida, T. Kunihiro, A. Ohnishi, T. T. Takahashi, in progress.
- Summary

QGP での本当の turbulence については
浅川さん、福嶋さんへ

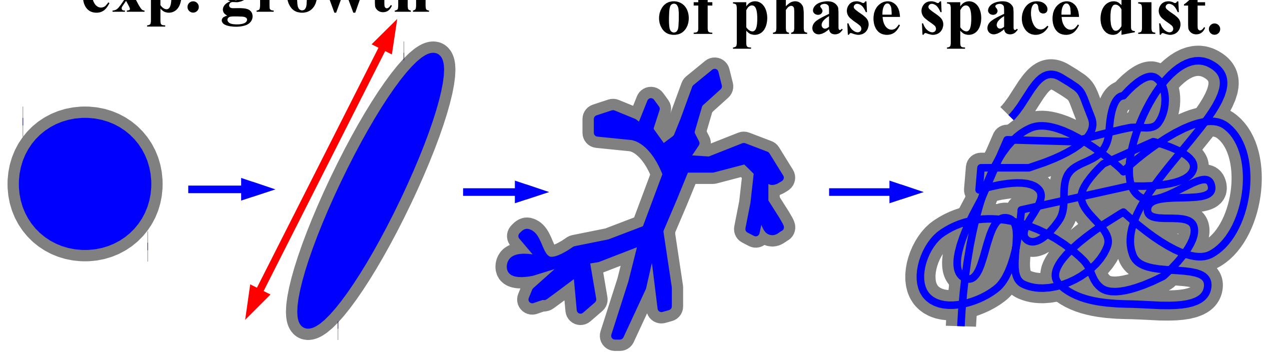
Chaoticity and Entropy

- **Kolmogorov-Sinai entropy rate h_{KS} = Entropy production rate**
  V. Latora and M.Baranger('99)

$$\frac{dS}{dt} = h_{\rm KS}, \quad h_{\rm KS} = \sum_{\lambda_i > 0} \lambda_i, \quad |\delta X_i(t)| = e^{\lambda_i t} |\delta X_i(0)|$$

λ_i : Lyapunov exponent

  - **Chaos = Initial state sensitivity & Complexity of phase space dist.**
    「引き延ばし」と「折りたたみ」(Sugita)
    → Exponential growth of "visited" phase space cell
    → Entropy prod.



Entropy production in quantum systems

- **Entropy in quantum mech.**
  - **Time evolution is unitary, then the von Neumann entropy is const.**

$$|\psi(t)\rangle = \exp(-iHt/\hbar)\,|\psi(0)\rangle$$

$$\rho = |\psi\rangle\langle\psi| \to |\psi(t)\rangle\langle\psi(t)|$$

$$S_{\rm vN} = -\mathrm{Tr}\left[\rho\log\rho\right] \to \mathrm{const.}$$

- **Two ways of entropy production at the quantum level**
  - **Entanglement entropy**

$$\rho_{\rm S} = \mathrm{Tr}_{\rm E}\left(\rho\right) \to S_{\rm S} = -\mathrm{Tr}\left(\rho_{\rm S}\log\rho_{\rm S}\right) > 0$$

Partial trace over environment → Loss of info. → entropy production

  - **Coarse grained entropy**

$$\rho \to \rho_z(\text{coarse grained}) \to S = -\int dz\ \rho_z \log\rho_z > 0$$

Coarse graining (粗視化) → entropy production
Yes, we can define it even in isolated systems such as HIC and early univ.!

Coarse graining in quantum mechanics

- **Wehrl entropy (Wehrl, 1978)** $S_{\rm W} = -\int \frac{dqdp}{2\pi\hbar} f(q,p) \log f(q,p)$
- **Wigner function (Wigner, 1932)**

$$f_{\rm W}(r,p) = \int ds\; e^{ips/\hbar}\, \langle r - s/2 \mid \rho \mid r + s/2 \rangle$$

  - **Quasi phase space dist. fn., but it can be negative.**
  - **Constant along the classical trajectory in semi-classical approx.**
    → No entropy prod.

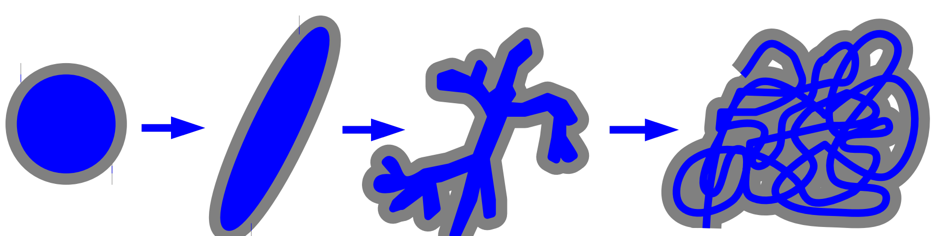
$$\partial f_{\rm W}/\partial t + v \cdot \nabla f_{\rm W} - \nabla U \cdot \nabla_p f_{\rm W} = 0$$

- **Husimi function (Husimi, 1940)**

$$f_{\rm H}(q,p) = \int \frac{dq'dp'}{\pi\hbar} e^{-\Delta(q-q')^2/\hbar - (p-p')^2/\Delta\hbar} f_{\rm W}(q',p')$$

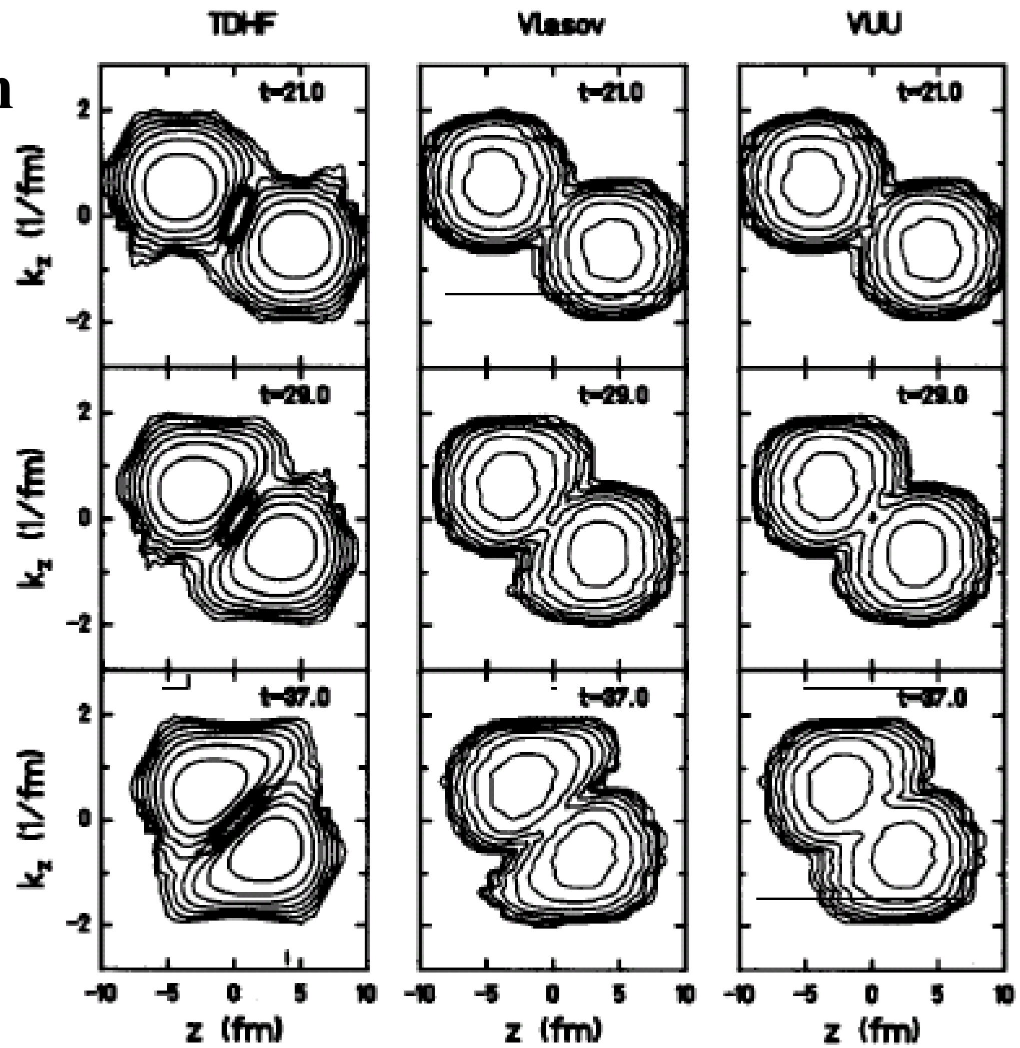
  - **Smeared with min. wave packet.**
  - **Exp. value under a coherent state.**

$$f_{\rm H} = \langle z \mid \rho \mid z \rangle\ , \quad z = (\Delta q + ip)/\sqrt{2\hbar\Delta}$$



Comarison of Semi-classical & Quantum Evolution

- **Comparison of the evolution in**
  Time-Dependent Hartree-Fock (TDHF, ~ TDDFT)
  and
  Vlasov Eq. (semi-classical)
  for a low energy heavy-ion collision
  Ca+Ca, 40 A MeV
  Cassing, Metag, Mosel, Niita,
  Phys. Rep. 188 (1990) 363.



Separation in phase space leads to acceleration of nuclei (deep inter-nuclear potential) e.g. AO, Horiuchi, Wada ('90).

Husimi Function

- **A simple example with instability**
Inverted Harmonic Oscillator

$$H = \frac{p^2}{2} - \frac{\lambda^2}{2} x^2$$

  - **exponential growth / shrink**

$$\dot{x} = p, \;\; \dot{p} = \lambda^2 x$$

$$\rightarrow p \pm \lambda x = \exp(\pm\lambda t)(p_0 \pm \lambda x_0)$$

  - **Wigner function**

$$f_W(x,p,t) = 2\exp[-K(x,p,t)/\hbar]$$

$$K = \omega x_0^2 + p_0^2/\omega$$

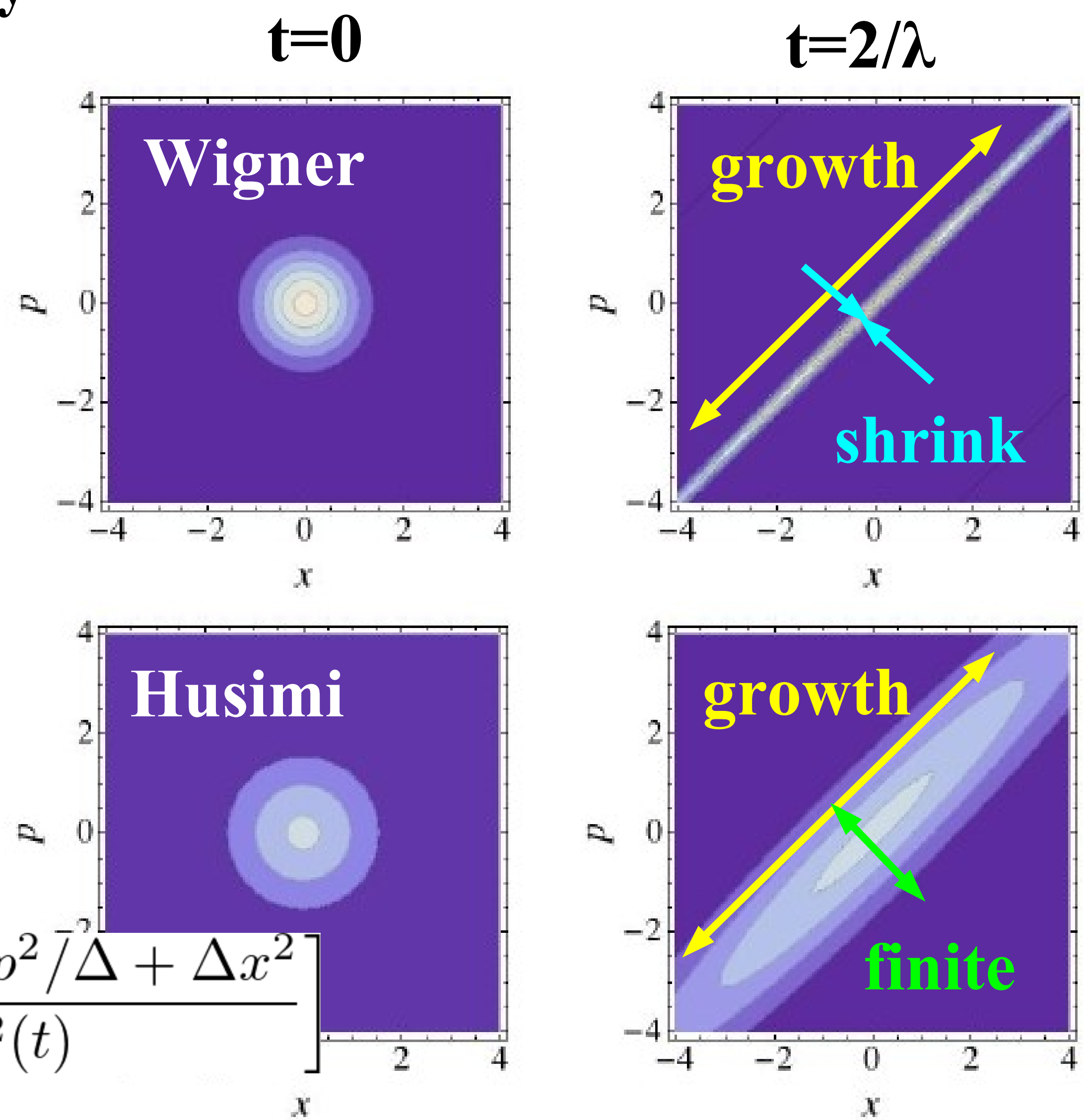
  - **Husimi function**

$$f_H(x,p,t) = \frac{2}{A(t)} \exp\left[-\frac{K(x,p,t) + p^2/\Delta + \Delta x^2}{\hbar A^2(t)}\right]$$

