Directed flow in heavy-ion collisions and softening of equation of state

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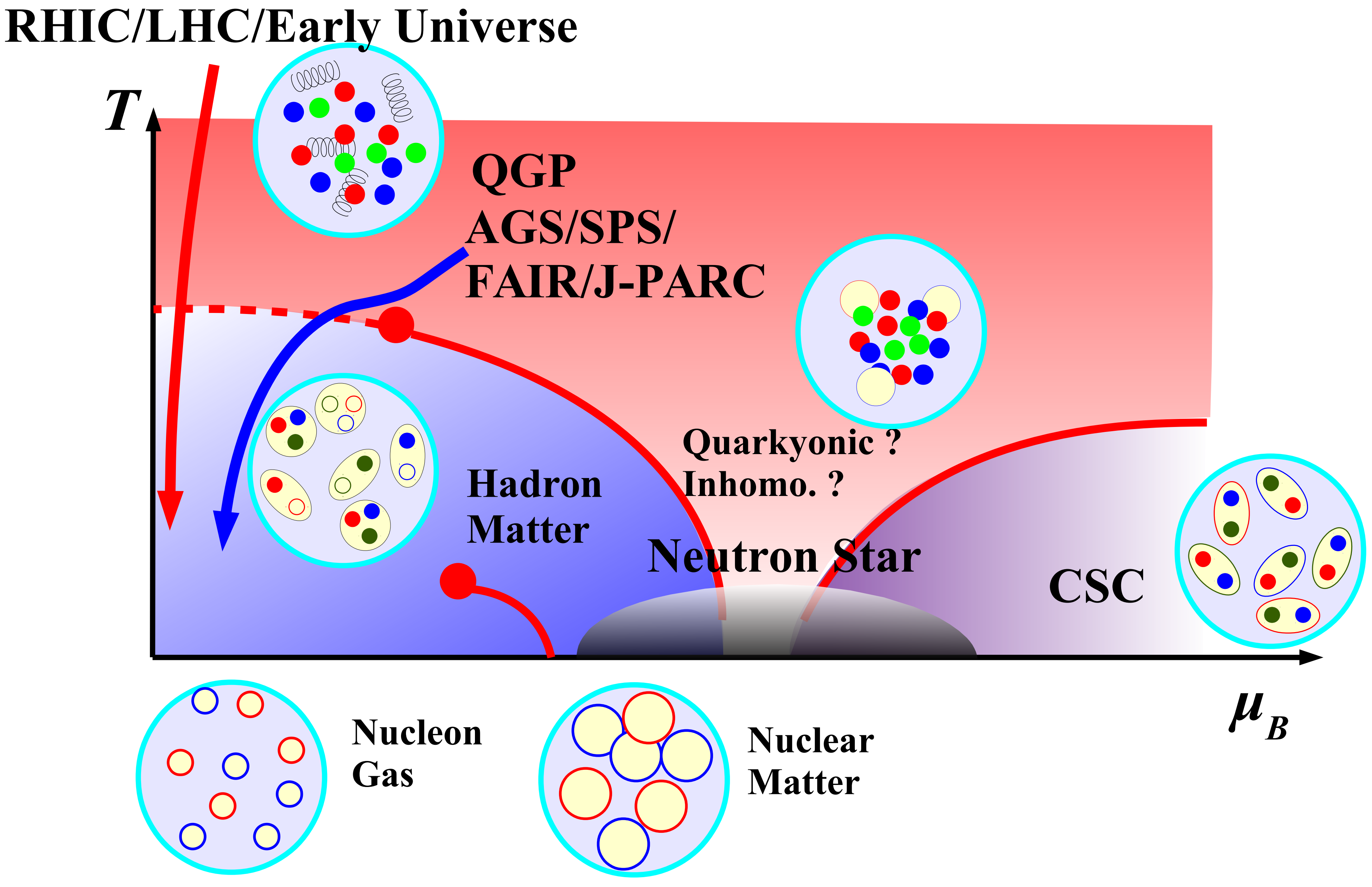
YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

Y. Nara, A. Ohnishi, arXiv:1512.06299 [nucl-th] (QM2015 proc.)
Y. Nara, A. Ohnishi, H. Stoecker, arXiv:1601.07692 [hep-ph]

QCD Phase Diagram

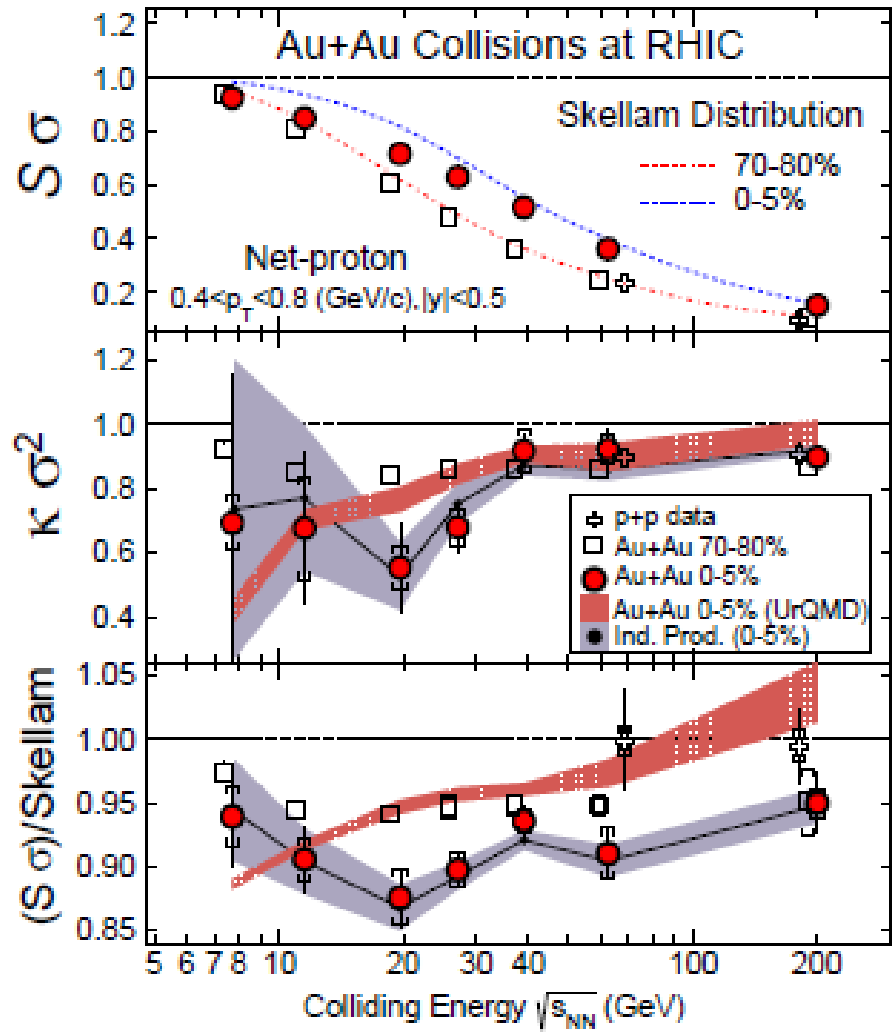


Signals of QGP formation & QCD phase transition

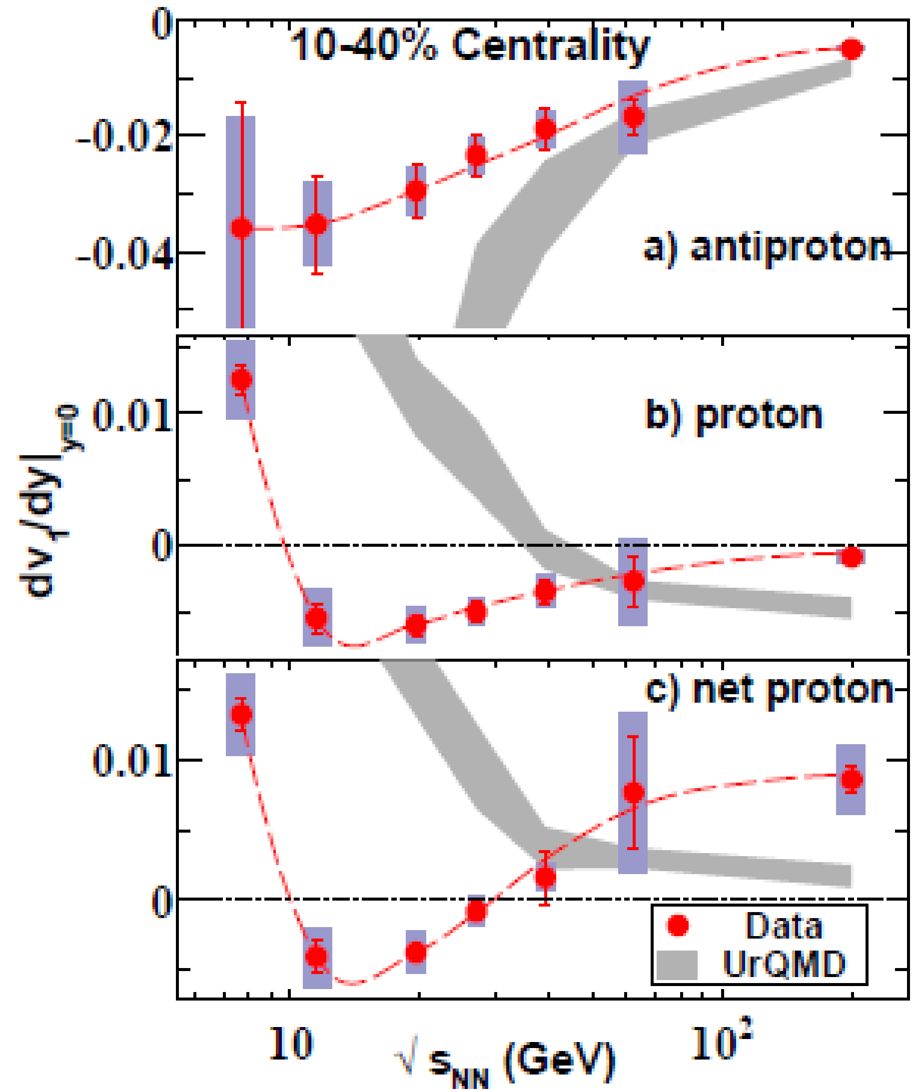
- **Signals of QGP formation at top RHIC & LHC energies**
  - **Jet quenching in AA collisions (not in dA)**
  - **Large elliptic flow (success of hydrodynamics)**
  - **Quark number scaling (coalescence of quarks)**
- **Next challenges**
  - **Puzzles: Early thermalization, Photon v2, Small QGP, …**
    → Complete understanding from initial to final states
  - **Discovery of QCD phase transition**
- **Signals of QCD phase transition at BES energies ?**
  - **Critical Point → Large fluctuation of conserved charges**
  - **First-order phase transition → Softening of EOS**

→ Non-monotonic behavior of
proton number moment ($\kappa\sigma^2$) and collective flow (dv_1/dy)

Net-Proton Number Cumulants & Directed Flow



STAR Collab. PRL 112('14)032302



STAR Collab., PRL 112('14)162301.

Signals of QGP formation & QCD phase transition

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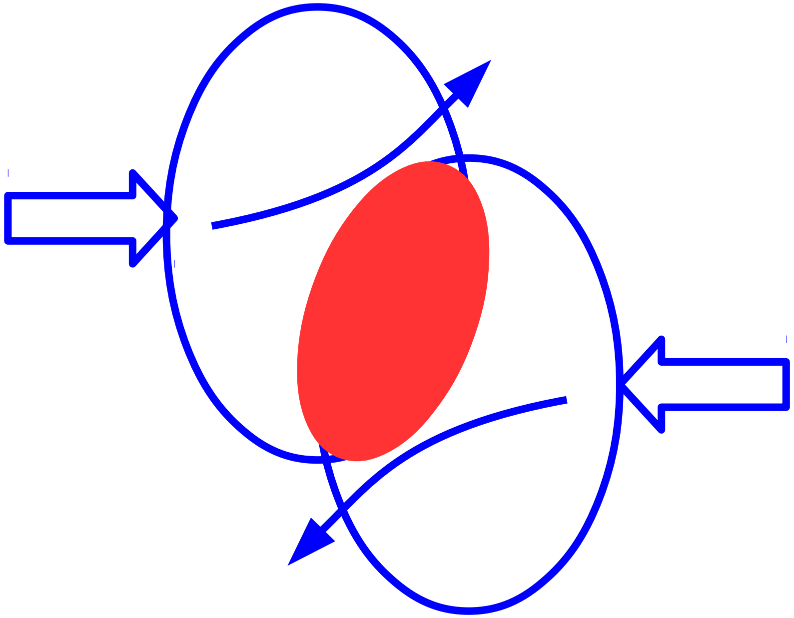
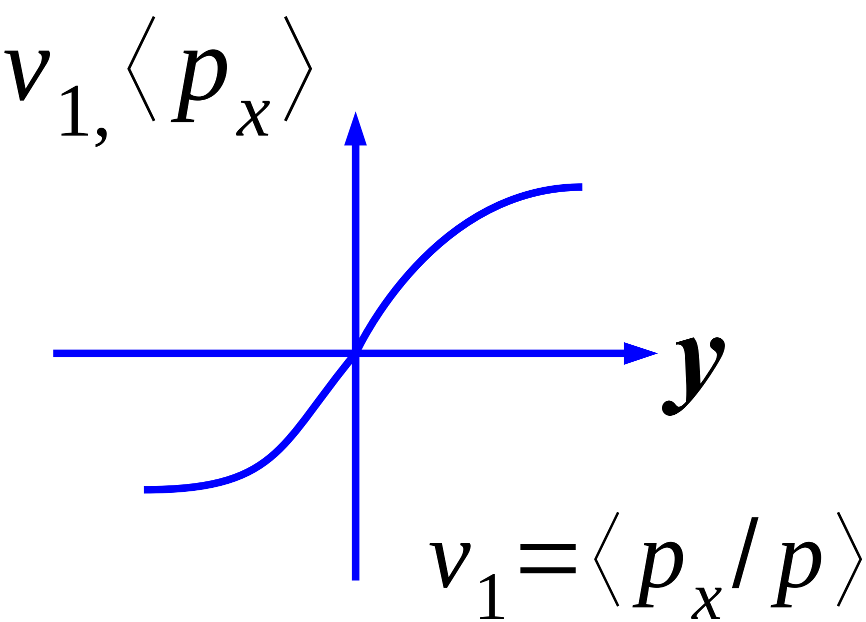
What is directed flow ?

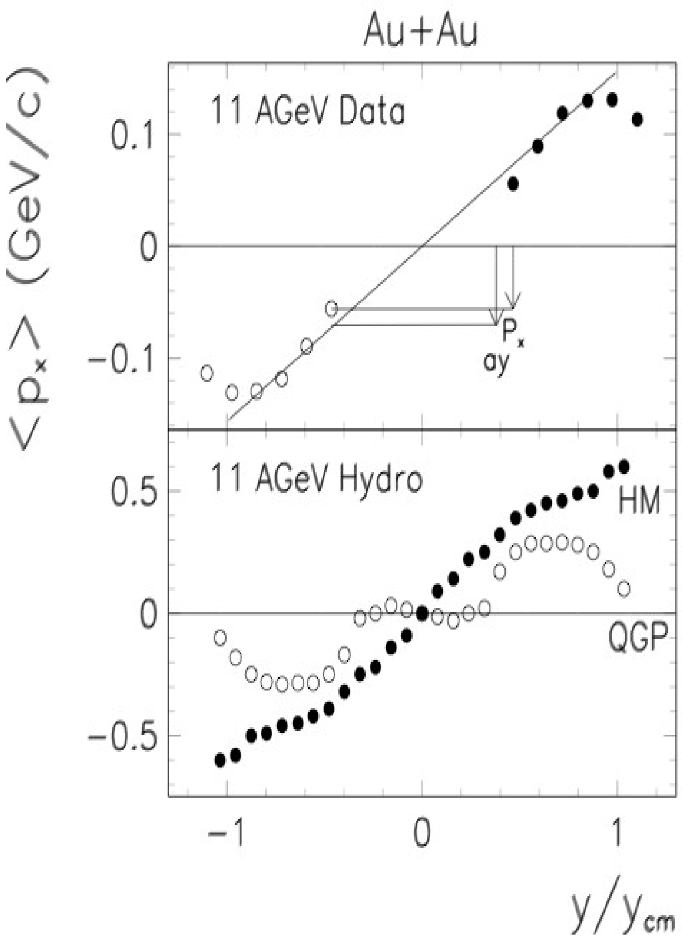


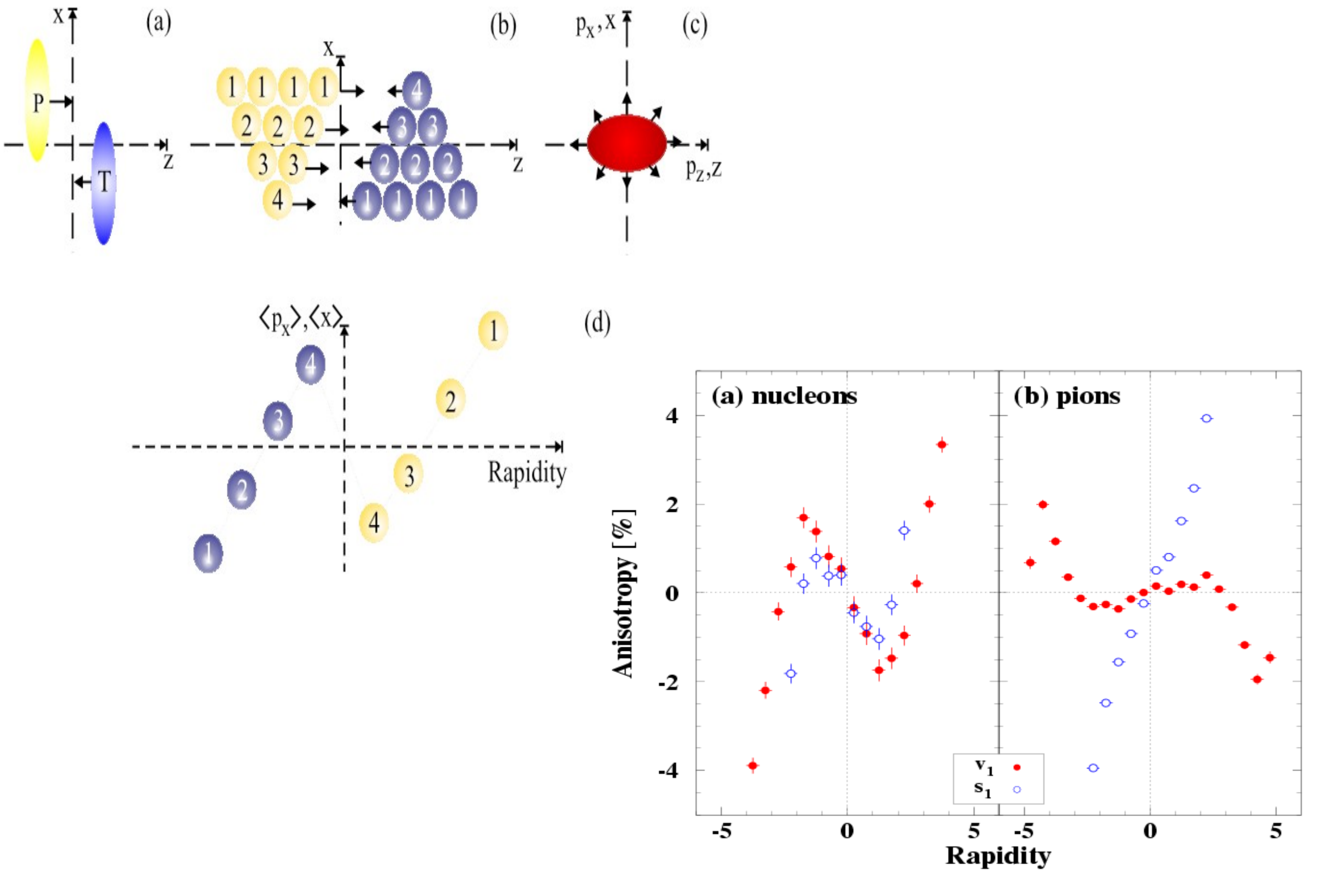
- v_1 or <p_x> as a function of y is called directed flow.
- Created in the overlapping stage of two nuclei
  → Sensitive to the EOS
  in the early stage.
- Becomes smaller at higher energies.
- How can we explain non-monotonic dependence of dv_1/dy ?



Does the "Wiggle" signal the QGP ?

- **Hydro predicts wiggle with QGP EOS.**
- **Baryon stopping + Positive space-momentum correlation leads wiggle (w/o QGP)**



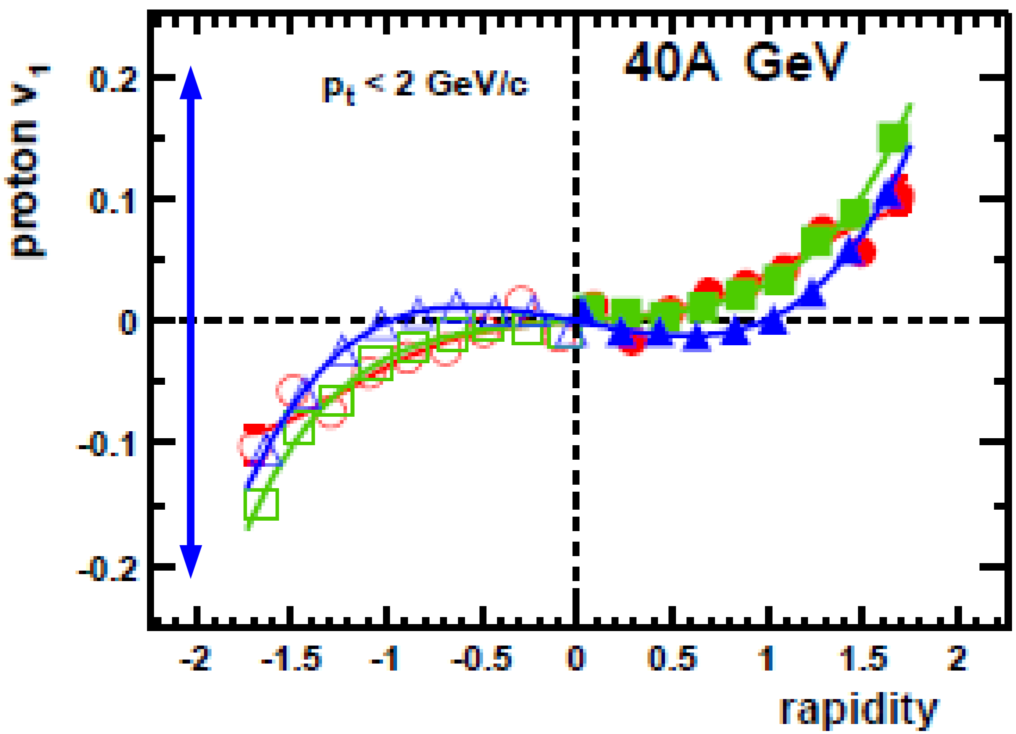


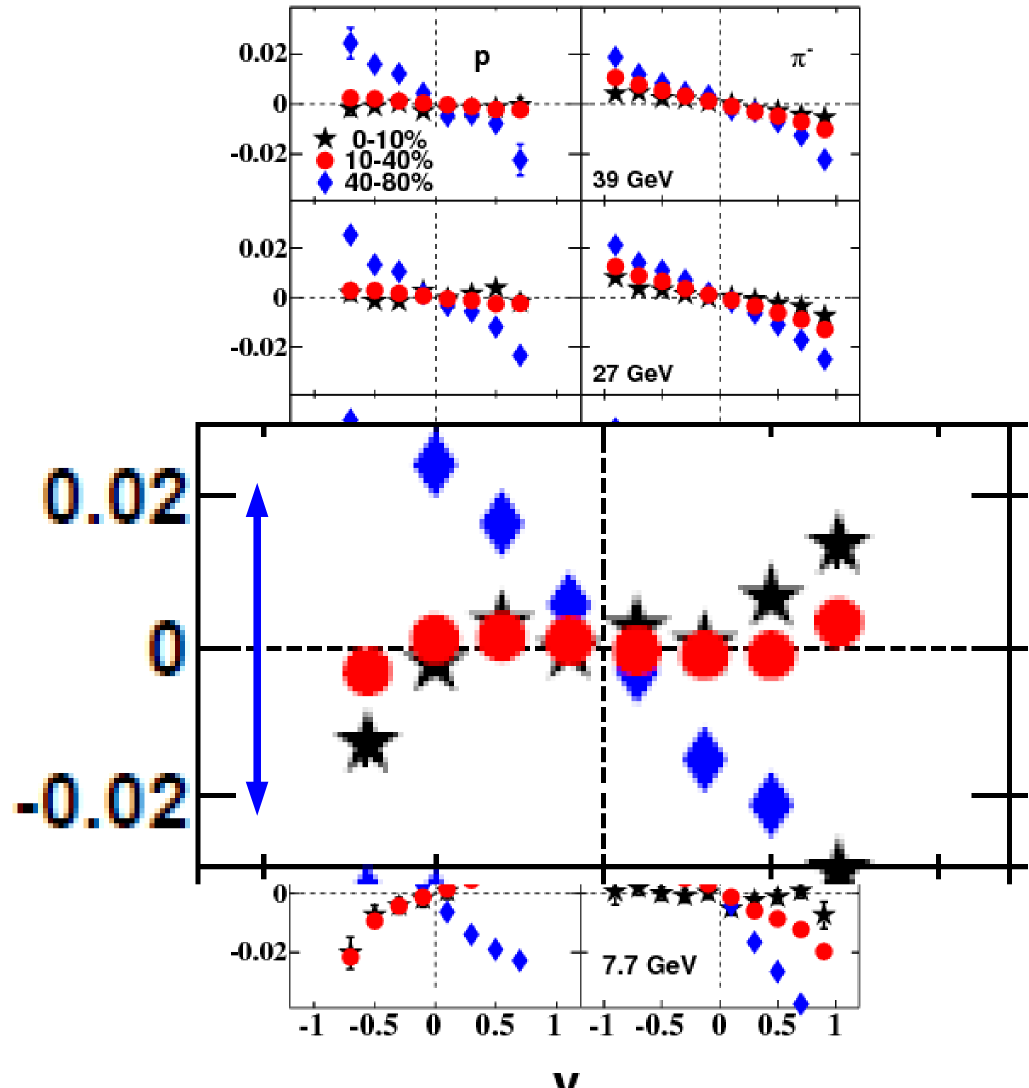
L. P. Csernai, D. Röhrich, PLB 45 (1999), 454.

R.Snellings, H.Sorge, S.Voloshin, F.Wang, N. Xu, PRL (84) 2803(2000)

SPS(NA49) vs RHIC(STAR)

- **SPS (NA49), $\sqrt{s_{NN}}$ = 8.9 GeV**
- **RHIC(STAR), $\sqrt{s_{NN}}$=11.5 GeV**



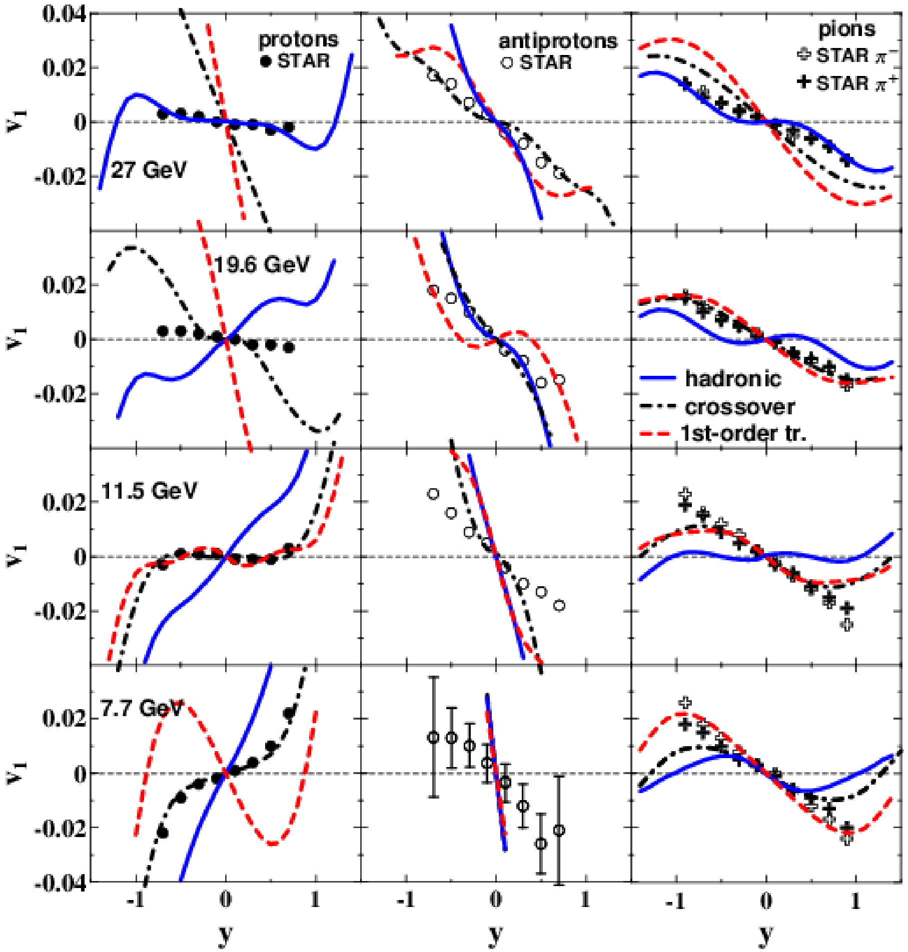


C. Alt et al. (NA49), PRC68 ('03) 034903

L. Adamczyk et al. (STAR), PRL 112(2014)162301

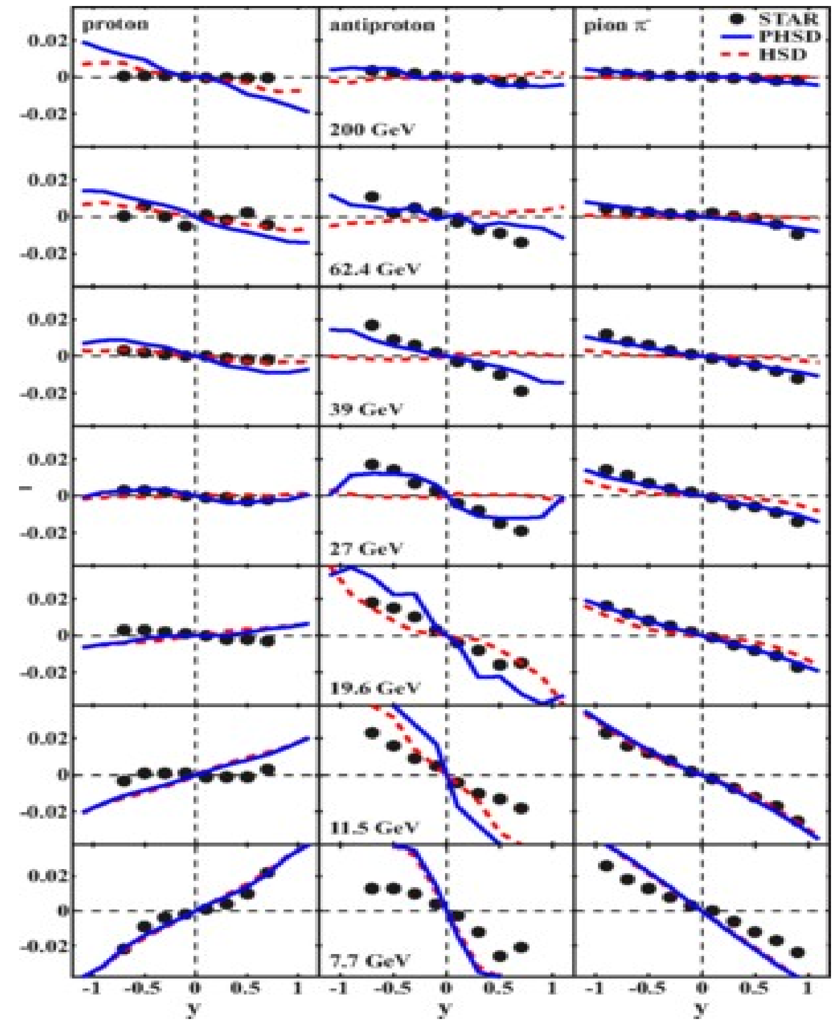
Negative dv_1/dy

- **Hydrodynamics**



Y. B. Ivanov and A. A. Soldatov, PRC91 (2015)024915

- **PHSD/HSD predictions**



V. P. Konchakovski, W. Cassing, Y. B. Ivanov, V. D. Toneev, PRC90('14)014903

Collapse of directed flow

- **Negative dv_1/dy at high-energy ($\sqrt{s_{NN}} > 20$ GeV)**
  - **Geometric origin (bowling pin mechanism), not related to FOPT**
    R.Snellings, H.Sorge, S.Voloshin, F.Wang, N. Xu, PRL84,2803(2000)
- **Negative dv_1/dy at $\sqrt{s_{NN}} \sim 10$ GeV**
  - **Yes, in three-fluid simulations.**
    Y. B. Ivanov and A. A. Soldatov, PRC91 (2015)024915
  - **No, in transport models incl. hybrid.**
    Exception: *B.A.Li, C.M.Ko ('98) with FOPT EOS*