$$A(t) = \sqrt{2(\sigma\rho\cosh 2\lambda t + 1 + \delta\delta')}$$ **~ exp(λt)**

$$\sigma = (\lambda^2+\omega^2)/2\lambda\omega > 1, \delta = (\lambda^2-\omega^2)/2\lambda\omega, \rho = (\Delta^2+\lambda^2)/2\Delta\lambda > 1, \delta' = (\Delta^2-\lambda^2)/2\Delta\lambda$$

Kunihiro, Muller, Schafer, AO ('09)



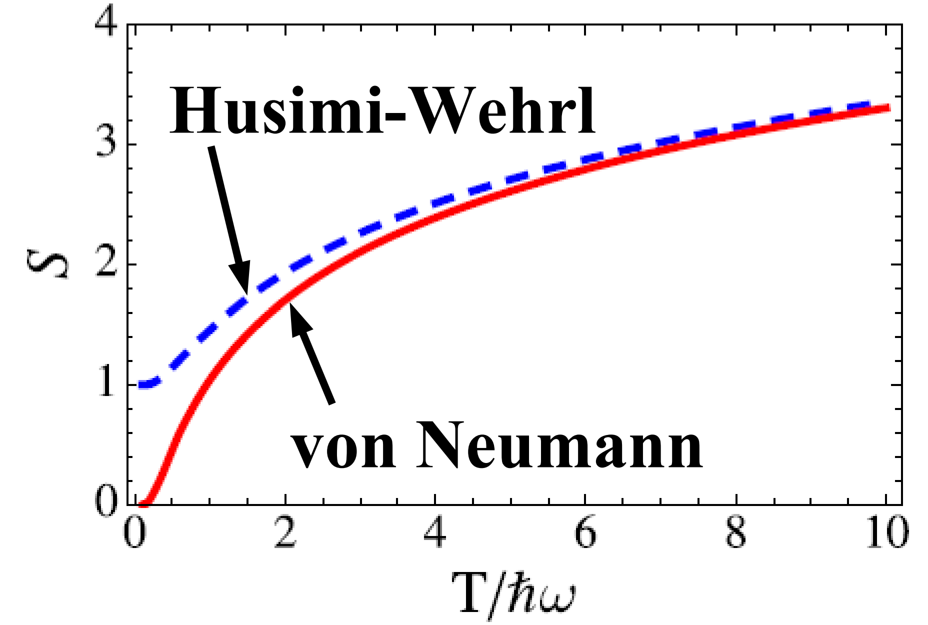
Husimi-Wehrl Entropy (1)

- **Husimi-Wehrl entropy = Wehrl entropy using Husimi function**
  Wehrl ('78), Husimi ('40), Anderson, Halliwell ('93), Kunihiro, Muller, AO, Schafer ('09).

$$S_{\mathrm{HW}} = -\int \frac{dqdp}{2\pi\hbar} f_{\mathrm{H}}(q,p) \log f_{\mathrm{H}}(q,p)$$

  - **Coarse grained entropy by minimum wave packet**

- **Harmonic oscillator in equilibrium**
  - **Min. value S_{HW}=1 (1 dim.) from smearing**
    Lieb ('78), Wehrl ('79)
  - **Husimi-Wehrl = von Neumann at high T ($T/\hbar\omega \gg 1$)**
    Anderson, Halliwell ('93),
    Kunihiro, Muller, AO, Schafer ('09).



Husimi-Wehrl Entropy (2)

- **Inverted Harmonic Oscillator**

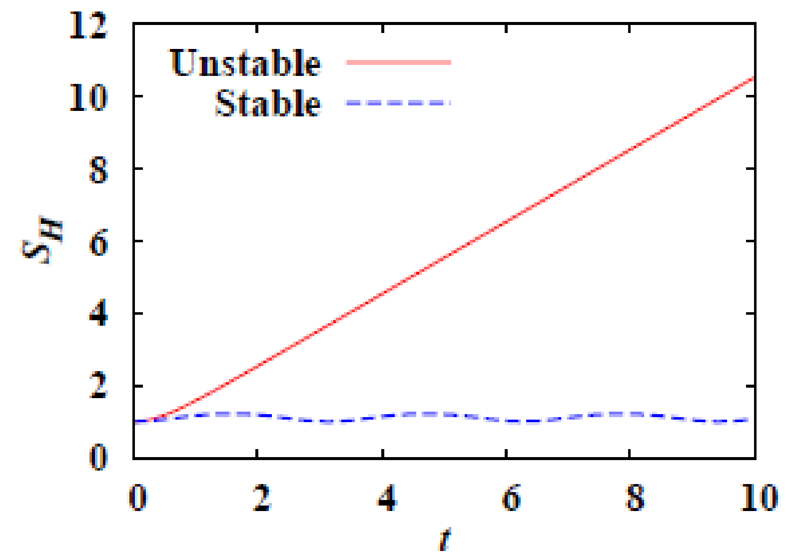
$$A(t) = \sqrt{2(\sigma\rho\cosh 2\lambda t + 1 + \delta\delta')}$$ **~ exp(λt), λ=Lyapunov exp.**

$$S_{\rm HW} = \log\frac{A(t)}{2} + 1\ , \quad \frac{dS_{\rm HW}}{dt} \to \lambda\ (t\to\infty)$$ **independent of Δ**

- **Many Harmonic & Inverted Harmonic Oscillators**

$$H = \sum_k \left(\frac{p_k^2}{2} - \frac{\lambda_k^2}{2}x_k^2\right) + \sum_i \left(\frac{p_i^2}{2} + \frac{\omega_i^2}{2}x_i^2\right)$$

$$\frac{dS_{\rm HW}}{dt} \to \sum_k \lambda_k\ (t\to\infty)$$



Classical unstable modes plays an essential role in entropy production at quantum level.

Husimi-Wehrl Entropy in Multi-Dimensions (1)

- **Challenge: Evolution of Husimi fn. & Multi-Dim. integral**

$$S_{\mathrm{HW}} = -\int \frac{d^D q d^D p}{(2\pi\hbar)^D} f_H(q,p) \log f_H(q,p)$$

$$f_H(q,p) = \int \frac{d^D q' d^D p'}{\pi\hbar} e^{-\Delta(q-q')^2/\hbar-(p-p')^2/\Delta\hbar} f_W(q,p)$$

- **Monte-Carlo + Semi-classical approx.**
  Tsukiji, Iida, Kunihiro, AO, Takahashi, in prep.
  - **Two-step Monte-Carlo method**
    Monte-Carlo integral + Liouville theorem [$f_W(q,p,t)=f_W(q_0,p_0,t=0)$]
  - **Test particle method: Test particle evol. + Monte-Carlo integral**

$$f_W(q,p,t) = \frac{2\pi\hbar}{N_{\mathrm{tp}}} \sum_{i=1}^{N_{\mathrm{tp}}} \delta(q-q_i(t))\,\delta(p-p_i(t)) \ ,$$

$$\frac{dq_i}{dt} = \frac{p_i}{m} \ , \qquad \frac{dp_i}{dt} = -\frac{\partial U}{\partial q_i} \ .$$

Husimi-Wehrl Entropy in Multi-Dimensions (2)

Tsukiji, Iida, Kunihiro, AO, Takahashi, in prep.

- **Two-step Monte-Carlo integral**

Liouville

$$S_{\rm HW}^{(\rm tsMC)} = -\int \frac{d^DQd^DP}{(\pi\hbar)^D}\, e^{-\Delta Q^2/\hbar - P^2/\Delta\hbar} \int \frac{d^Dqd^Dp}{(2\pi\hbar)^D}\, f_{\rm W}(q,p,t)$$

$$\times \log\left[\int \frac{d^DQ'd^DP'}{(\pi\hbar)^D}\, e^{-\Delta(Q')^2/\hbar-(P')^2/\Delta\hbar} f_{\rm W}(q+Q+Q', p+P+P', t)\right]$$

$$= -\frac{1}{N_{\rm out}}\sum_{k=1}^{N_{\rm out}} \log\left[\frac{1}{N_{\rm in}}\sum_{l=1}^{N_{\rm in}} f_{\rm W}(q_k+Q_k+Q'_l, p_k+P_k+P'_l, t)\right]$$

Outside MC → S **Inside MC → f_H**

- **Test particle method: test particle evolution + MC integral**

$$S_{\rm HW}^{(\rm tp)} = -\frac{1}{N_{\rm tp}}\sum_{i=1}^{N_{\rm tp}} \int \frac{d^Dqd^Dp}{(\pi\hbar)^D}\, e^{-\Delta(q-q_i(t))^2/\hbar-(p-p_i(t))^2/\Delta\hbar} \log f_H(q,p,t)$$

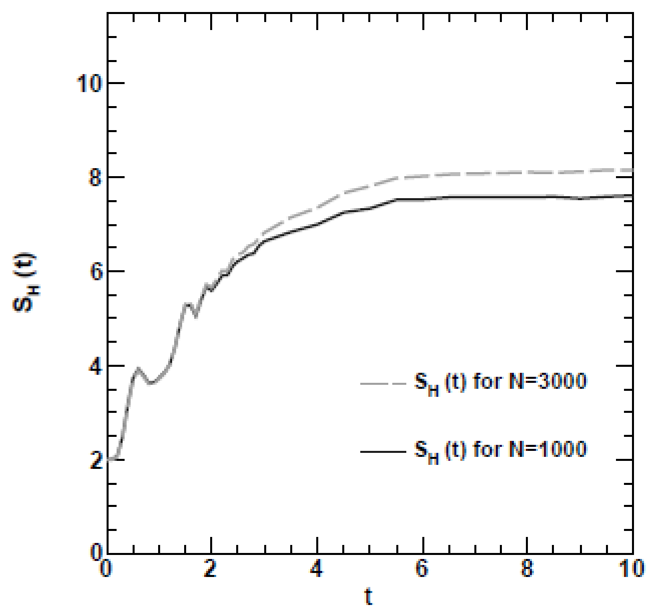
$$= -\frac{1}{MN_{\rm tp}}\sum_{k=1}^{M}\sum_{i=1}^{N_{\rm tp}} \log\left[\frac{2^D}{N_{\rm tp}}\sum_{j=1}^{N_{\rm tp}} e^{-\Delta(Q_k+q_i(t)-q_j(t))^2/\hbar-(P_k+p_i(t)-p_j(t))^2/\Delta\hbar}\right]$$

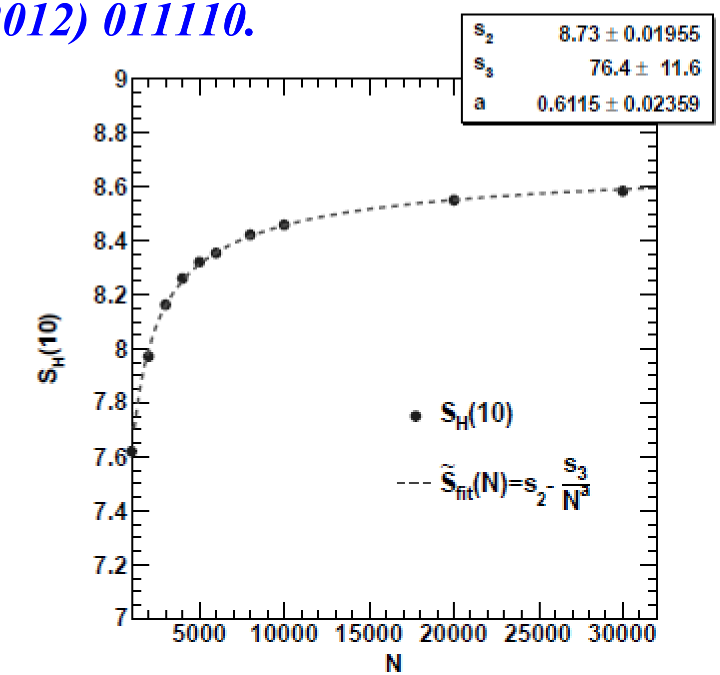
“Yang-Mills” Quantum Mechanics

- **Yang-Mills quantum mechanics**

$$H = \frac{p_1^2 + p_2^2}{2m} + \frac{1}{2} g^2 q_1^2 q_2^2$$

  - **Quartic interaction term → almost globally chaotic**
    S. G. Matinyan, G. K. Savvidy, N. G. Ter-Arutunian Savvidy, Sov. Phys. JETP 53, 421 (1981); A. Carnegie and I. C. Percival, J. Phys. A: Math. Gen. 17, 801 (1984); P. Dahlqvist and G. Russberg, Phys. Rev. Lett. 65, 2837 (1990).
  - **Husimi-Wehrl entropy in a test particle method for the Husimi fn. (w/ $\hbar^2$ corrections, EOM with a moment method)**
    H.-M. Tsai, B. Muller, Phys.Rev. E85 (2012) 011110.