***We investigate the directed flow at BES energies
in hadronic transport model
with / without mean field effects
with / without softening effects via attractive orbit.***

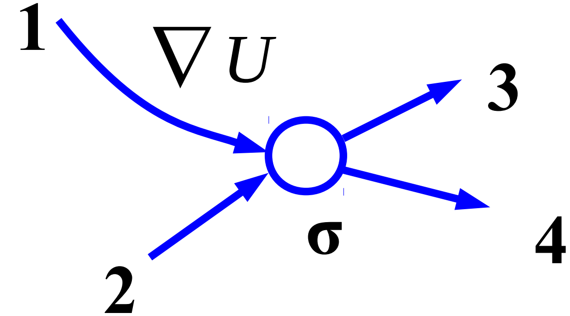
Hadronic Transport Approach
Cascade / Cascade + Mean Field

Microscopic Transport Models

- UrQMD 3.4 Frankfurt public
  resonance model N*,D*, string pQCD, PYTHIA6.4
- PHSD Giessen (Cassing) upon request
  D(1232),N(1440),N(1530), string, pQCD, FRITIOF7.02
- GiBUU 1.6 Giessen (Mosel) public
  resonance model N*,D*, string, pQCD,PYTHIA6.4
- AMPT public
  HIJING+ZPC+ART
- JAM (Y. Nara) public
  resonance model N*,D*, string, pQCD, PYTHIA6.1

Transport Model

- **Boltzmann equation with (optional) potential effects**
  E.g. Bertsch, Das Gupta, Phys. Rept. 160(88), 190



$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla U \cdot \nabla_p f = I_{\rm coll}$$

$$I_{\rm coll}(\mathbf{r}, \mathbf{p}) = -\frac{1}{2} \int \frac{d\mathbf{p}_2}{(2\pi)^3} d\Omega \; v_{12} \, \frac{d\sigma}{d\Omega} \; [f f_2 (1 - f_3)(1 - f_4)) - (12 \leftrightarrow 34)]$$

(NN elastic scattering case)

- **Hadron-string transport model JAM**
  - **Collision term → Hadronic cascade with resonance and string excitation**
    Nara, Otuka, AO, Niita, Chiba, Phys. Rev. C61 (2000), 024901.
  - **Potential term → Mean field effects in the framework of RQMD/S**
    Sorge, Stocker, Greiner, Ann. of Phys. 192 (1989), 266.
    Tomoyuki Maruyama et al., Prog. Theor. Phys. 96(1996), 263.
    Isse, AO, Otuka, Sahu, Nara, Phys.Rev. C 72 (2005), 064908.

Relativistic QMD/Simplified (RQMD/S)

- **RQMD is developed based on constraint Hamiltonian dynamics**
  H. Sorge, H. Stoecker, W. Greiner, Ann. Phys. 192 (1989), 266.
  - **8N dof → 2N constraints → 6N (phase space)**
  - **Constraints = on-mass-shell constraints + time fixation**
- **RQMD/S uses simplified time-fixation**
  Tomoyuki Maruyama, et al. Prog. Theor. Phys. 96(1996),263.
  - **Single particle energy (on-mass-shell constraint)**

$$p_i^0 = \sqrt{\boldsymbol{p}_i^2 + m_i^2 + 2 m_i V_i}$$

  - **EOM after solving constraints**

$$\dot{\boldsymbol{r}}_i = \frac{\boldsymbol{p}_i}{p_i^0} + \sum_j \frac{m_j}{p_j^0} \frac{\partial V_j}{\partial \boldsymbol{p}_i} \qquad \dot{\boldsymbol{p}}_i = -\sum_j \frac{m_j}{p_j^0} \frac{\partial V_j}{\partial \boldsymbol{r}_i}$$

  - **Relative distances $(r_i - r_j)^2$ are replaced with those in the two-body c.m.**
    → Potential becomes Lorentz scalar

Mean Field Potential

- **Skyrme type density dependent + momentum dependent potential**

$$V = \sum_i V_i = \int d^3r \left[\frac{\alpha}{2} \left(\frac{\rho}{\rho_0} \right)^2 + \frac{\beta}{\gamma+1} \left(\frac{\rho}{\rho_0} \right)^{\gamma+1} \right] + \sum_k \int d^3\boldsymbol{r} d^3\boldsymbol{p} d^3\boldsymbol{p}' \frac{C_{ex}^{(k)}}{2\rho_0} \frac{f(\boldsymbol{r},\boldsymbol{p}) f(\boldsymbol{r},\boldsymbol{p}')}{1 + (\boldsymbol{p}-\boldsymbol{p}')^2/\mu_k^2}$$

Type	α (MeV)	β (MeV)	γ	$C_{\rm ex}^{(1)}$ (MeV)	$C_{\rm ex}^{(2)}$ (MeV)	μ_1 (fm^{-1})	μ_2 (fm^{-1})	K (MeV)
MH1	-12.25	87.40	5/3	-383.14	337.41	2.02	1.0	371.92
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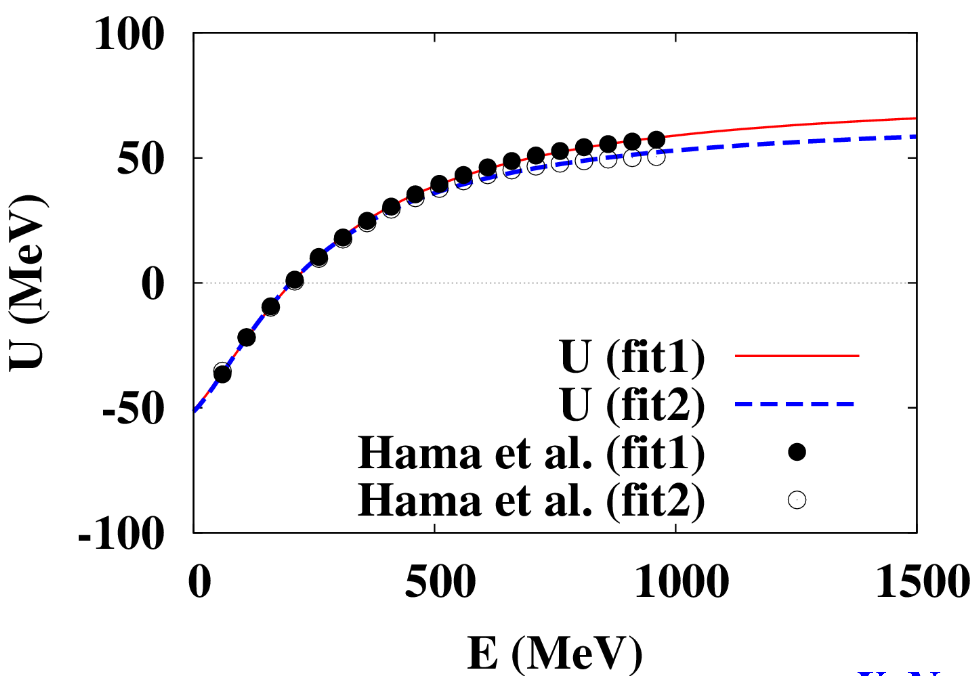
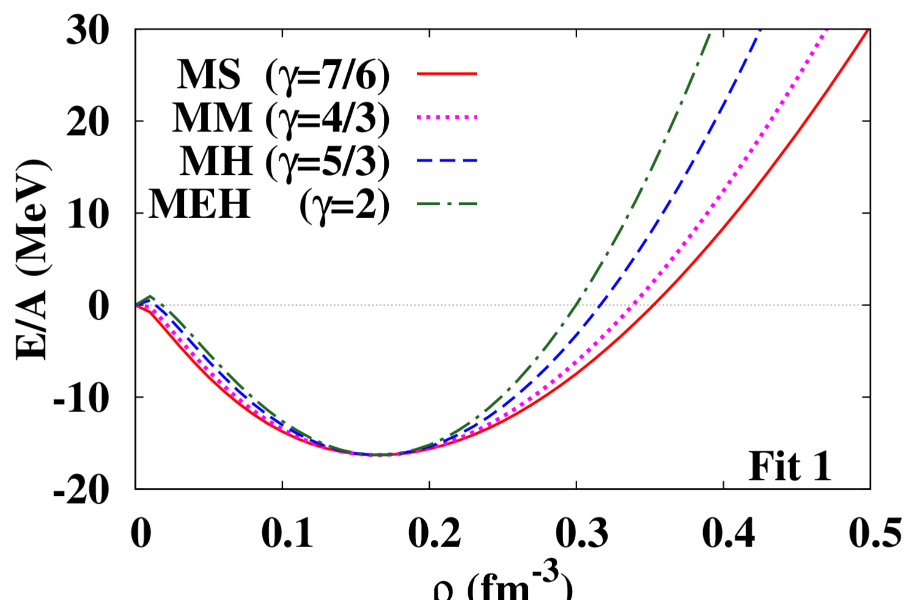
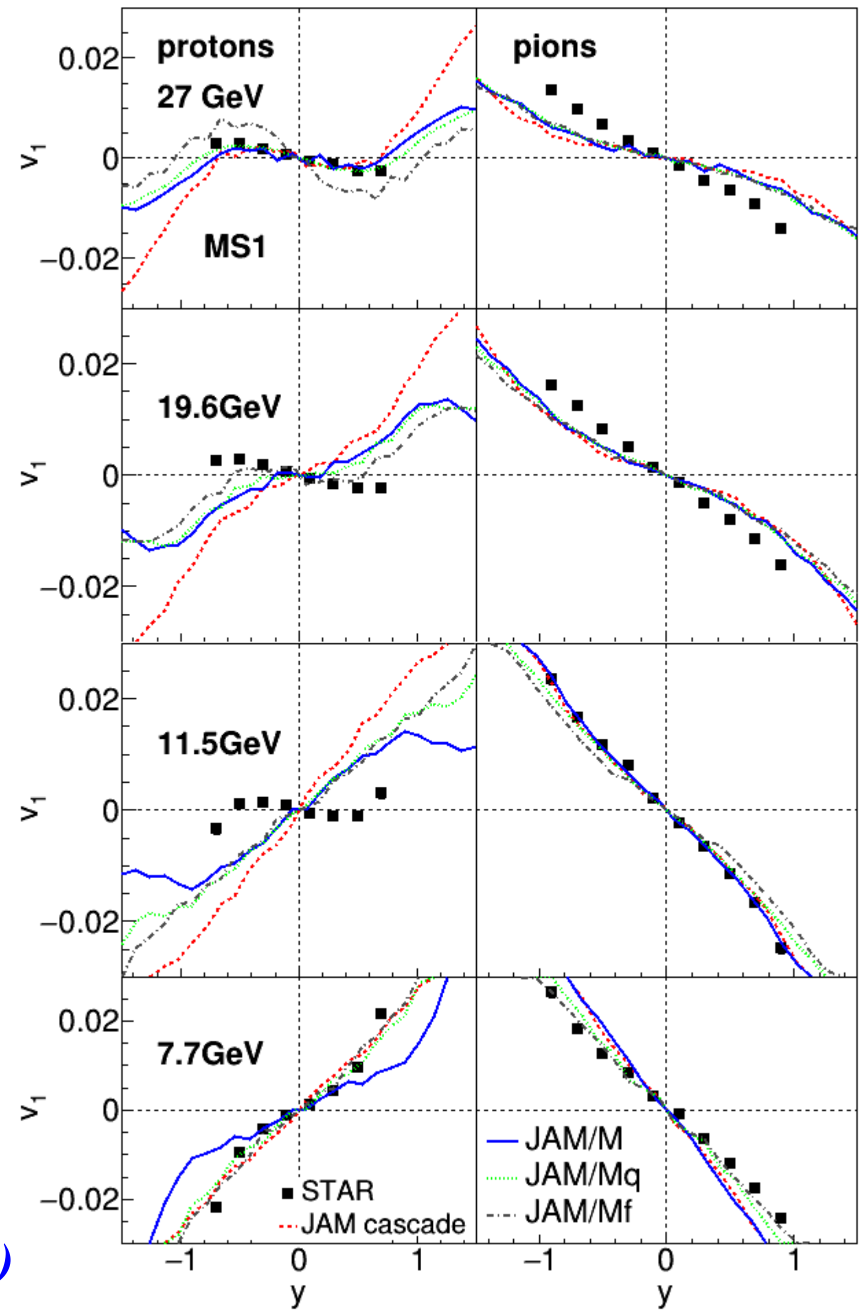




Y. Nara, AO, arXiv:1512.06299 [nucl-th] (QM2015 proc.)
Isse, AO, Otuka, Sahu, Nara, PRC 72 (2005), 064908.

Comparison with RHIC data on v_1

- **Pot. Eff. on the v1 is significant**
- **Hadronic approach does not reproduce the beam energy dependence of the directed flow.**
  → Something happens around 10-20GeV?



JAM/M: only formed baryons feel potential forces
JAM/Mq: pre-formed hadron feel potential with factor 2/3 for diquark, and 1/3 for quark
JAM/Mf: both formed and pre-formed hadrons feel potential forces.

Y. Nara, AO, arXiv:1512.06299 [nucl-th] (QM2015 proc.)

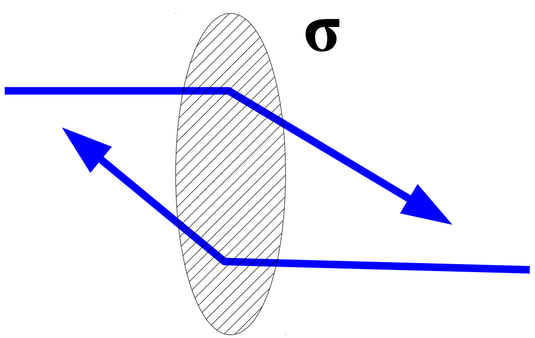
Hadronic Transport Approach with Softening Effects

Softening Effects via Attractive Orbit Scattering

- **Attractive orbit scattering simulates softening of EOS**
  P. Danielewicz, S. Pratt, PRC 53, 249 (1996)
  H. Sorge, PRL 82, 2048 (1999).

$$P = P_f + \frac{1}{3TV}\sum_{(i,j)} (\boldsymbol{q}_i \cdot \boldsymbol{r}_i + \boldsymbol{q}_j \cdot \boldsymbol{r}_j)$$

(Virial theorem)



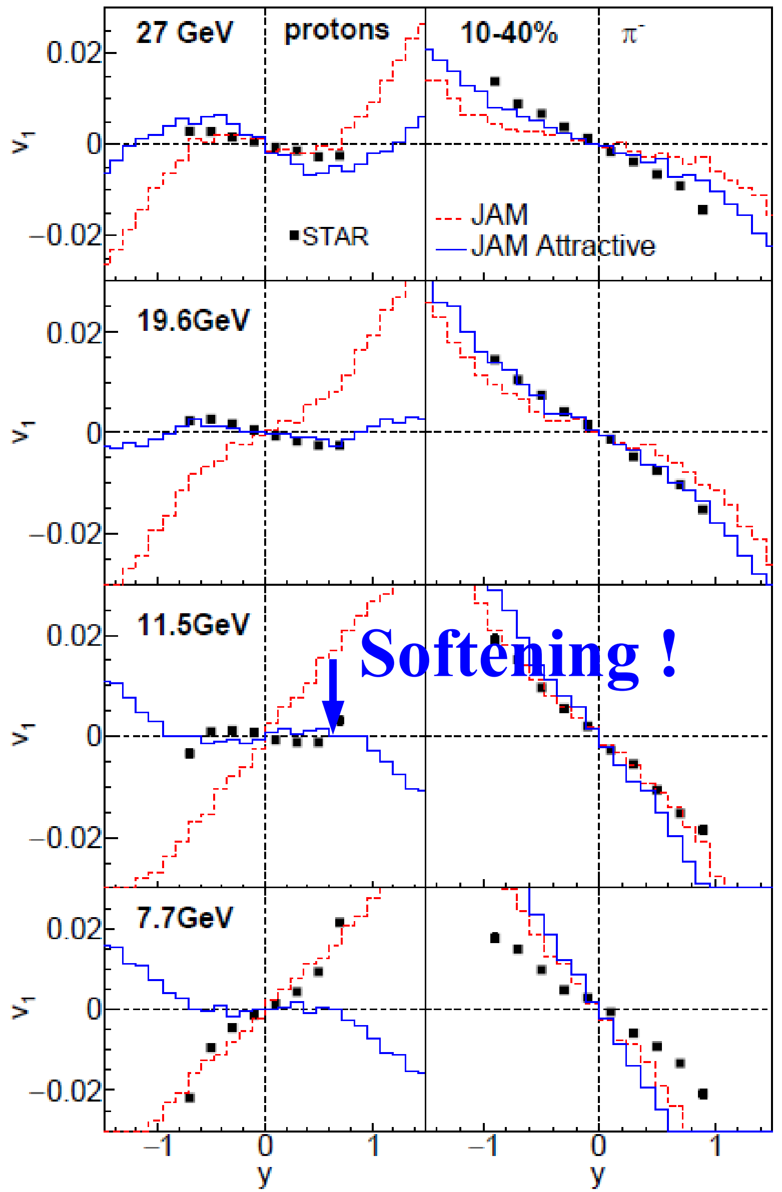
  - **Attractive orbit → particle trajectory are bended in denser region**

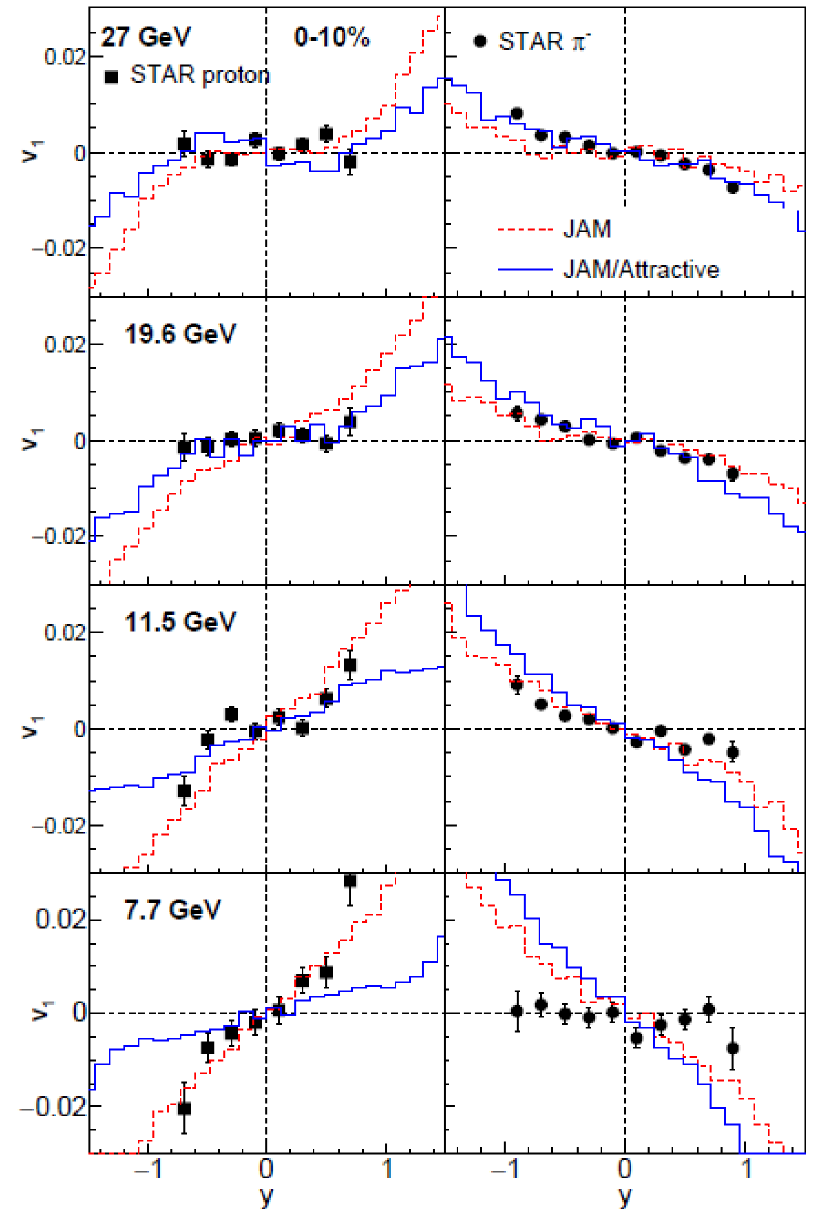
Let us examine the EOS softening effects,
which cannot be explained in hadronic mean field potential,
by using attractive orbit scatterings !

Y. Nara, AO, H. Stöcker, arXiv:1601.07692 [hep-ph]

Directed Flow with Attractive Orbits

Nara, AO, Stöcker ('16)

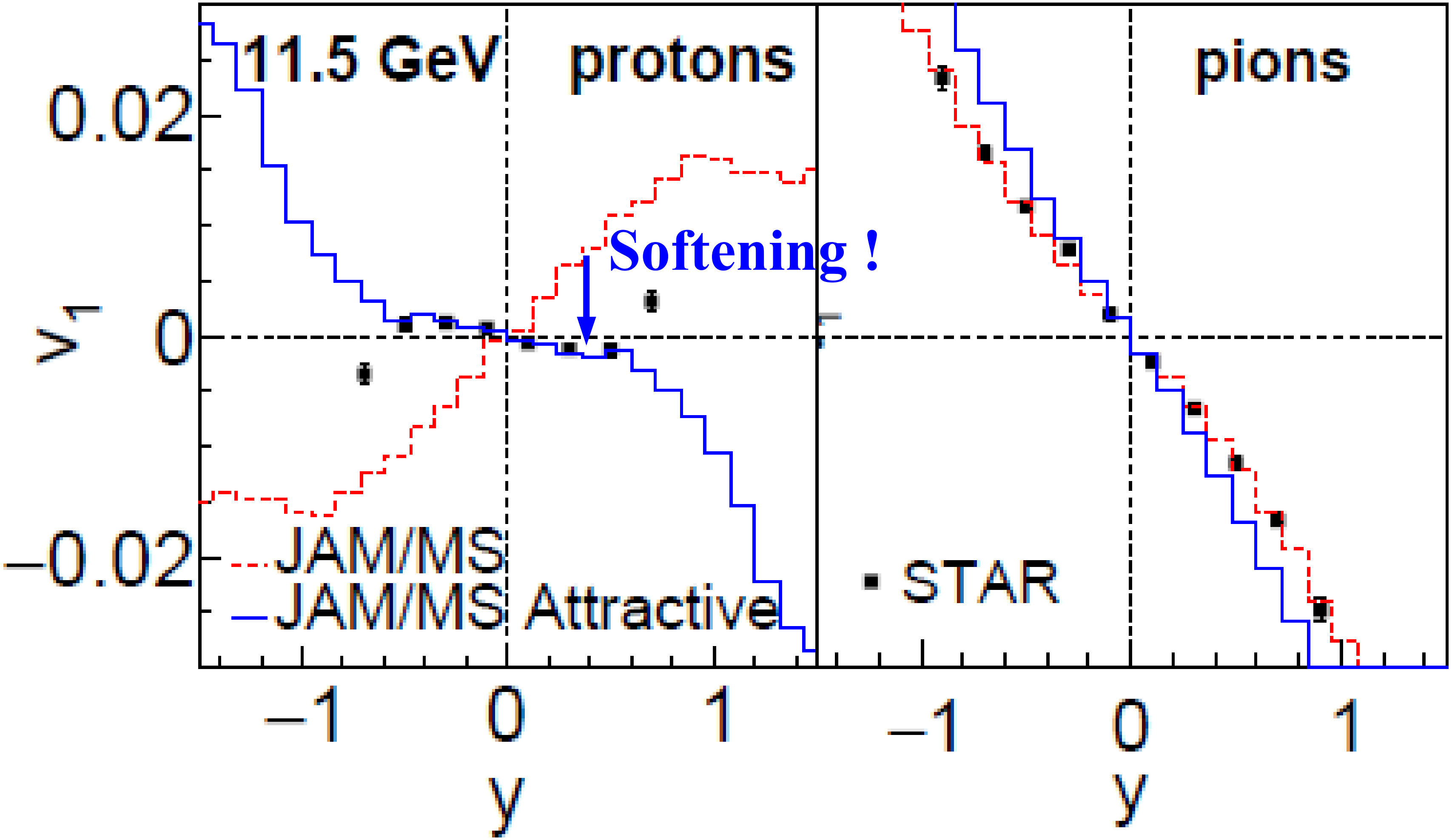




mid-central (10-40 %) **central (0-10 %)**

Mean Field + Attractive Orbit

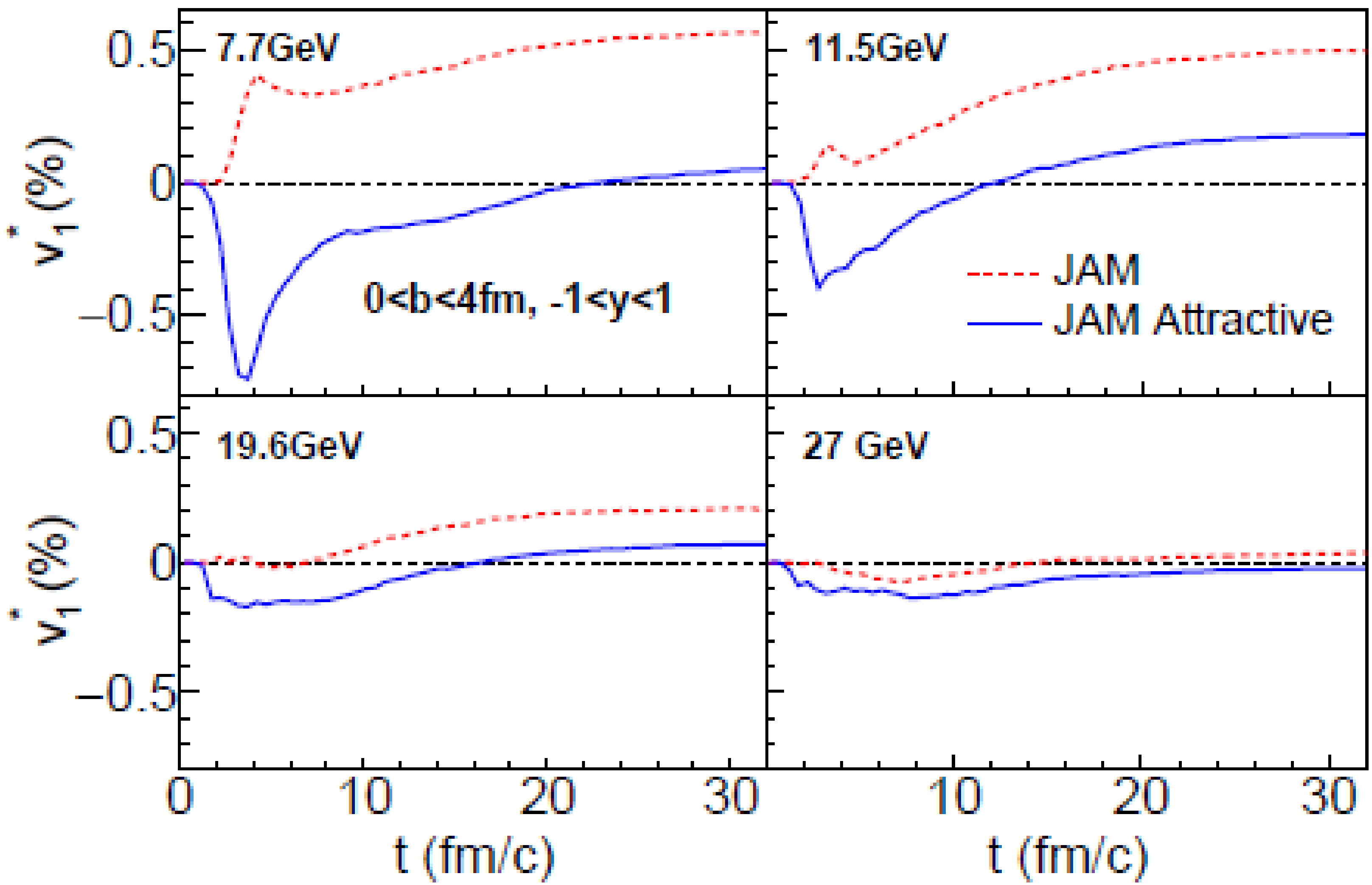
Nara, AO, Stöcker ('16)



MF+Attractive Orbit make dv_1/dy negative at $\sqrt{s_{NN}}$ ~ 10 GeV

When is negative v_1 slope generated ?

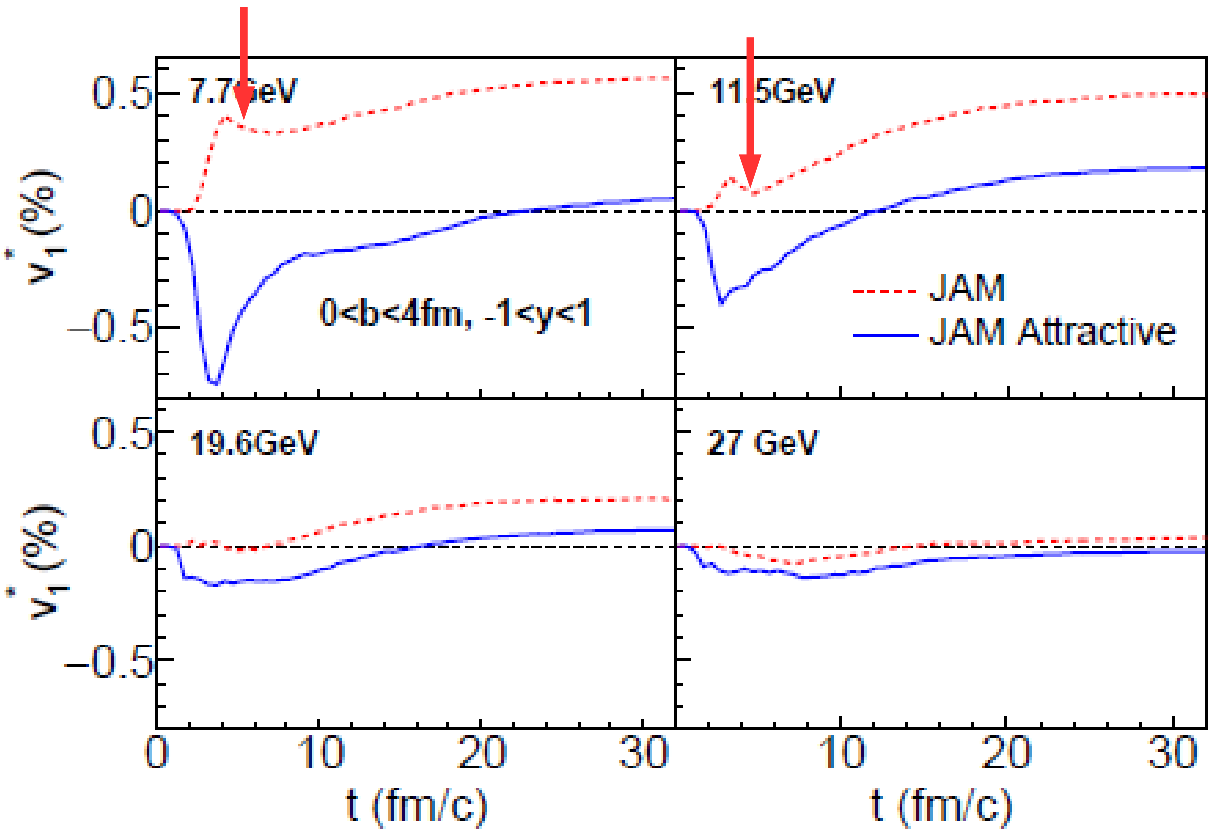
Nara, AO, Stöcker ('16)

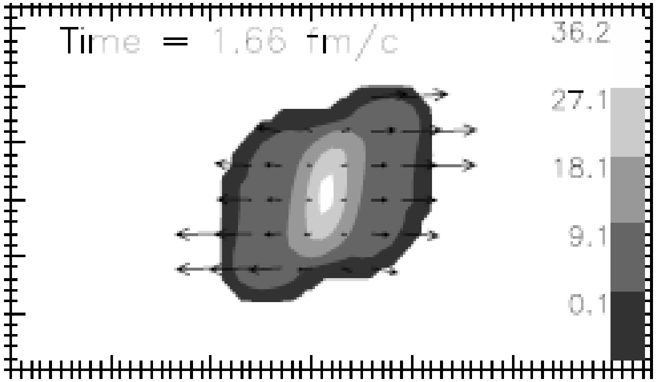


We need to make v1 slope negative in the compressing stage.