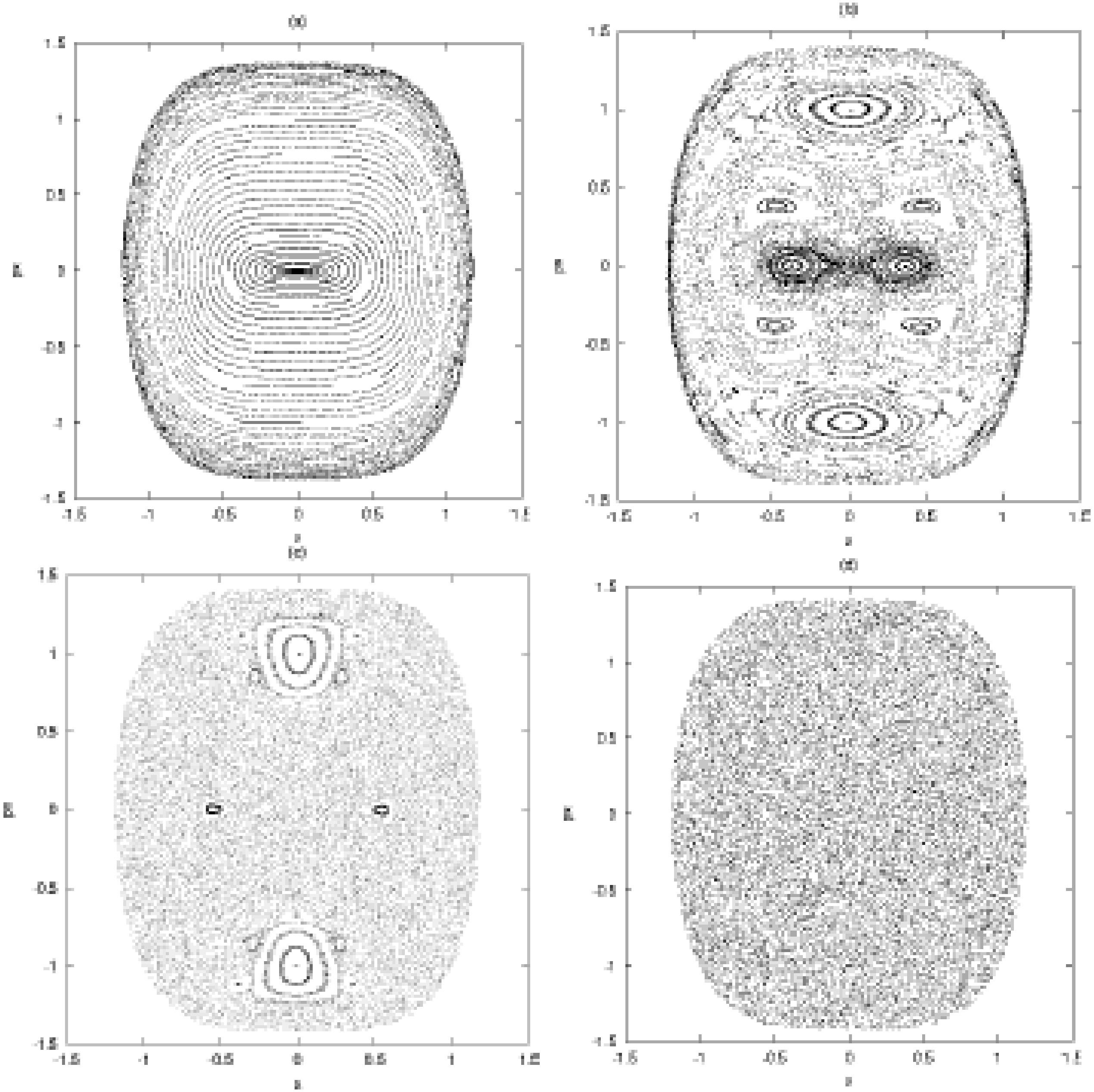




Poincare Map of 2D Quartic Oscillator

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^4 + y^4) - k^2 x^2 y^2$$

k=0, 0.2, 0.4, 0.6

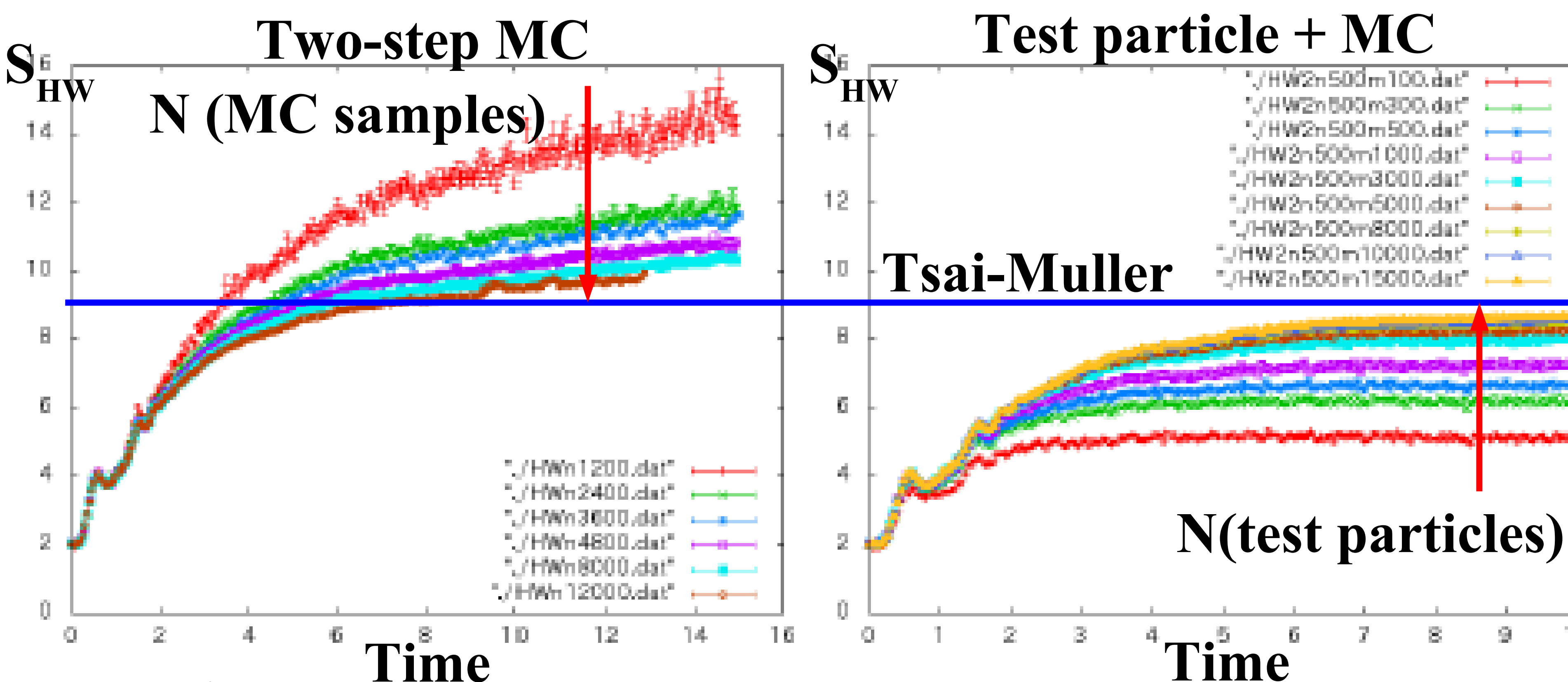


Sugita, Aiba,
Phys.Rev. A65 ('02) 036205.

Monte-Carlo + Semi-Classical Approx.

Tsukiji, Iida, Kunihiro, AO, Takahashi, in prep.

- **Semi-Classical + MC methods reproduce mesh integral values of S_{HW}.**
  - **Two-step MC results converge from above.**
  - **Test particle + MC results converge from below.**



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    T. Kunihiro, B. Müller, A. Ohnishi, A. Schäfer, PTP 121 ('09), 555.
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    T. Kunihiro, B. Müller, AO, A. Schäfer, T.T. Takahashi, A. Yamamoto, PRD82('10),114015.
    H.Iida, T.Kunihiro, B.Müller, AO, A.Schäfer, T.T.Takahashi, PRD88('13),094006.
    H.Iida, T.Kunihiro, AO, T. T. Takahashi, arXiv:1410.7309 [hep-ph].
    S. Tsutsui, H. Iida, T. Kunihiro, AO, arXiv:1411.3809.
    H. Tsukiji, H. Iida, T. Kunihiro, A. Ohnishi, T. T. Takahashi, in progress.
- **Summary**

Classical Yang-Mills Field

- **Yang-Mills action**

$$S_{\rm YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = \frac{1}{g^2} S_{\rm CYM}(A_{cl}) + \mathcal{O}(g^0) \quad (A_{cl} = \langle gA \rangle)$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu = \frac{1}{g}\left[\partial_\mu (gA)^a_\nu - \partial_\nu (gA)^a_\mu + f^{abc} (gA)^b_\mu (gA)^c_\nu\right]$$

- **CYM Hamiltonian in temporal gauge (A_0=0)**

$$H = \frac{1}{2} \sum_{a,i,x} \left[E^a_i(x)^2 + B^a_i(x)^2\right], \quad B^a_i(x) = \varepsilon_{ijk} F^a_{jk}(x)/2$$

$$\frac{dA^a_i(x)}{dt} = E^a_i(x), \quad \frac{dE^a_i(x)}{dt} = -\frac{\partial H}{\partial A^a_i(x)}$$

- **Wigner functional and Husimi functional**

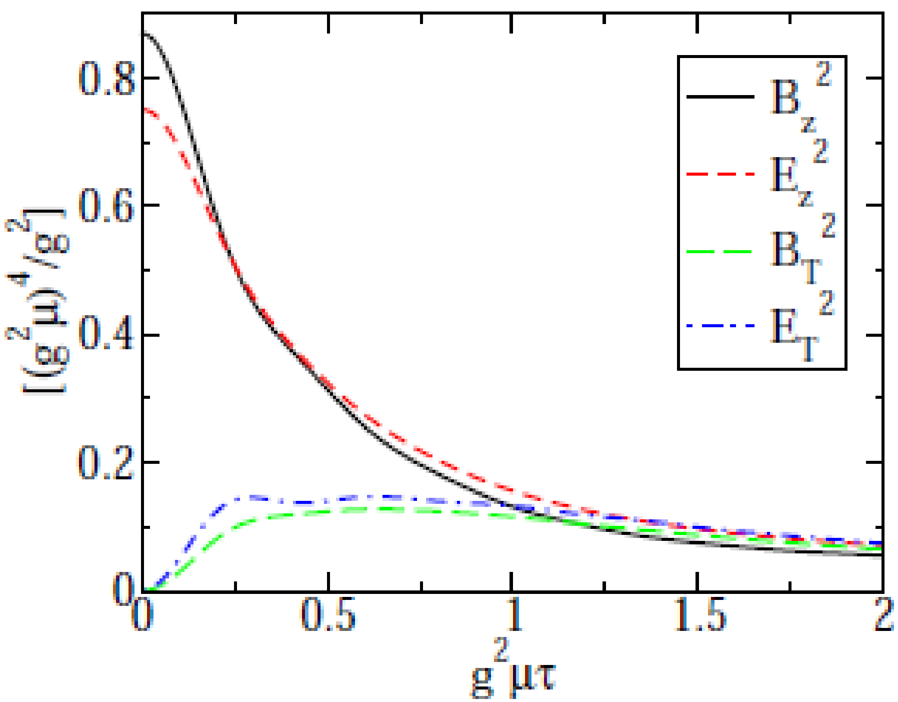
S. Mrowczynski and B. M¨uller, PRD 50('94)7542.
T. Kunihiro, B. Muller, A. Schafer, A. Ohnishi, PTP 121('09)555.