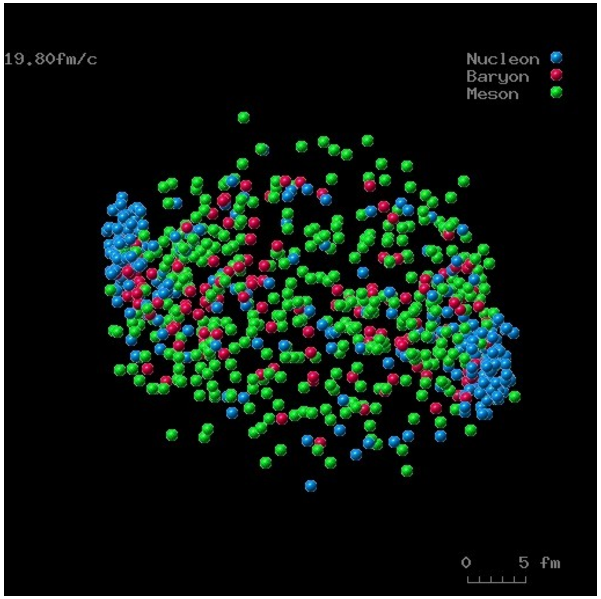
Tilted Ellipsoid ?

Nara, AO, Stöcker ('16)





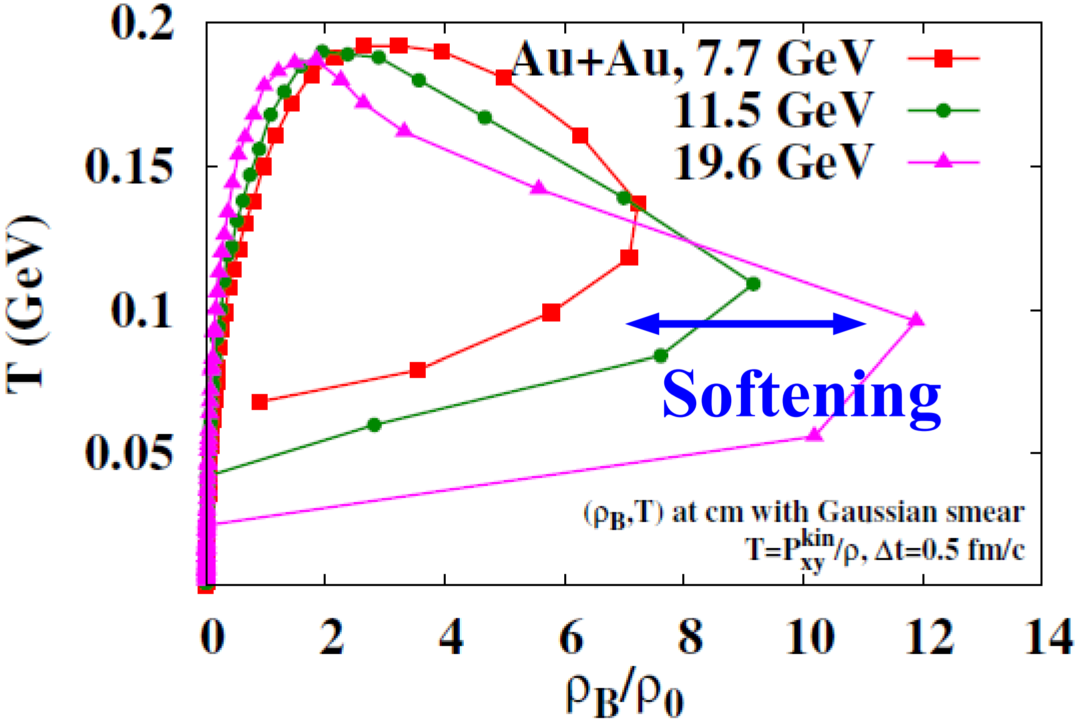
18 GeV, 3-fluid
Toneev et al. ('03)

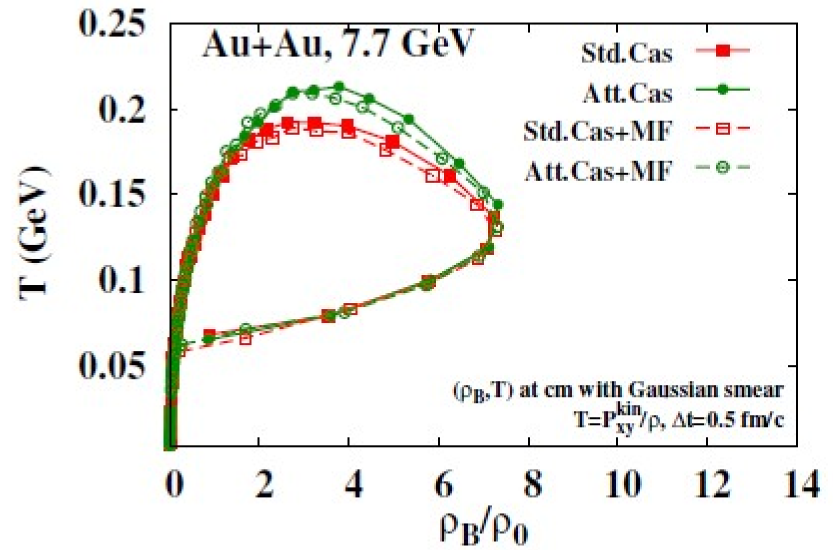


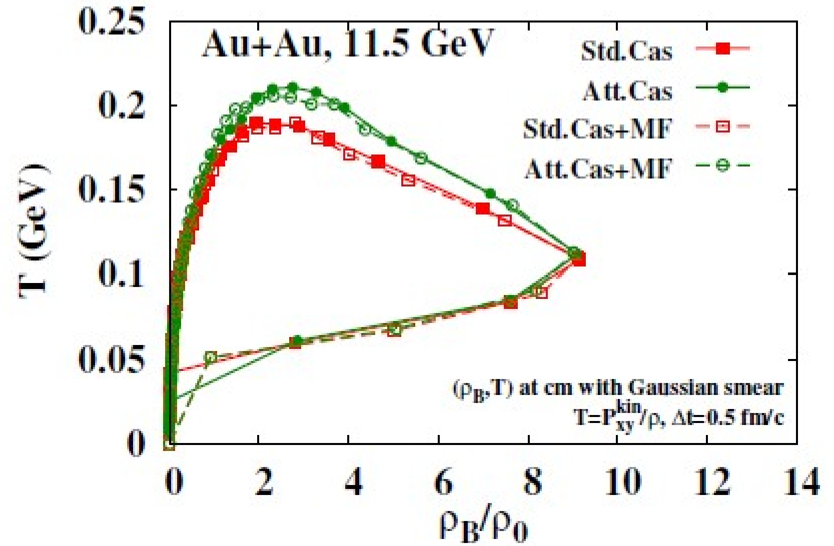
Transport model results also show tilted-ellipsoid-like behavior, but it is not enough.

Softening of EOS: Where and How much ?

- "Softening" should take place at $\sqrt{s_{NN}}$=11.5 GeV → $\rho/\rho_B \sim$ (6-10)
- Attractive orbit
  → Larger interactions
  & Higher T at later times





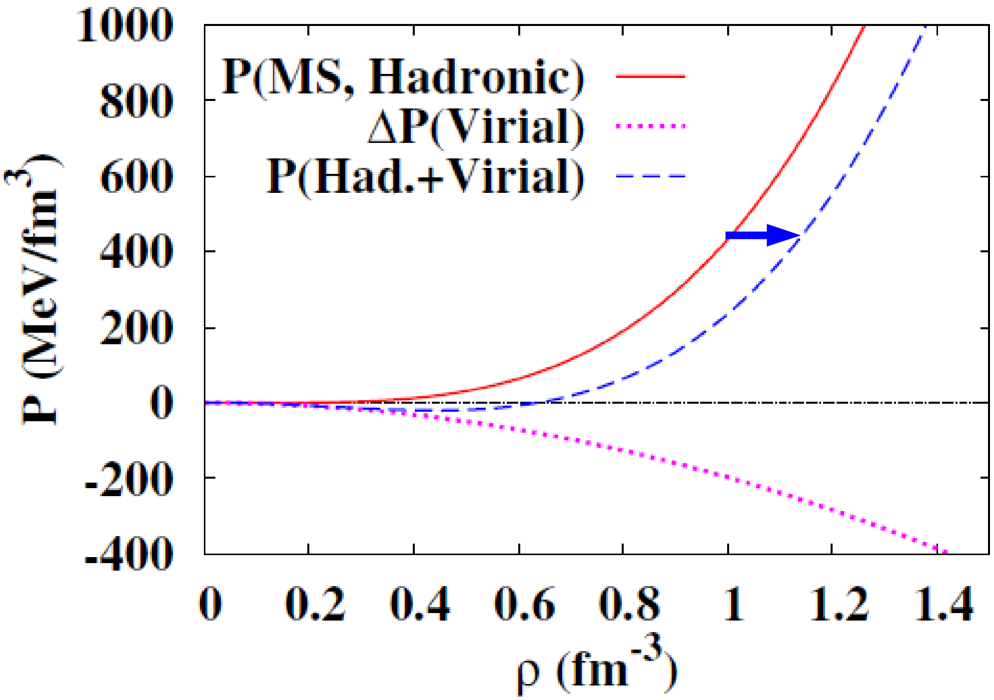


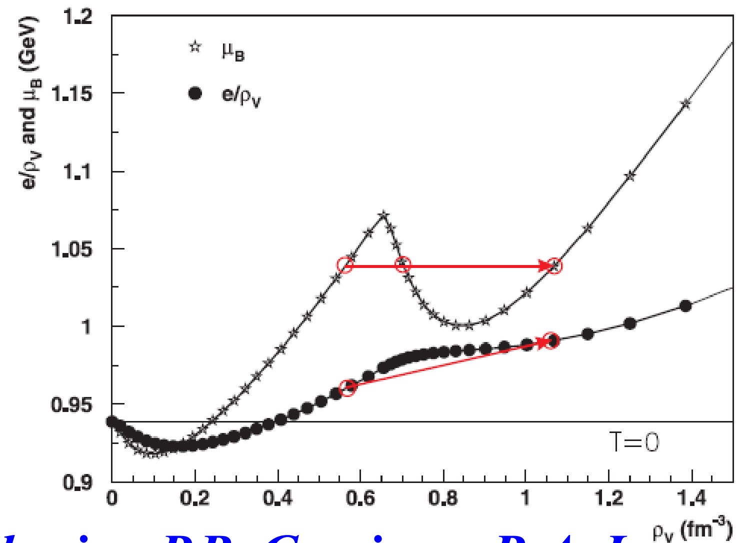
How much softening do we need ?

- **Virial theorem**

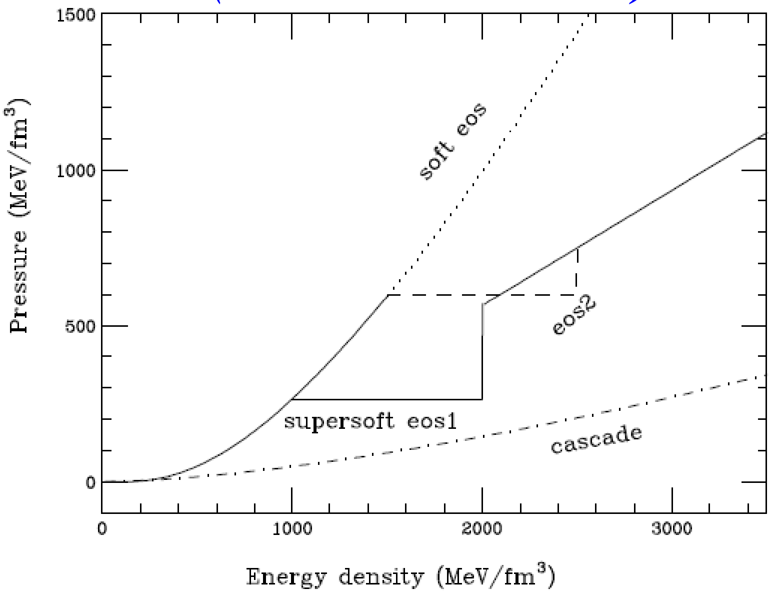
$$\Delta P = \frac{1}{3}\langle v\rho^2\sigma \mathbf{q}\cdot\mathbf{\Delta r}\rangle$$

Simple esitmate: σ= 30 mb, <q ΔR> ~ 1





P. Danielewicz, P.B. Gossiaux, R.A. Lacey, nucl-th/9808013 (Les Houches 1998)



B. A. Li, C. M. Ko, PRC58 ('98) 1382

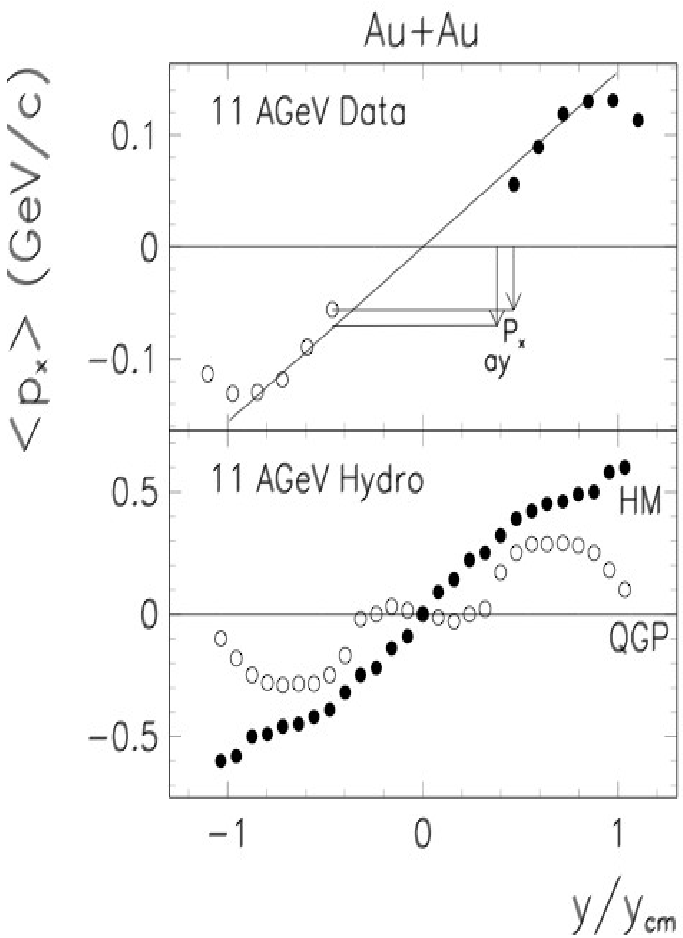
Summary

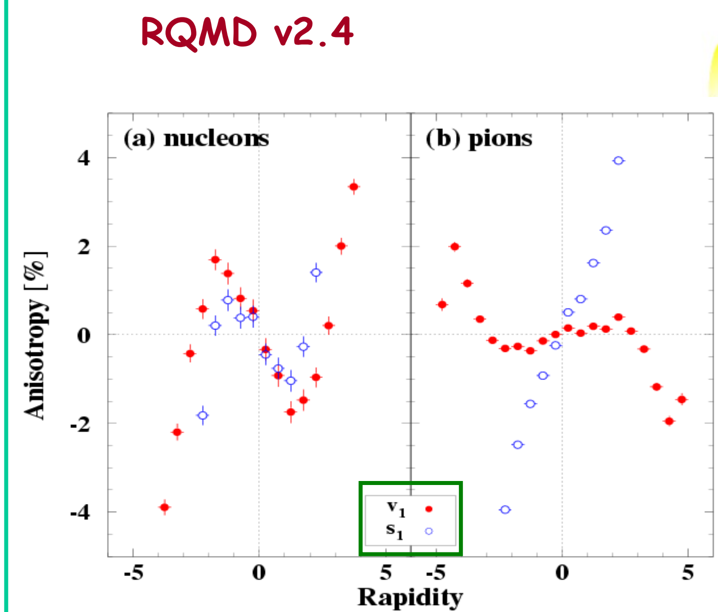
- **We may see QCD phase transition (1st or 2nd) signals at BES (or J-PARC) energies in baryon number cumulants and v_1 slope.**
- **Hadronic transport models cannot explain negative v_1 slope below $\sqrt{s_{NN}}$ = 20 GeV.**
  - **Geometric (bowling pin) mechanism becomes manifest at higher energies (JAM, JAM-MF, HSD, PHSD, UrQMD, ….).**
- **Hadronic transport with EOS softening can describe negative v_1 slope below $\sqrt{s_{NN}}$ = 20 GeV.**
  - **Attractive orbit scattering simulates EOS softening (virial theorem).**
  - **We need more studies to confirm its nature.**
    First-order phase transition ? Crossover ? Forward-backward rapidities ? MF leading to softer EOS ?
- ***We need "re-hardening" at higher energies, e.g. $\sqrt{s_{NN}}$ = 27 GeV.***

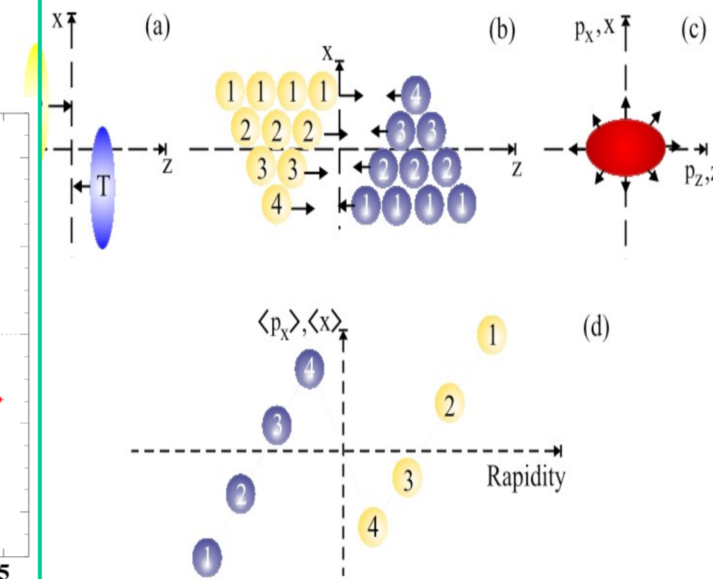
Thank you !

Wiggle: QGP signal in the directed flow?

Baryon stopping + Positive space-momentum correlation leads wiggle (no QGP)







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L. P. Csernai, D. Röhrich, PLB 45 (1999), 454.

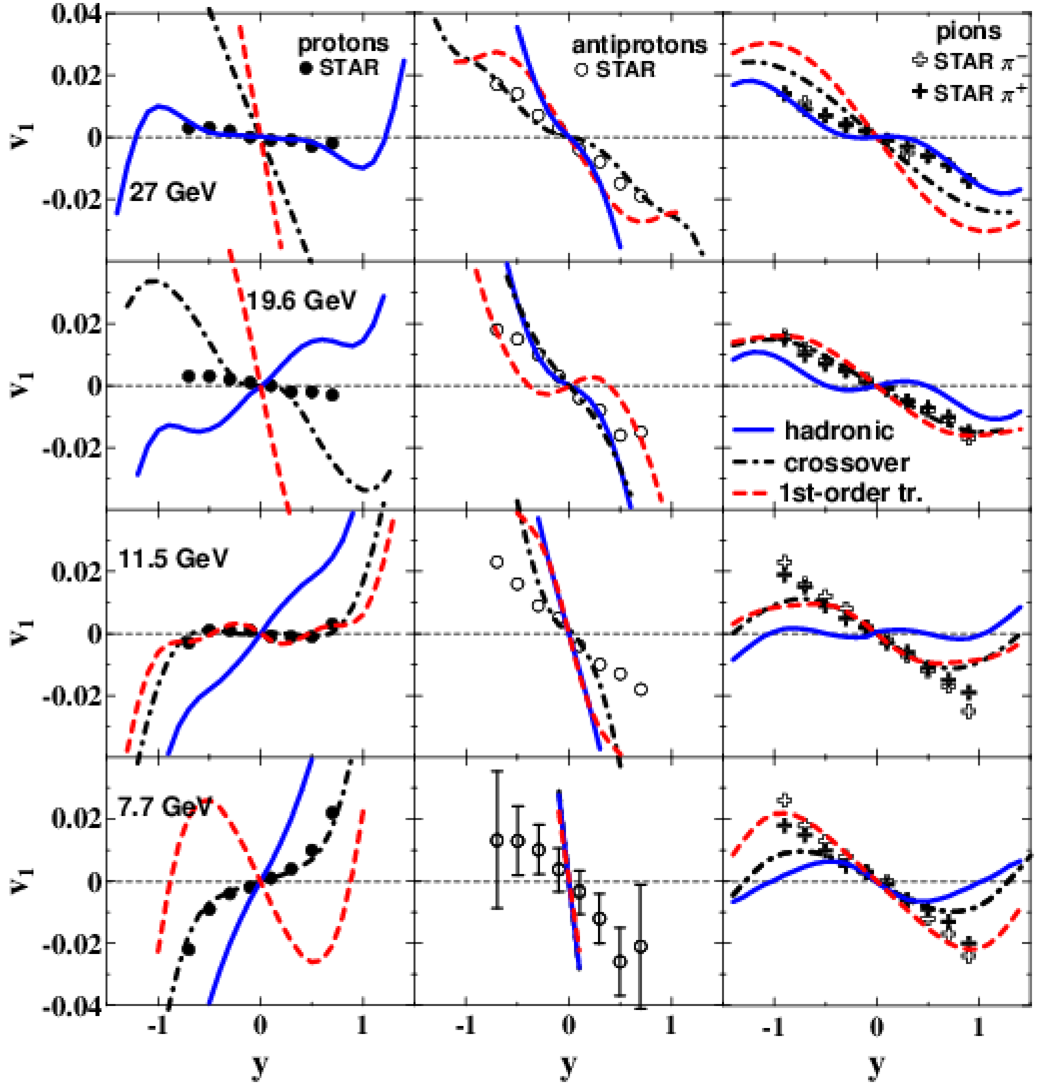
QGP EoS predicts wiggle in hydro

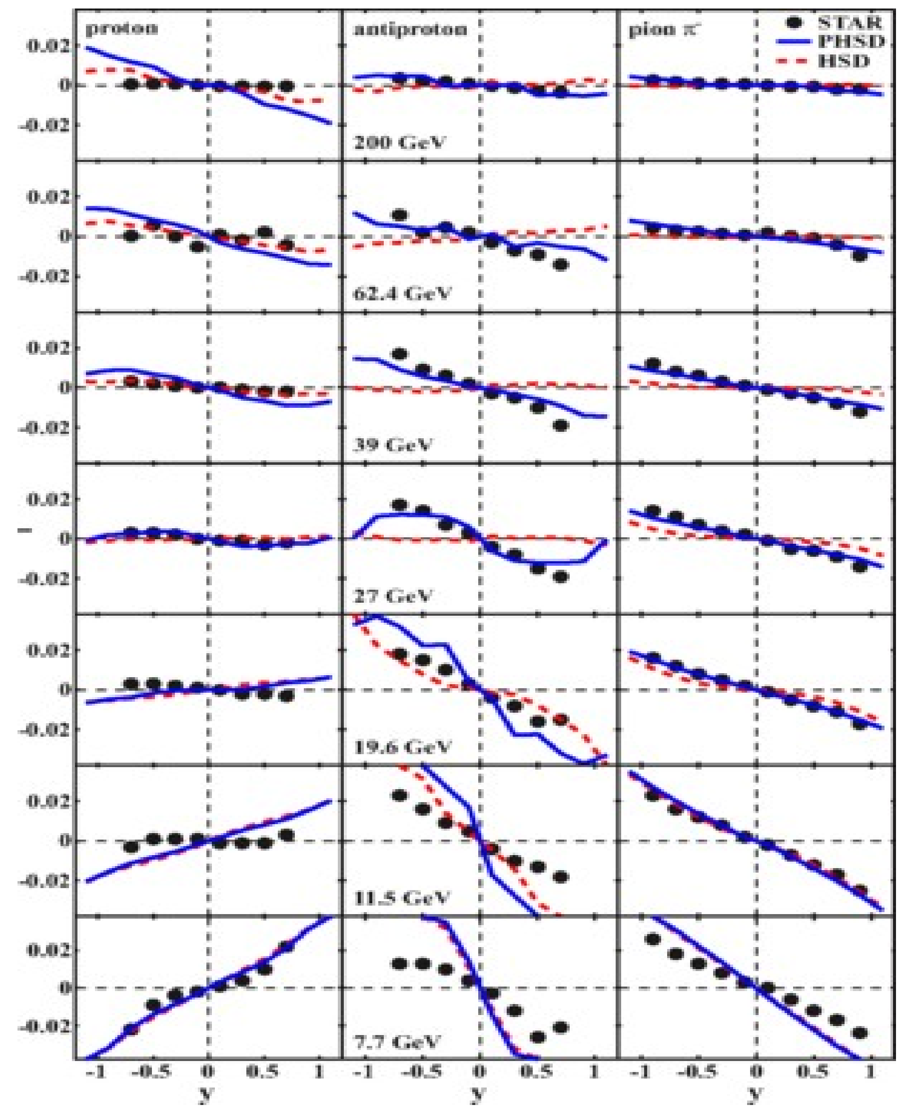
V1 from hydrodynamics

PHSD/HSD predictions

Y. B. Ivanov and A. A. Soldatov, Phys. Rev. C91, no. 2, 024915 (2015)

V. P. Konchakovski, W. Cassing, Y. B. Ivanov and V. D. Toneev, Phys. Rev. C90, no. 1, 014903 (2014)





Microscopic transport models (event generator for nuclear collisions)

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  resonance model N*,D*, string pQCD, PYTHIA6.4

- **PHSD Giessen (Cassing) upon request**
  D(1232),N(1440),N(1530), string, pQCD, FRITIOF7.02

- **GiBUU 1.6 Giessen (Mosel) public**
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- **AMPT public**
  HIJING+ZPC+ART

- **JAM Japan (Y. Nara) public**
  resonance model N*,D*, string, pQCD, PYTHIA6.1

Mean field potential

Skyrme type density dependent + Lorentzian momentum dependent potential

$$V = \sum_i V_i = \int d^3r \left[\frac{\alpha}{2}\left(\frac{\rho}{\rho_0}\right)^2 + \frac{\beta}{\gamma+1}\left(\frac{\rho}{\rho_0}\right)^{\gamma+1}\right] + \sum_k \int d^3\boldsymbol{r} d^3\boldsymbol{p} d^3\boldsymbol{p}' \frac{C_{ex}^{(k)}}{2\rho_0} \frac{f(\boldsymbol{r},\boldsymbol{p}) f(\boldsymbol{r},\boldsymbol{p}')}{1+(\boldsymbol{p}-\boldsymbol{p}')^2/\mu_k^2}$$

Type	α (MeV)	β (MeV)	γ	$C_{\rm ex}^{(1)}$ (MeV)	$C_{\rm ex}^{(2)}$ (MeV)	μ_1 ($\rm fm^{-1}$)	μ_2 ($\rm fm^{-1}$)	K (MeV)
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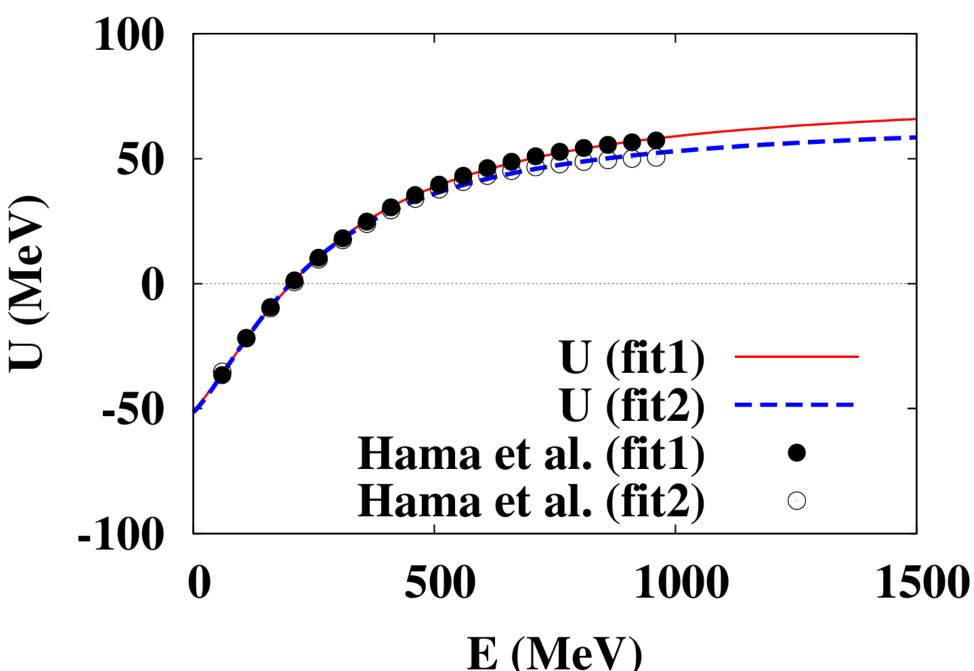

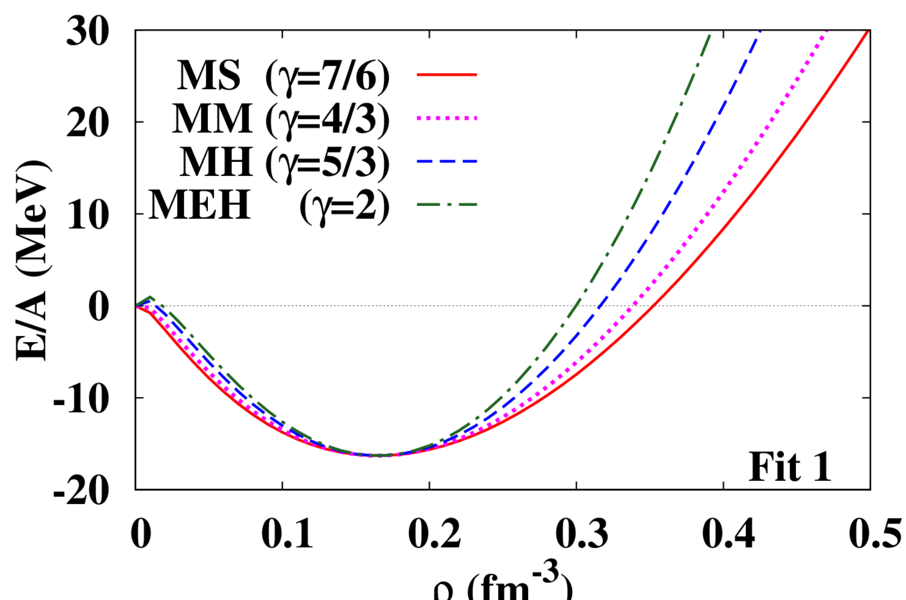

Y. Nara, AO, arXiv:1512.06299 [nucl-th] (QM2015 proc.)
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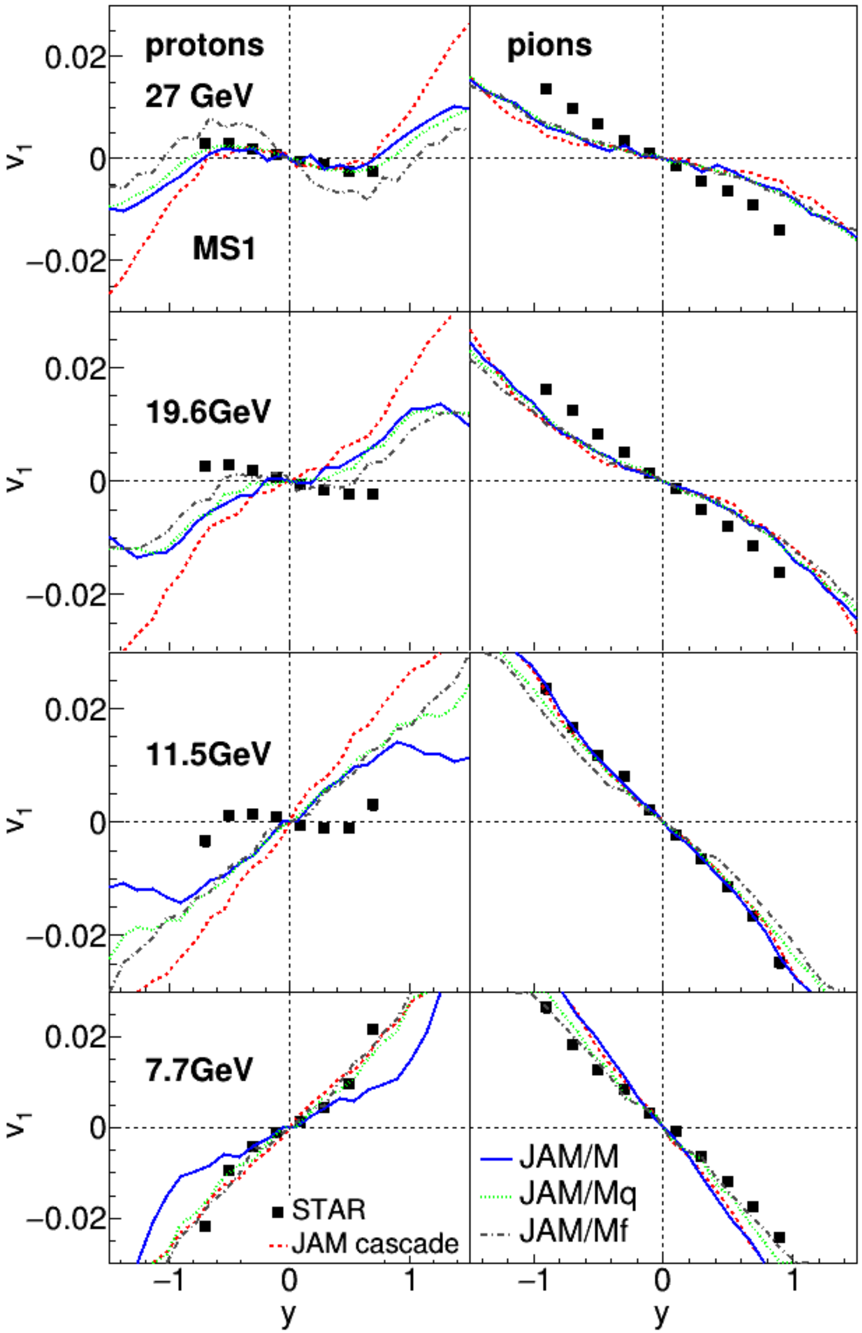
Comparison of v1



Effects of potential on the v1 is significant

Hadronic approach does not reproduce the correct beam energy dependence of the directed flow.