$$f_{\rm W}[A, E] = \int ds\ e^{iEs/\hbar}\ \langle A - s/2 \mid \rho \mid A + s/2 \rangle$$

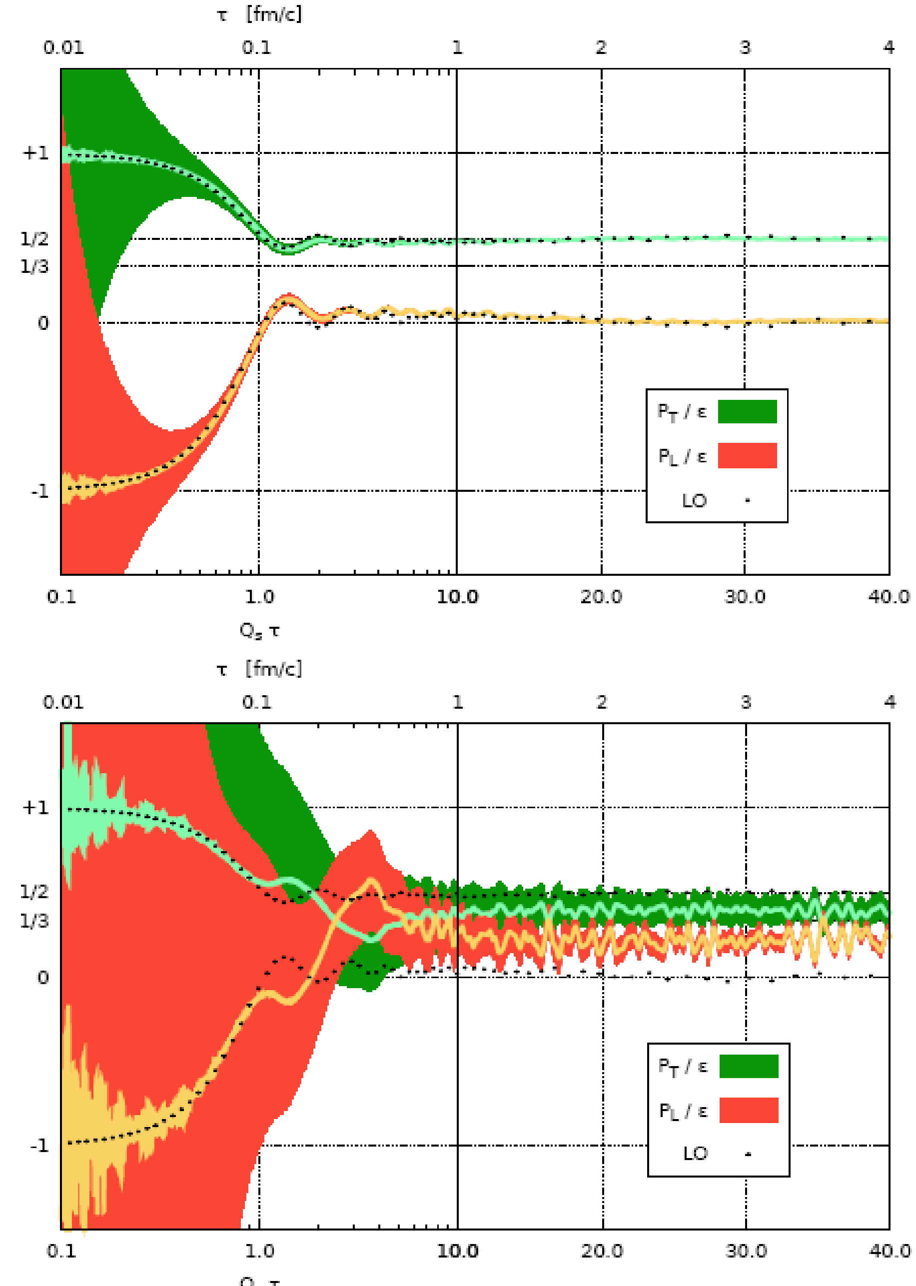
$$f_{\rm H}[A, E] = \int \frac{dA' dE'}{\pi \hbar} e^{-\Delta(A-A')^2/\hbar - (E-E')^2/\Delta\hbar} f_{\rm W}[A', E']$$

Classical Yang-Mills evolution

- **Classical Statistical Simulation (CGC initial cond.+fluc. +CYM eq.)**
  McLerran, Venugopalan ('94), Romatschke, Venugopalan ('06), Lappi, McLerran ('06), Berges, Scheffler, Sexty ('08), Fukushima ('11), Fukushima, Gelis ('12), Epelbaum, Gelis ('13)



Lappi, McLerran ('06)



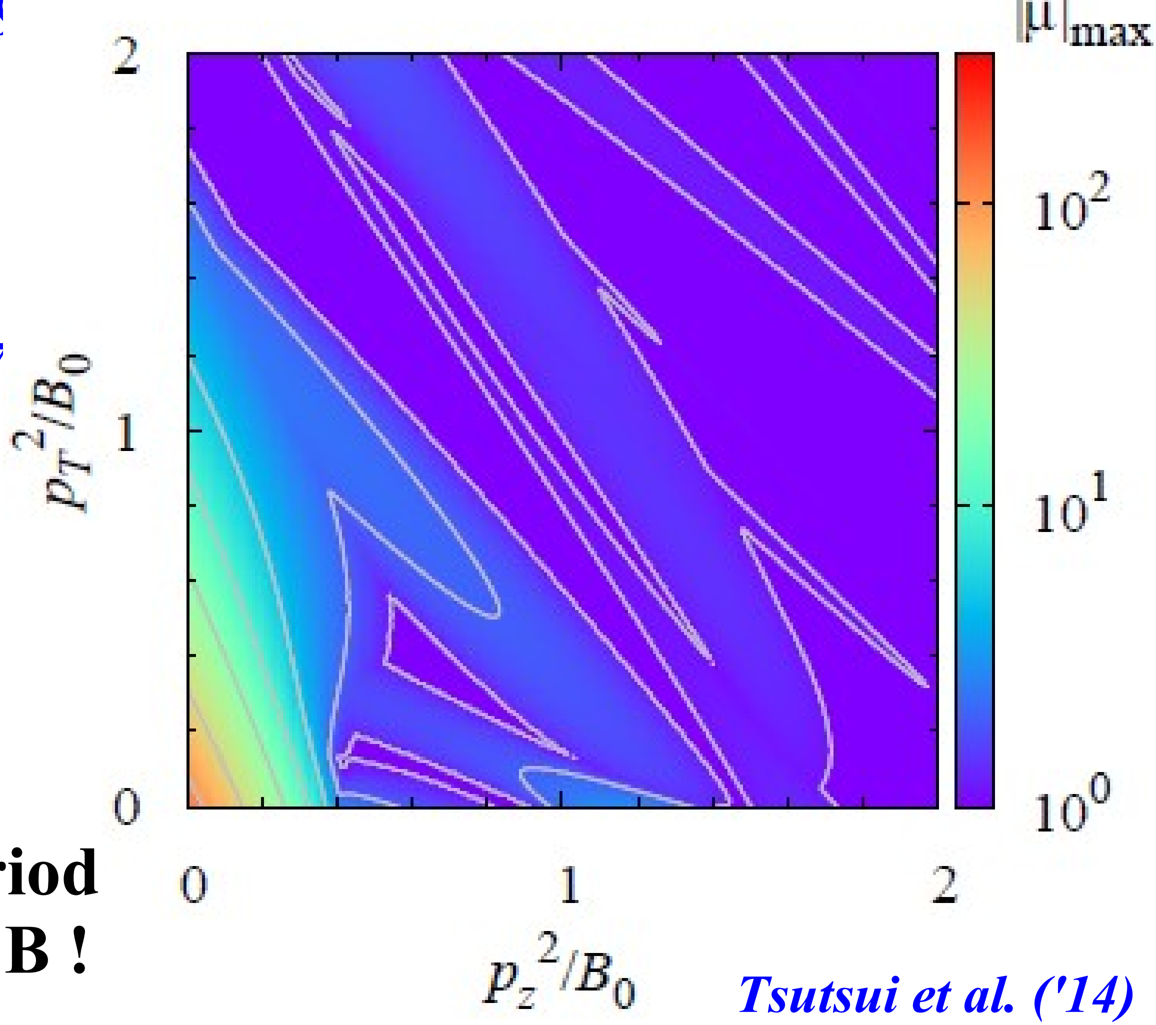
Epelbaum, Gelis ('13)

CYM Instabilities under color-magnetic background

- **Weibel instability**
  E.S.Weibel, PRL 2 ('59),83; S. Mrowczynski, PLB 214 ('88),587.
- **Nielsen-Olesen instability**
  N. Nielsen, P. Olesen, NPB 144 ('78), 376;
  H. Fujii, K. Itakura, NPA 809 ('08), 88
  H. Fujii, K. Itakura, A. Iwazaki,
  NPA 828 ('09), 178.
- **Parametric instability**
  J. Berges, S. Scheffler, S. Schlichting,
  D. Sexty (BSSS), PRD 85 ('12),034507
  S. Tsutsui, H. Iida, T. Kunihiro, AO,
  arXiv:1411.3809.

$$A_i^a = \tilde{A}(t)\left(\delta^{a2}\delta_{ix} + \delta^{a1}\delta_{iy}\right)$$

μ = enh. factor in one period



Tsutsui et al. ('14)

Enh. by >100 times in one period under homogeneous-periodic B !

How to obtain Lyapunov exponents

- **Kolmogorov-Sinai entropy rate h_{KS} = Entropy production rate**

$$\frac{dS}{dt} = h_{\mathrm{KS}}, \quad h_{\mathrm{KS}} = \sum_{\lambda_i > 0} \lambda_i, \quad |\delta X_i(t)| = e^{\lambda_i t} |\delta X_i(0)|$$

λ_i = Lyapunov exponent

- **EOM of δX → Integral (Trotter formula)**

$$\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} H_p \\ -H_x \end{pmatrix} \rightarrow \delta \dot{X} = \begin{pmatrix} H_{px} & H_{pp} \\ -H_{xx} & -H_{xp} \end{pmatrix} \delta X \equiv \tilde{H} \delta X$$

$$\left(H_{px} \equiv \partial^2 H / \partial p \partial x \quad \text{etc} \right)$$

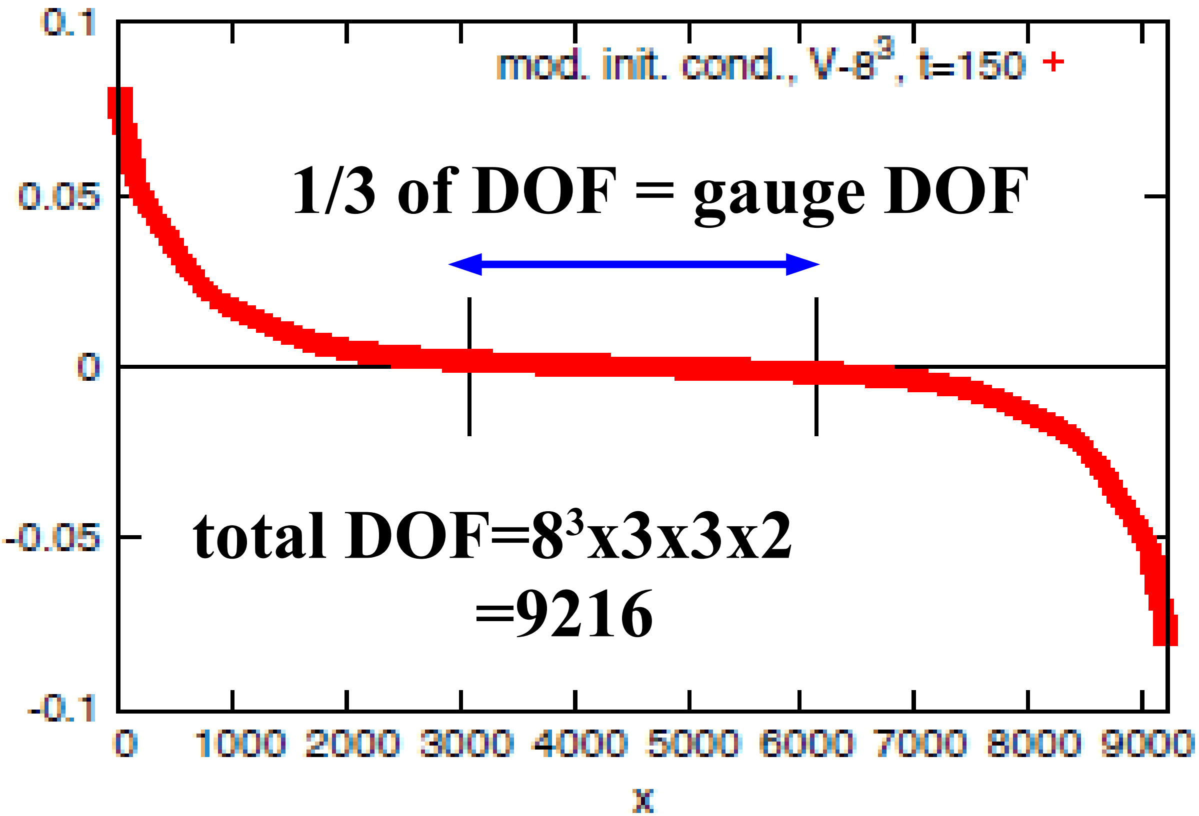
$$\delta X(t) = T \exp\left(\int_0^t dt' \tilde{H}(t') \right) \delta X(t=0) \simeq T \prod_{k=1,N} (1 + \tilde{H} \Delta t) \delta X(t=0)$$

$$= U(0,t) \delta X(t=0)$$

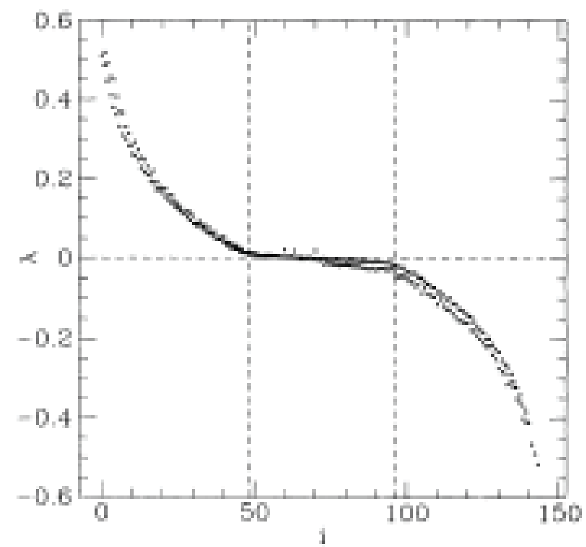
  - **Diagonalizing U and the eigen value becomes λt.**
  - **Matrix size = 3 (xyz) x (N_c^2-1) x L^3 x 2 (A,E)**

Typical Lyapunov spectrum

- **Sum of all Lyapunov exponent = 0 (Liouville theorem)**
- **1/3 Positive, 1/3 negative, and 1/3 zero (or pure imag.)**



Iida, Kunihiro, B.Müller, AO, Schäfer, Takahashi ('13)



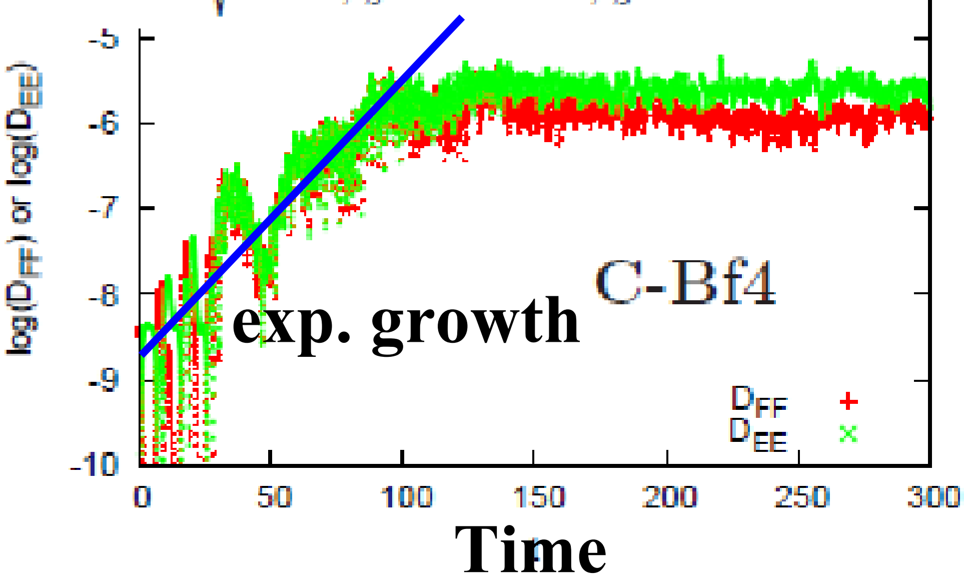
cf) Lyapunov spectrum ($V=2^3$)
Gong, Phys.Rev.D49, 2642 (1994).

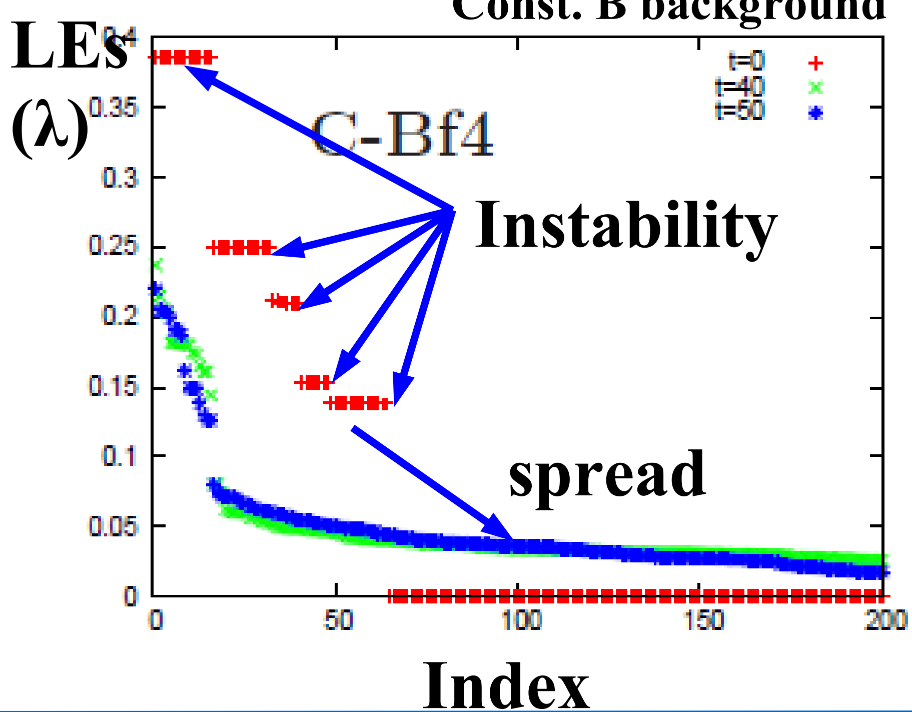
Chaoticity of CYM

Kunihiro, Müller, AO, Schäfer, Takahashi, Yamamoto ('10)
Iida, Kunihiro, B.Müller, AO, Schäfer, Takahashi ('13)

- **Chaoticity in CYM**
  T. S. Biro, S. G. Matinyan, B. Muller, Lect. Notes Phys. 56 ('94), 1; S. G. Matinyan, E. B. Prokhorenko, G. K. Savvidy, JETP Lett. 44 ('86) 138; NPB 298 ('88), 414; B. Muller, A. Trayanov, PRL 68 ('92), 3387; T. S. Biro, C. Gong, B. Muller, PRD 52 ('95),1260; C. Gong, PRD 49 ('94), 2642.
  - **Exponential growth of distance from adjacent init. cond.**
  - **Rapid spread of positive Lyapunov exponents**

$$D_{FF} = \sqrt{\sum_x \{\sum_{a,i,j} F^a_{ij}(x)^2 - \sum_{a,i,j} F'^a_{ij}(x)^2\}^2}$$





Conformal Property

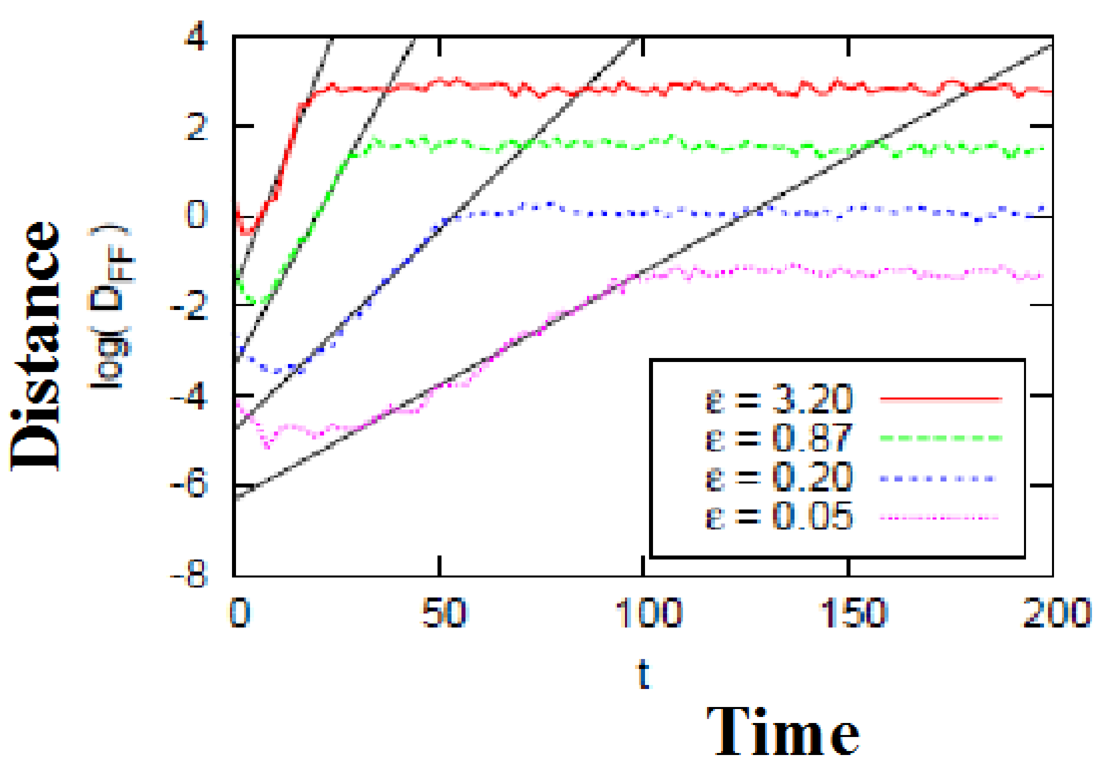
Kunihiro, Müller, AO, Schäfer, Takahashi, Yamamoto ('10)