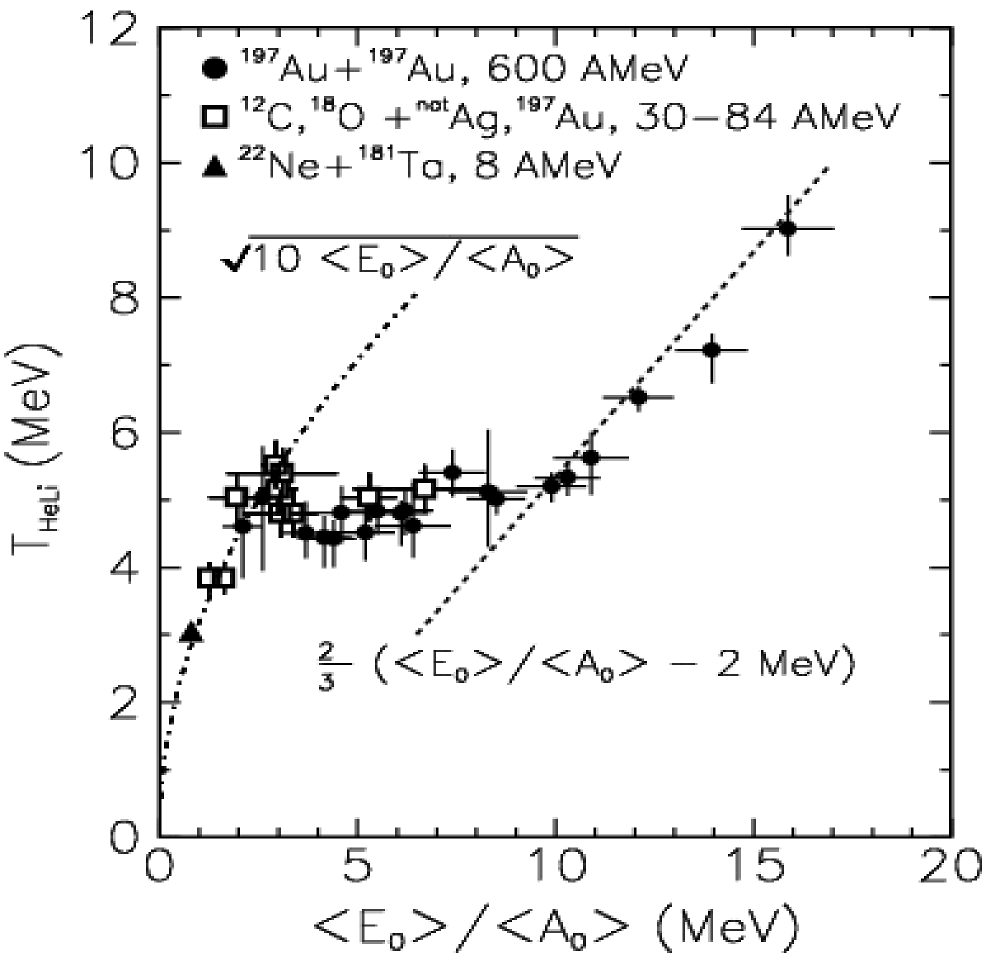
Something happens around 10-20GeV?

JAM/M: only formed baryons feel potential forces
JAM/Mq: pre-formed hadron feel potential with factor 2/3 for diquark, and 1/3 for quark
JAM/Mf: both formed and pre-formed hadrons feel potential forces.

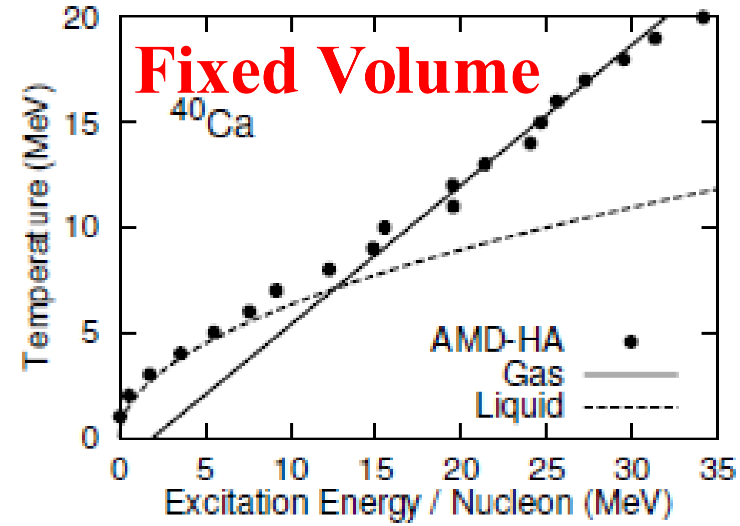
Y. Nara, AO, arXiv:1512.06299 [nucl-th] (QM2015 proc.)

Nuclear Liquid-Gas Phase Transition

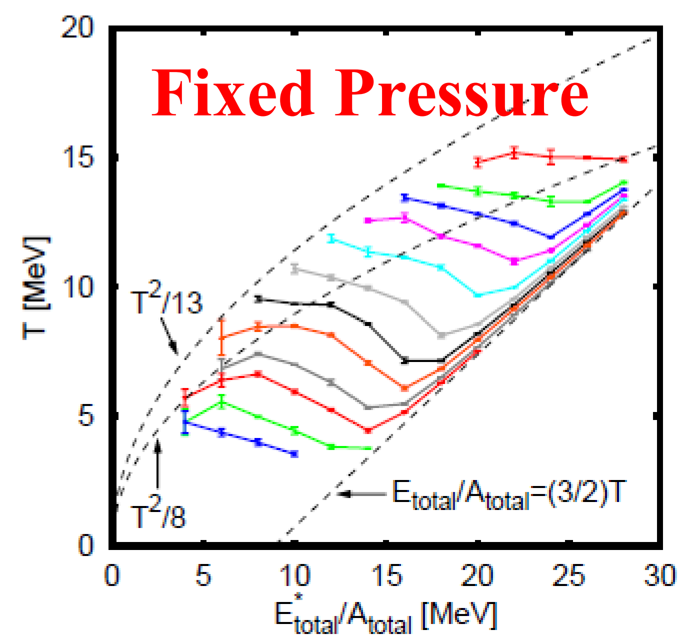
- **Caloric curve → LG phase transition (Smoking gun)**



J. Pochadzalla et al. (GSI-ALLADIN collab.), PRL 75 (1995) 1040.



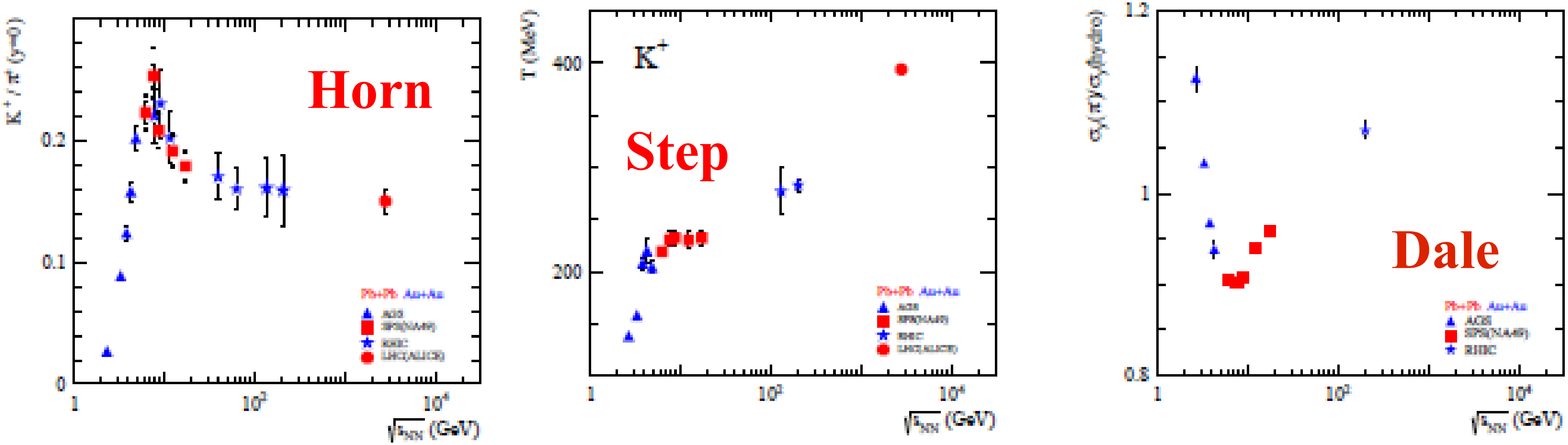
AO, Randrup ('98)



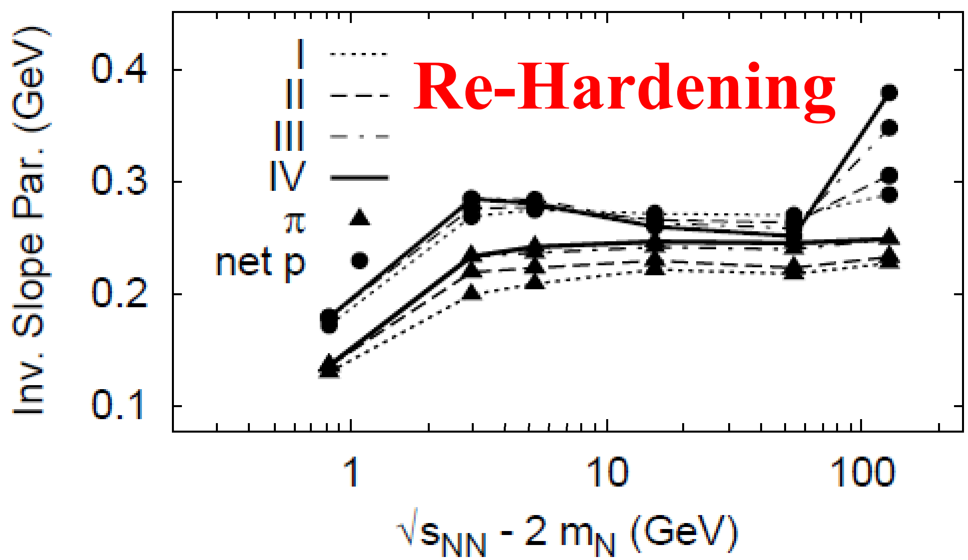
T. Furuta, A.Ono ('09)

Horn, Step and Dale

- **Non-monotonic behavior in K^+/π^+ ratio (Horn), m slope par. (Step or re-hardening), rapidity dist. width of π**



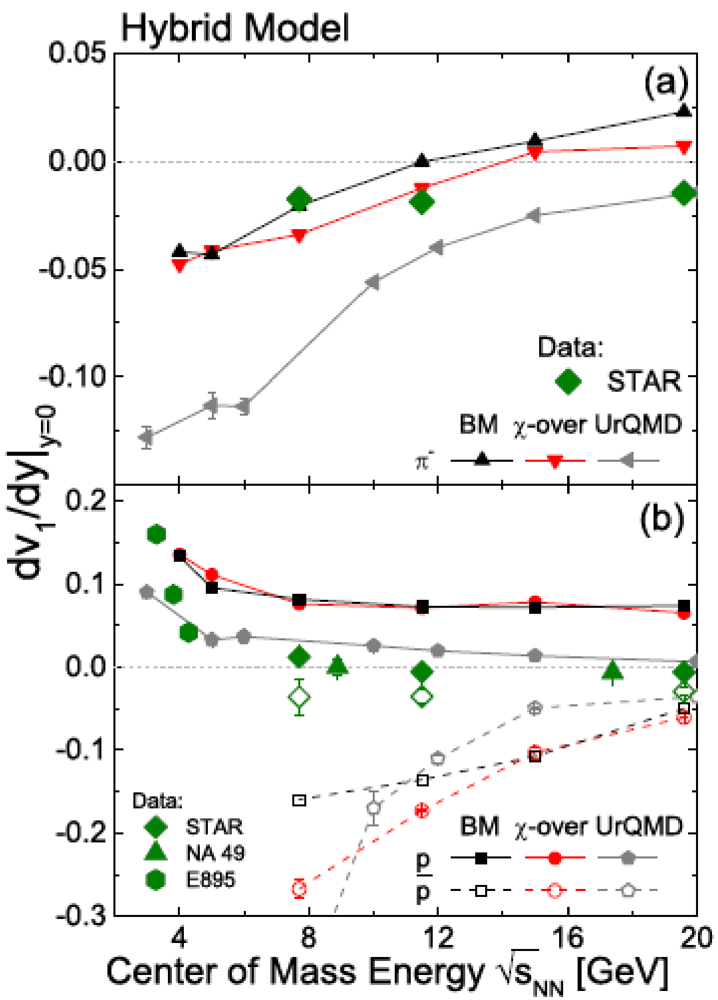
E.g. A. Rustamov (2012)