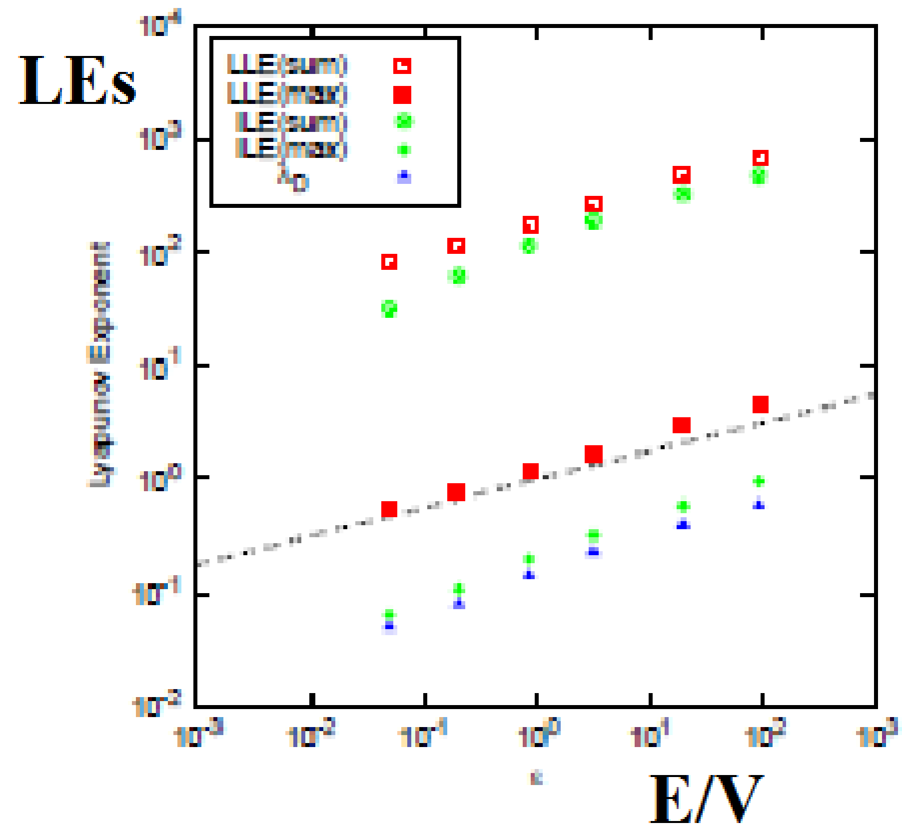
- **No conformal anomaly in CYM**
  → Any average quantity scales as $\varepsilon^{n/4}$ (ε: energy density, n: mass dim.)

$$\lambda^{\rm LLE}_{\rm sum}/L^3 \simeq 3\times\varepsilon^{1/4}\ ,\quad \lambda^{\rm LLE}_{\rm max} \simeq 1\times\varepsilon^{1/4}$$

$$\lambda^{\rm ILE}_{\rm sum}/L^3 \simeq 2\times\varepsilon^{1/4}\ ,\quad \lambda^{\rm ILE}_{\rm max} \simeq 0.2\times\varepsilon^{1/4}$$

  - **LLE: temporally local, ILE: integral during exp. growing period**





Husimi-Wehrl entropy of CYM

Tsukiji, Iida, Kunihiro, AO, Takahashi, in progress

- **Husimi-Wehrl entropy of CYM on the lattice**

$$S_{\rm HW} = -\int \frac{d^D A d^D E}{(2\pi\hbar)^D} f_H[A,E] \log f_H[A,E]$$

$$f_H[A,E] = \int \frac{d^D A' d^D E'}{\pi\hbar} e^{-\Delta(A-A')^2/\hbar-(E-E')^2/\Delta\hbar} f_W[A,E]$$

  - **D=576 on 4^3 lattice for N_c=2 → 1152 dim. integral, average exponent ~ D (problem with large deviation !)**

- **Hartree approximation**

$$f_H[A,E] \simeq \prod_{x,i,a} f_H^{iax}(A,E)$$

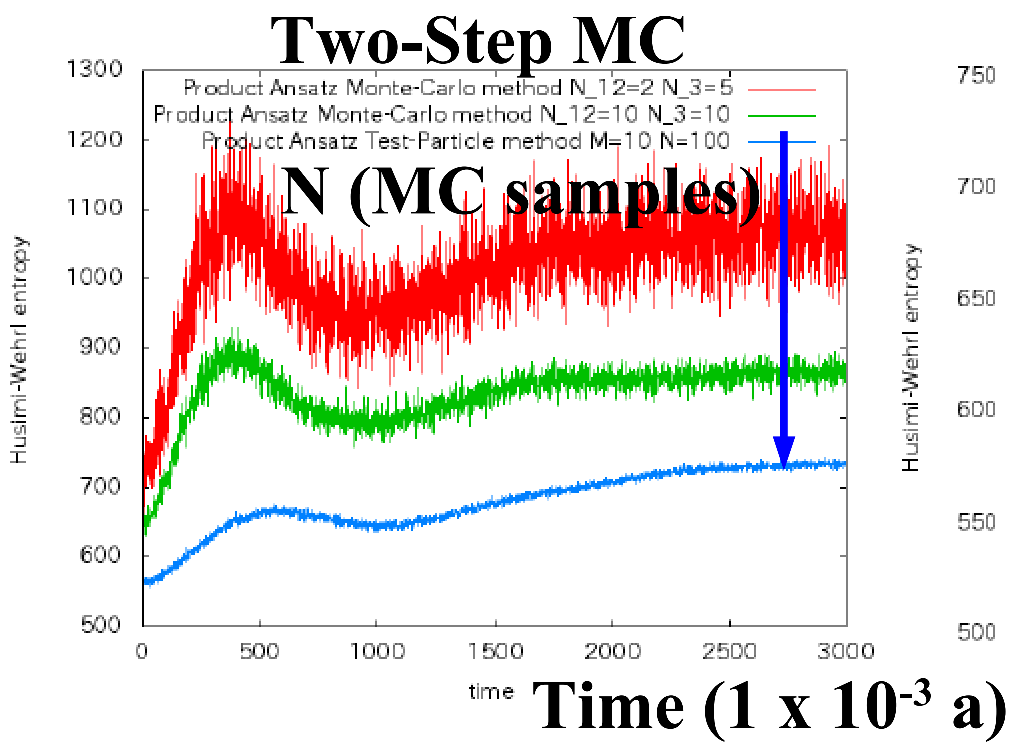
$$\to S_{\rm HW} = -\sum_{x,i,a} \int \frac{dAdE}{2\pi\hbar} f_H^{iax}(A,E) \log f_H^{iax}(A,E)$$

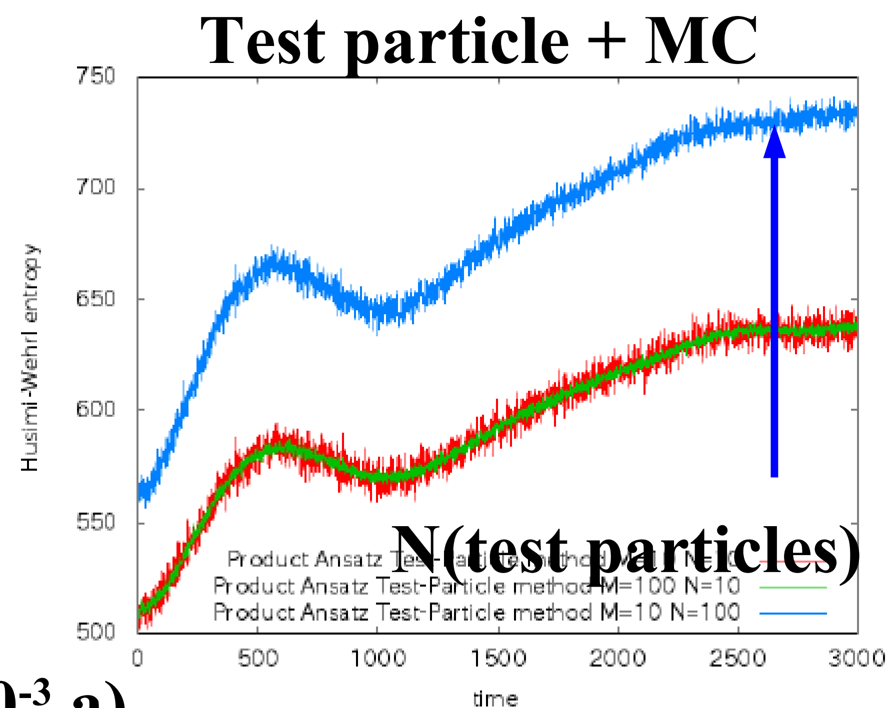
  - **Hartree approx. gives error of 10-20 % in HW entropy for 2d quantum mech.**

Preliminary Results

Tsukiji, Iida, Kunihiro, AO, Takahashi, in progress

- **Preliminary numerical results of SU(2) CYM on a 4^3 lattice**
  - **Initial cond. = min. wave packet (gaussian) → $S_{HW} \sim 576$**
  - **Entropy production is observed !**





まとめ

- 高エネルギー重イオン衝突物理において「早い熱平衡化」は大きな未解決問題のひとつであり、「エントロピー生成」機構解明が望まれている。
- 位相空間の複雑さにより生まれるエントロピー生成は、不安定モードの強さと数により記述され、系のカオス性と密接にかかわる。
(Kolmogorov-Sinaii entropy rate= 正の Lyapunov exponents の和)
- 多自由度系における Husimi-Wehrl entropy の評価は現在でも研究が進む課題。
  - 不安定調和振動子、Yang-Mills 量子力学系において半古典近似 +Smearing を用いた Husimi 関数の時間発展によりエントロピー生成を議論 [Kunihiro et al.('09), Tsukiji et al.(in prep.)]
- 古典ヤンミルズ理論に基づく高エネルギー重イオン衝突初期のダイナミクスが活発に議論されている。
  - 場の変数と共役運動量を正準変数として Wigner 汎関数、Husimi 汎関数を定義し、不安定性、Kolmogorov-Sinai rate 、Husimi-Wehrl entropy を議論 [Kunihiro et al.('10), Iida et al.('13, '14), Tsutsui et al.('14)]
- 乱流との関連 ....

Thank you for your attention !

Decoherence Entropy

Muller, Schafer ('03,'06), Fries, Muller, Schafer ('09), Iida, Kunihiro, AO, Takahashi ('14)

- **Coherent State**

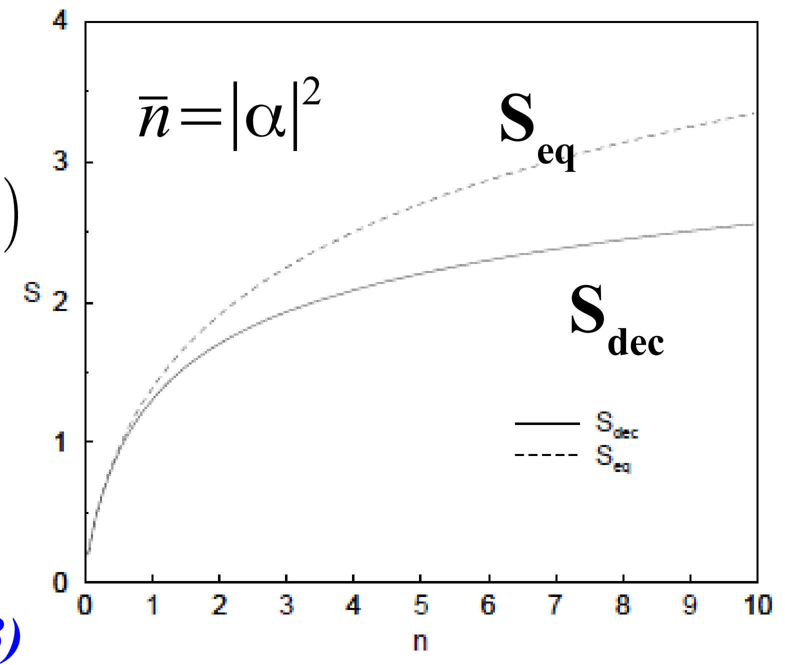
$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

$$|\alpha\rangle = N\exp(\alpha\hat{a}^{+})|0\rangle = \exp(-|\alpha|^2/2)\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

  - **n-quanta states are coherently superposed in a coherent state.**
  - **When this coherence is broken, entropy is generated (decoherence entropy)**

$$P_n = \frac{|\alpha|^{2n}}{n!}\exp(-|\alpha|^2) \quad (\text{Poisson dist.})$$

$$\rightarrow \quad S_{\rm dec} = -\sum_{n=0}^{\infty} P_n \log P_n > 0$$



Muller, Schafer ('03)

CYM as a Coherent State

Muller, Schafer ('03,'06), Fries, Muller, Schafer ('09), Iida, Kunihiro, AO, Takahashi ('14)

- **What kind of state does the CYM correspond to ?**
  → Natural guess = Coherent State

$$|\,\mathrm{CYM}\,\rangle \simeq \prod_{\boldsymbol{k},a,i} |\alpha_{\boldsymbol{k}ai}\rangle$$

- **Decoherence entropy from CYM**

$$S_{\mathrm{dec}} = -\sum_{\boldsymbol{k},a,i}\sum_{n} P_n(\alpha_{\boldsymbol{k}ai}) \log P_n(\alpha_{\boldsymbol{k}ai})$$

$$\alpha_{\boldsymbol{k}ai} = \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}}\left[\omega_{\boldsymbol{k}} A_{ai}(\boldsymbol{k},t) + i E_{ai}(\boldsymbol{k},t)\right], \quad \omega_{\boldsymbol{k}} = \sqrt{\sin^2 k_x + \sin^2 k_y + \sin^2 k_z}$$

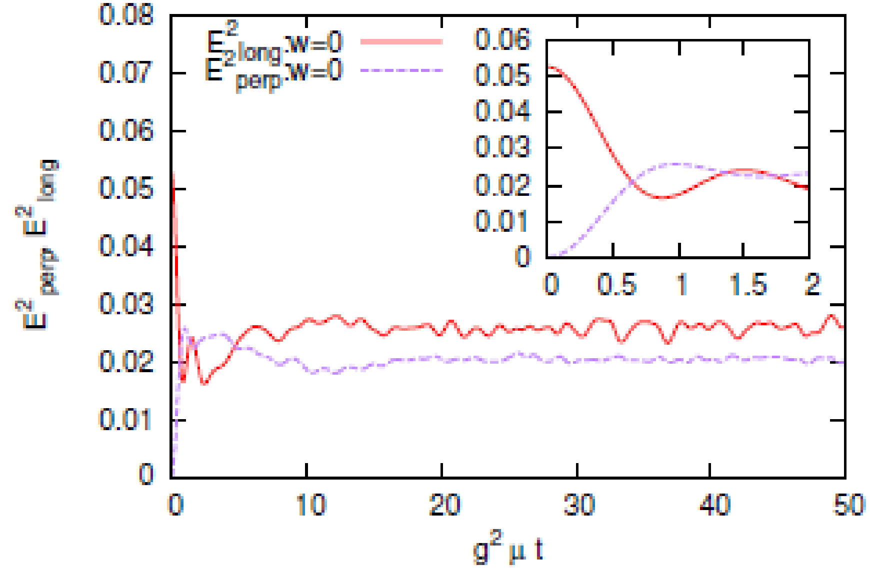
- **Is the above assignment unique ?**
  - **Coherent state in each "coherent domain"** *Fries, Muller, Schafer ('09)*
  - **Deviation from Poisson dist. with coupled oscillator**
    Glauber ('66), Gelis, Venugopalan ('06)

Initial Condition and Time Evolution

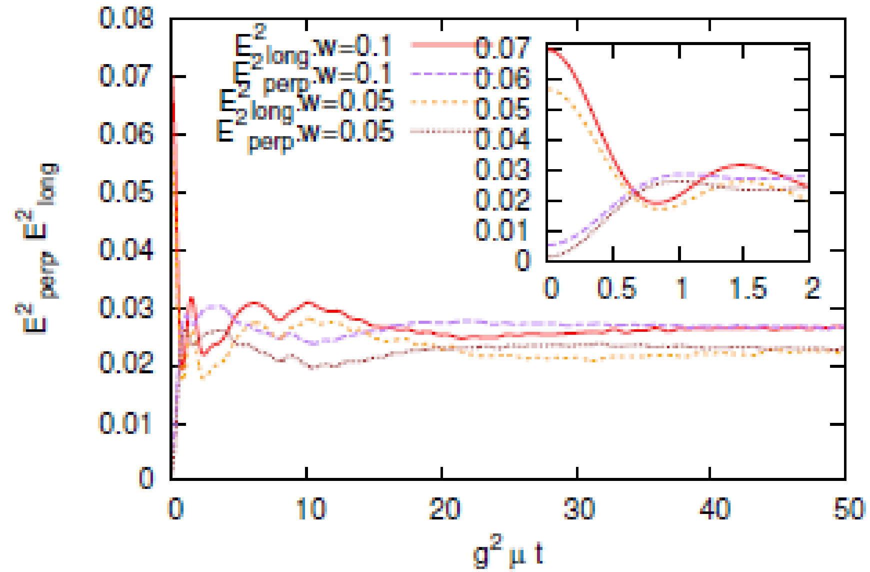
- **"Glasma-like" init. cond.**
  - **MV model (boost inv.)**
    + Longitudinal fluctuations

    → Bx,y, Ex,y, Bη, Eη
    McLerran, Venugopalan ('94), Romatschke, Venugopalan ('06), Fukushima, Gelis ('12)
  - **Non-expanding geometry is assumed, Substitute Bη and Eη in MV model into Bz and Ez at t=0.**

20^3 lattice

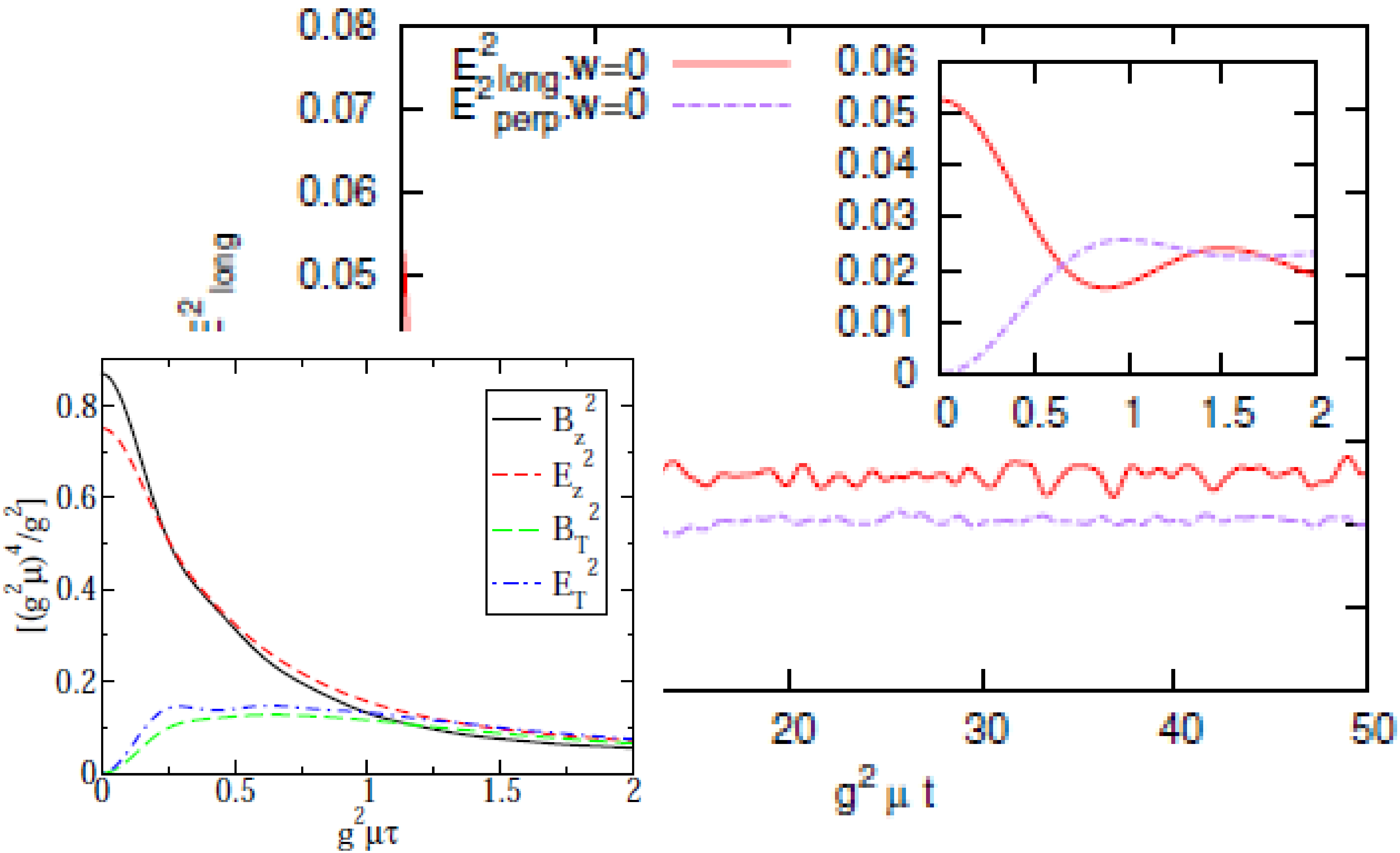


- **Time-evolution**
  - **Short time behavior of E^2 does not depend on the fluctuation strength.**
    (and similar to expanding geo. results.)
    E.g. Lappi, McLerran ('06)
  - **Long-time behavior:**
    Earlier "isotropization" in perp. and long. directions of E^2.



Iida, Kunihiro, AO, Takahashi ('14)

Initial Condition and Time Evolution



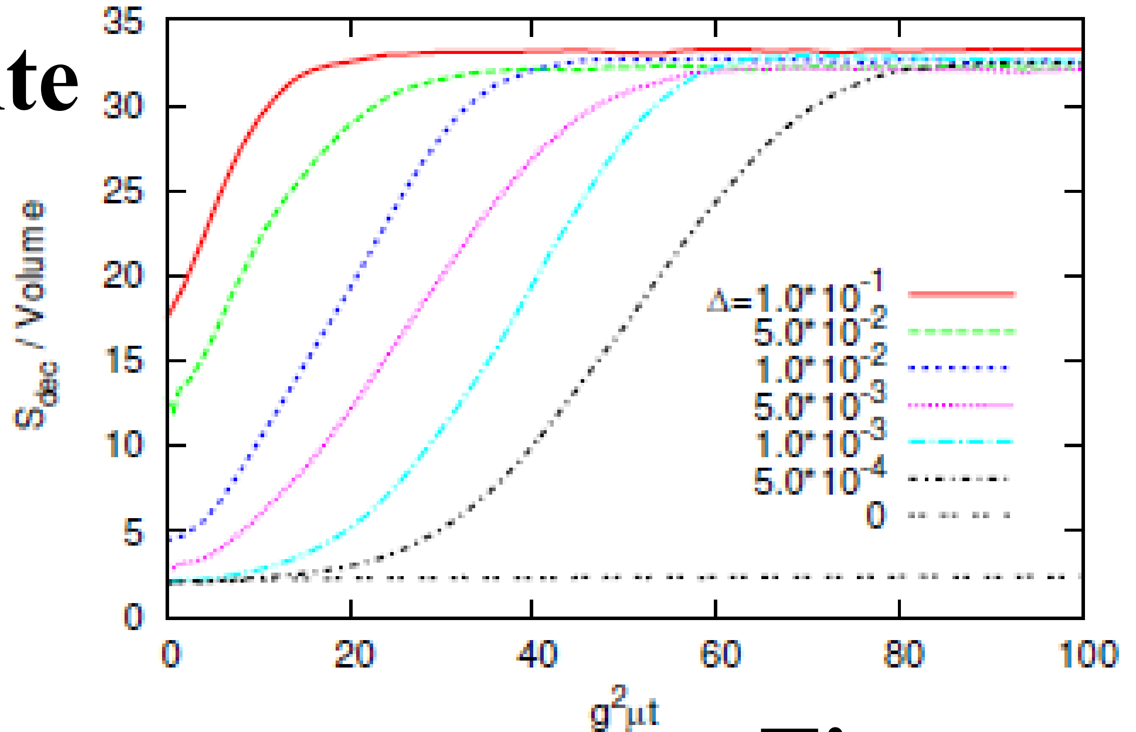
Lappi, McLerran ('06)

Iida, Kunihiro, AO, Takahashi ('14)

Decoherence Entropy of CYM

- **How about the decoherence entropy ?**
  - **< δE^2>/<E^2> ~ 0.1 (Δ=0.05) and 0.3 (Δ=0.1)**
  - **S_{dec} ~ 2.3 (Δ=0) and 33 (Δ=0.05, 0.1)**
  - **Entropy from initial state fluc. and chaoticity**
  - **No long. fluc. results in 2D (pz=0 mode) entropy, while 3D entropy is realized with finite long. fluc. (non-zero Δ).**

S_{dec}/site

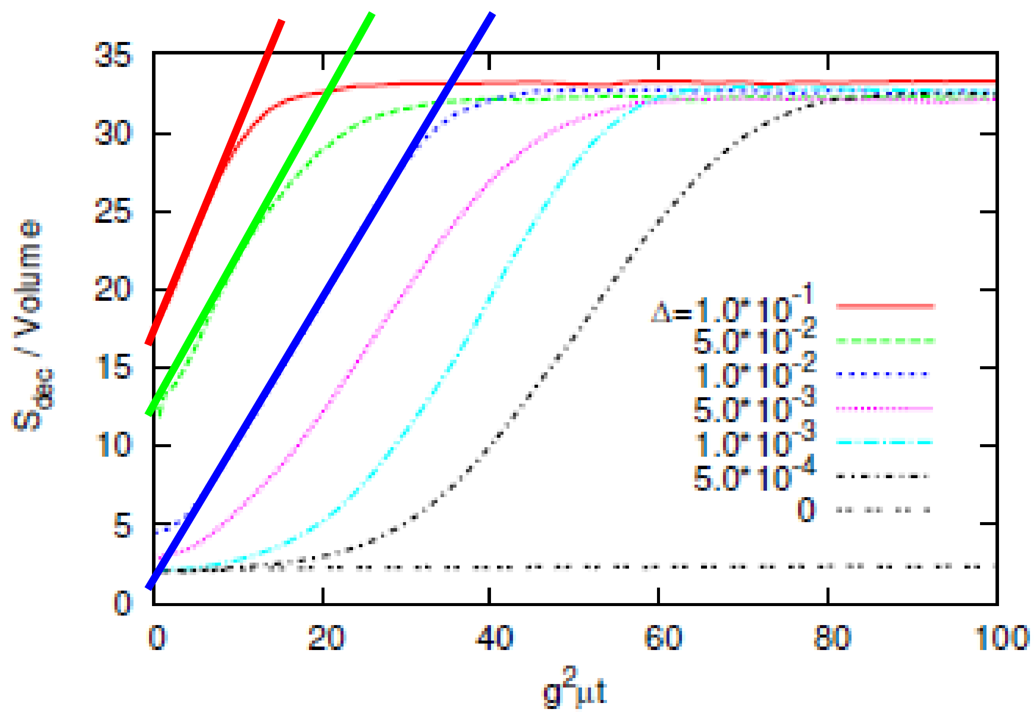


Time

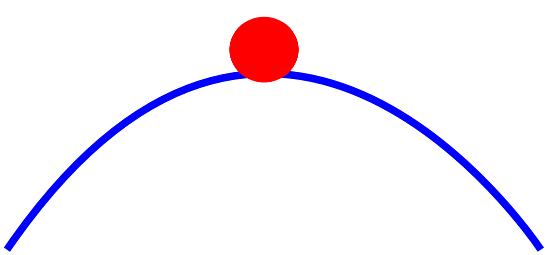
Decoherence Entropy Production Rate

- **Decoherence entropy growth rate should be compared with KS entropy**
  - **$dS_{dec}/dt \sim 0.88\ (\Delta=0.01),\ 1.05\ (\Delta=0.05),\ 1.36\ (\Delta=0.1)$**
  - **KS entropy estimate: $S_{KS} \sim c_{KS}\ \varepsilon^{1/4}$, $c_{KS} \sim 2$ (conformal chaotic value)**
  - **Energy density: $\varepsilon = 0.17\ (\Delta=0.01),\ 0.18\ (\Delta=0.05),\ 0.21\ (\Delta=0.1)$**
    $\rightarrow c_{KS} = dS^{dec}/dt/\varepsilon^{1/4} = 1.4\ (\Delta=0.01),\ 1.6\ (\Delta=0.05),\ 2.0\ (\Delta=0.1)$

$$\frac{1}{S_{\rm KS}}\frac{dS_{\rm dec}}{dt} \sim (0.7 - 1.0)$$



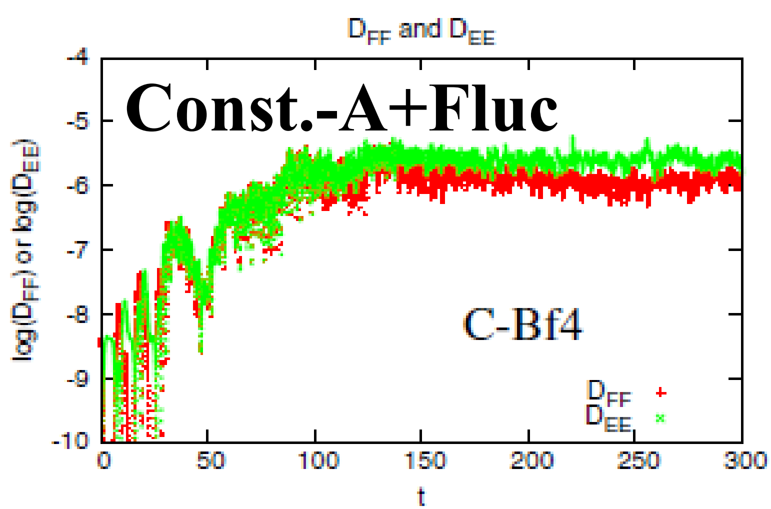
  - **KS entropy**
    = Potentially realized
    growth rate

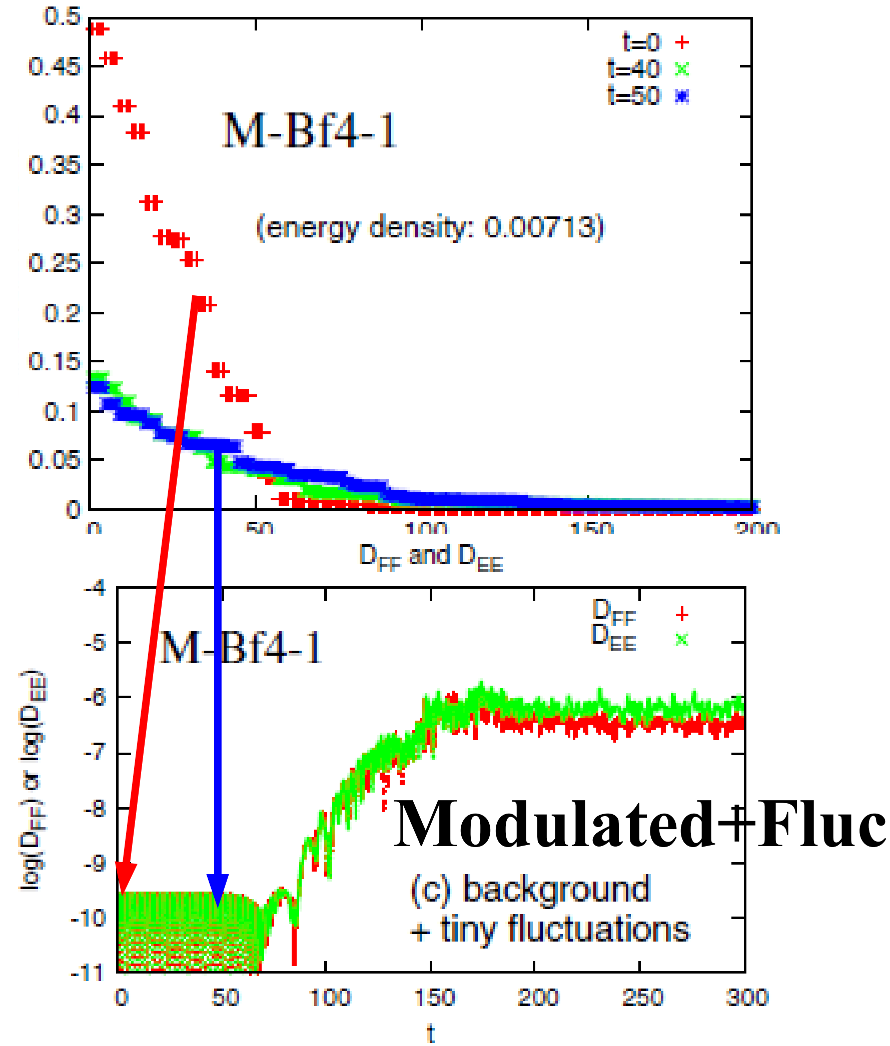


Δ=0: unstable
but stationary

KS entropy in CYM from glasma-like init. cond.

- **Instability under strong color-magnetic field**
  Nielsen, Olesen ('78), Fujii, Itakura ('08), Berges, Scheffler, Schlichting, Sexty ('12)
  - **No chaotic behavior is observed with sine waves and constant-A w/o fluctuations.**
  - **Small fluctuations activate instability and chaoticity.**
  - **Chaoticity emerges after instability spreads to many modes.**





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- *Chaotic behavior in classical Yang-Mills dynamics*, **T. Kunihiro, B. Müller, A. Ohnishi, A. Schäfer, T. T. Takahashi, A. Yamamoto, Phys. Rev. D 82 (2010), 114015 [arXiv:1008.1156].**
- *Entropy production in classical Yang-Mills theory from Glasma initial conditions*, **H. Iida, T. Kunihiro, B. Müller, A. Ohnishi, A. Schäfer, T. T. Takahashi, Phys. Rev. D 88 (2013), 094006 [arXiv:1304.1807].**
- *Time evolution of gluon coherent state and its von Neumann entropy in heavy-ion collisions*, **H. Iida, T. Kunihiro, A. Ohnishi, T. T. Takahashi, arXiv:1410.7309 [hep-ph].**
- *Parametric Instability of Classical Yang-Mills Fields under Color Magnetic Background*, **S. Tsutsui, H. Iida, T. Kunihiro, A. Ohnishi, arXiv:1411.3809.**

Chaoticity, Lyapunov exponent, and KS entropy

- **Entropy in classical dynamics = Wehrl entropy**

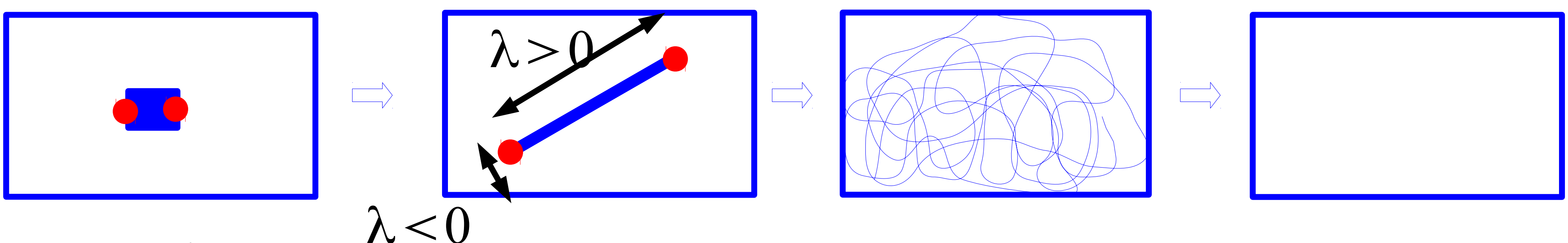
$$S = -\int d\Gamma\, H \log H$$

($d\Gamma$= dx dp = phase space, H= phase space dist. fn., e.g. Husimi fn.)

- **Lyapunov exponent and Kolmogorov-Sinaii entropy**

$$\delta X_i(t) = \delta X_i(t_0) \exp[\lambda_i (t - t_0)] \quad (X = (x, p)),$$
$$dS/dt = S_{\mathrm{KS}} \equiv \sum_{i, \lambda_i > 0} \lambda_i$$

  - **δ X = difference of two trajectories from adjacent initial conditions**
    λ = initial state sensitivity (Lyapunov exponent, measure of chaoticity)
  - **When λ > 0, exponentially growing number of phase space cells are visited**
    → phase space dist. fn. becomes smooth after proper coarse graining
    → entropy production (Kolmogorov-Sinaii entropy)



Classical Yang-Mills dynamics on the lattice

- Lattice CYM Hamiltonian in temporal gauge (A_0=0) in the lattice unit

$$H=\frac{1}{2}\sum_{x,a,i}\left[E_i^a(x)^2+B_i^a(x)^2\right]$$

$$F_{ij}^a(x)=\partial_i A_j^a(x)-\partial_j A_i^a(x)+\sum_{b,c} f^{abc}\boxed{A_i^b(x)A_j^c(x)}=\varepsilon_{ijk}B_k^a(x)$$

Non-linear & coupling

- Non-compact (A, E) form !
  - Demerit: Gauge invariance is not fully satisfied at finite lattice spacing.
  - Merit: Easy to consider the coherent state, and conformality is manifest.
- Initial conditions ($E_i^a(x)=0$ is assumed here.)
  - Random initial condition: $A_i^a(x)$ = random in [-η, η],
  - Modulated init. cond.: $A_i^a(\vec{r})=\delta_{i2}\left[\epsilon_1\sin\left(\frac{2x\pi}{N_x}\right)+\epsilon_2\sin\left(\frac{2nx\pi}{N_x}\right)\sin\left(\frac{2mz\pi}{N_z}\right)\right]$
  - Constant-A init. cond. $A_i^a(x)=\sqrt{B/g}\left(\delta_{i2}\delta^{a3}+\delta_{i3}\delta^{a2}\right)$ *Berges et al.('12)*

magnetic field ~ z direction ($\varepsilon_1 >> \varepsilon_2$), w and w/o fluc.