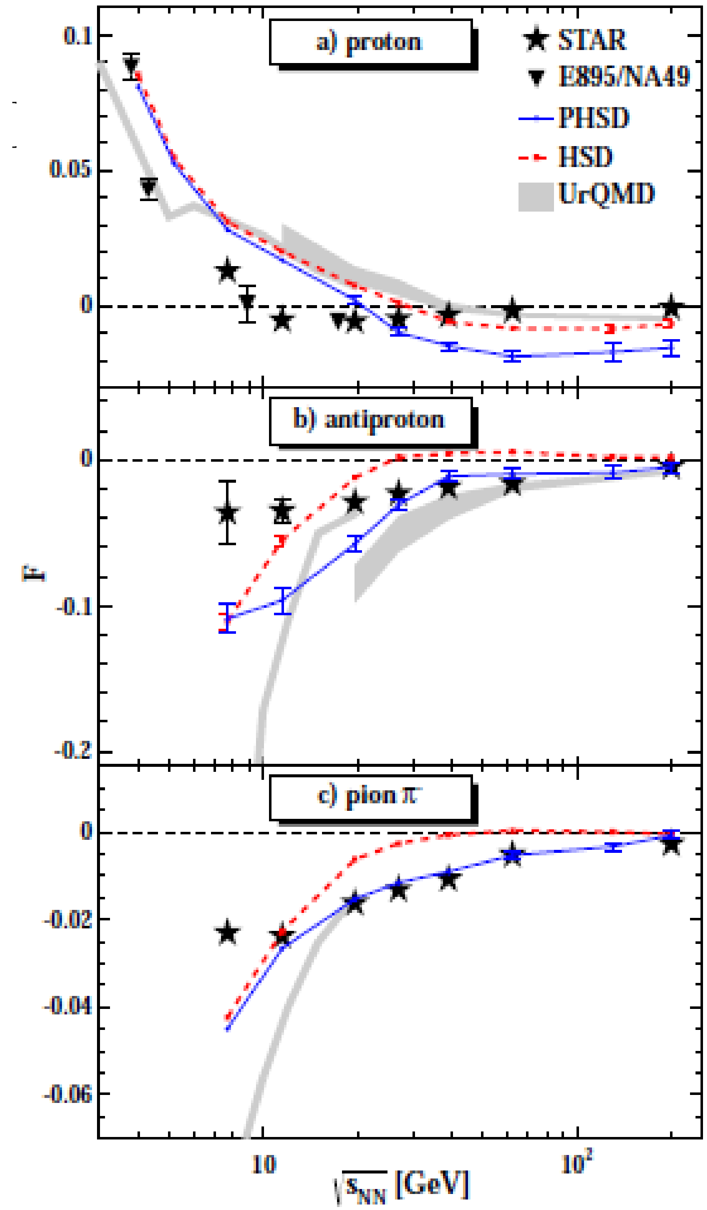


N. Otuka, P.K.Sahu, M. Isse, Y. Nara, AO, nucl-th/010205

Hybrid Approaches

- **Both Hybrid model (Frankfurt) and PHSD (Giessen) show higher balance**





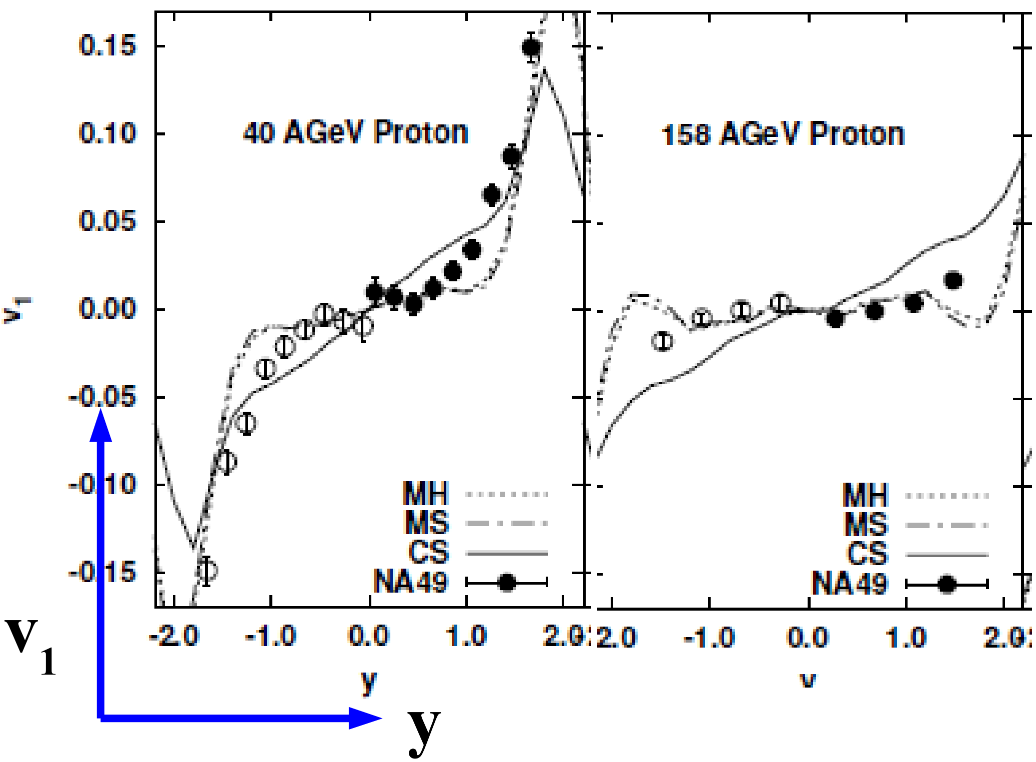
J. Steinheimer, J. Auvinen, H. Petersen, M. Bleicher, H. Stöcker, PRC89 ('14) 054913

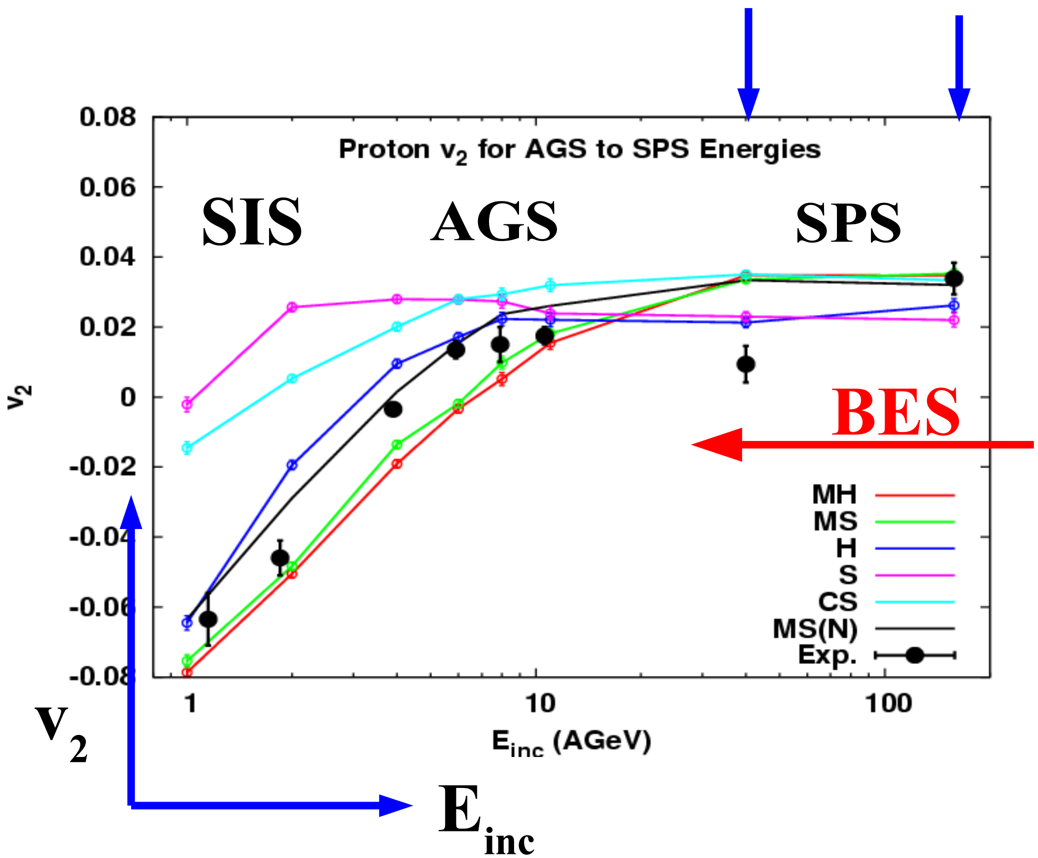
V. P. Konchakovski, W. Cassing, Yu. B. Ivanov, V. D. Toneev, PRC90('14)014903

JAM results at AGS and SPS Energies

- **JAM w/ Mean-Field effects roughly explains v_1 and v_2 at AGS & SPS (1-158 A GeV → $\sqrt{s_{NN}}$ = 2.5-20 GeV)**

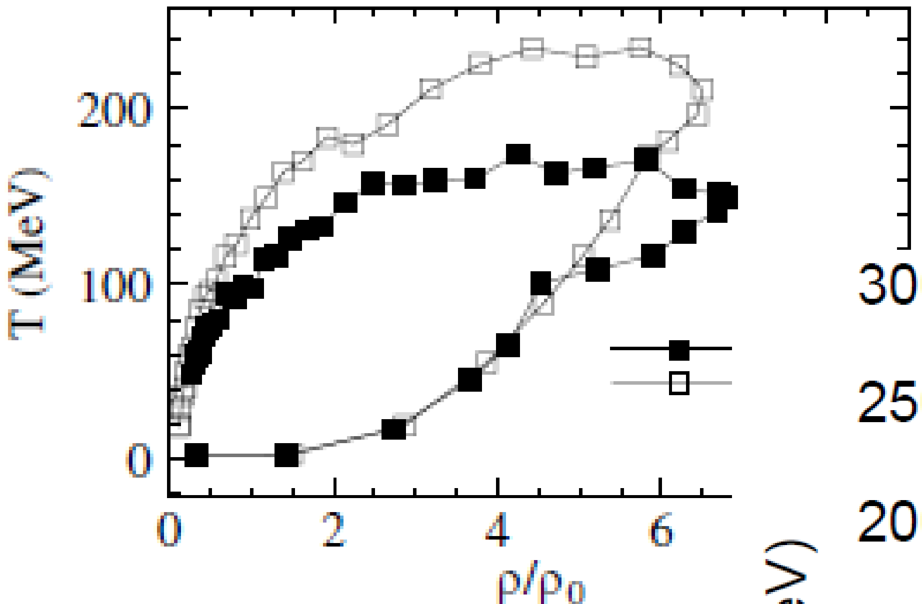
$\sqrt{s_{NN}}$=8.9 GeV $\sqrt{s_{NN}}$=17.3 GeV



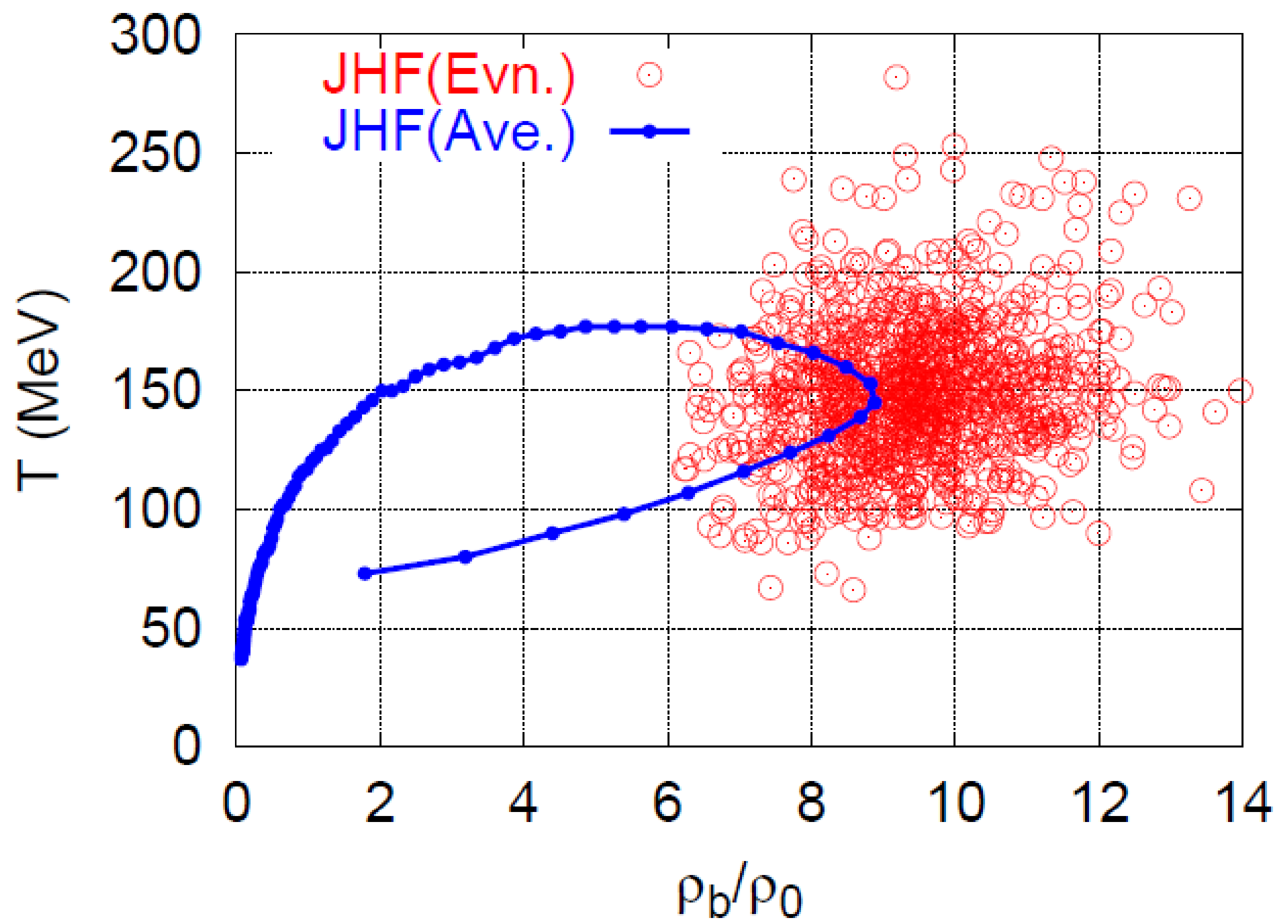


M. Isse, AO, N. Otuka, P. K. Sahu, Y. Nara, PRC72('05)064908

Highest Density Matter at J-PARC ?



Nara, Otuka, AO, Maruyama ('97)

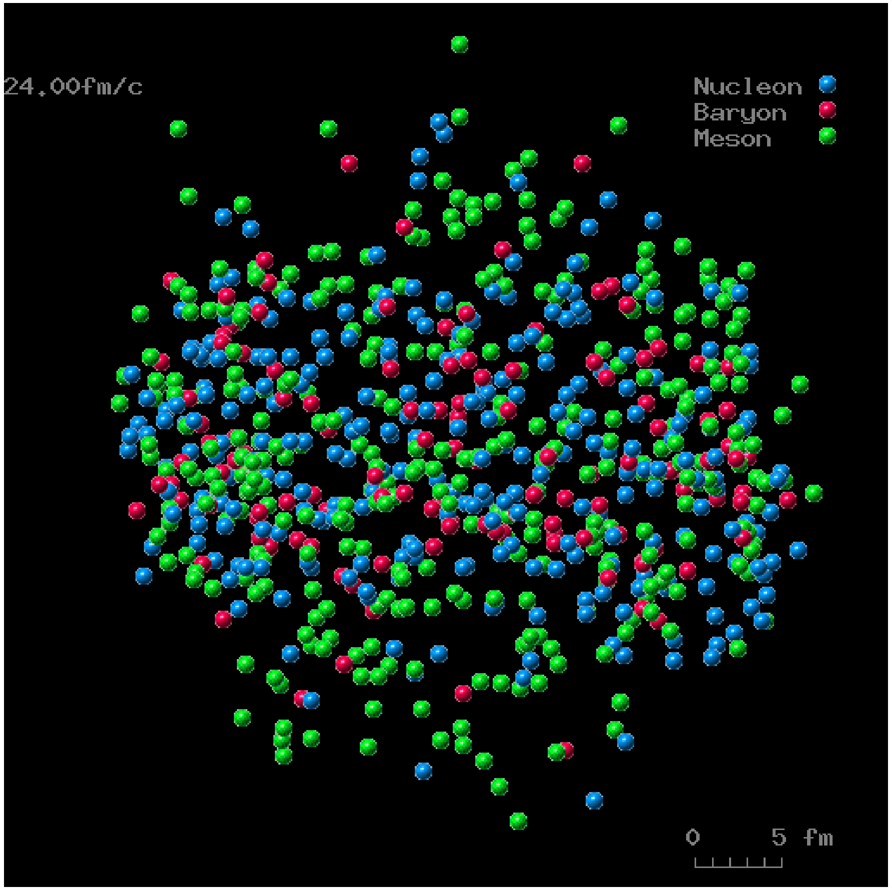


Central 1 fm^3 cube.

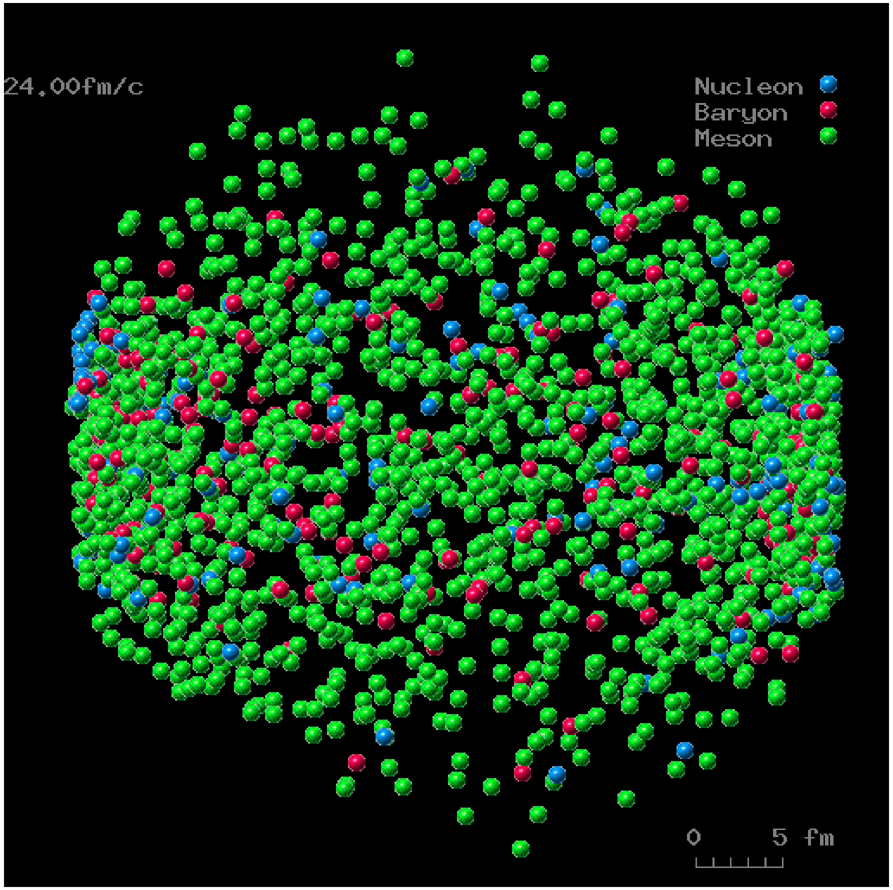
大西、JHF workshop (2002)

How do heavy-ion collisions look like ?

Au+Au, 10.6 A GeV

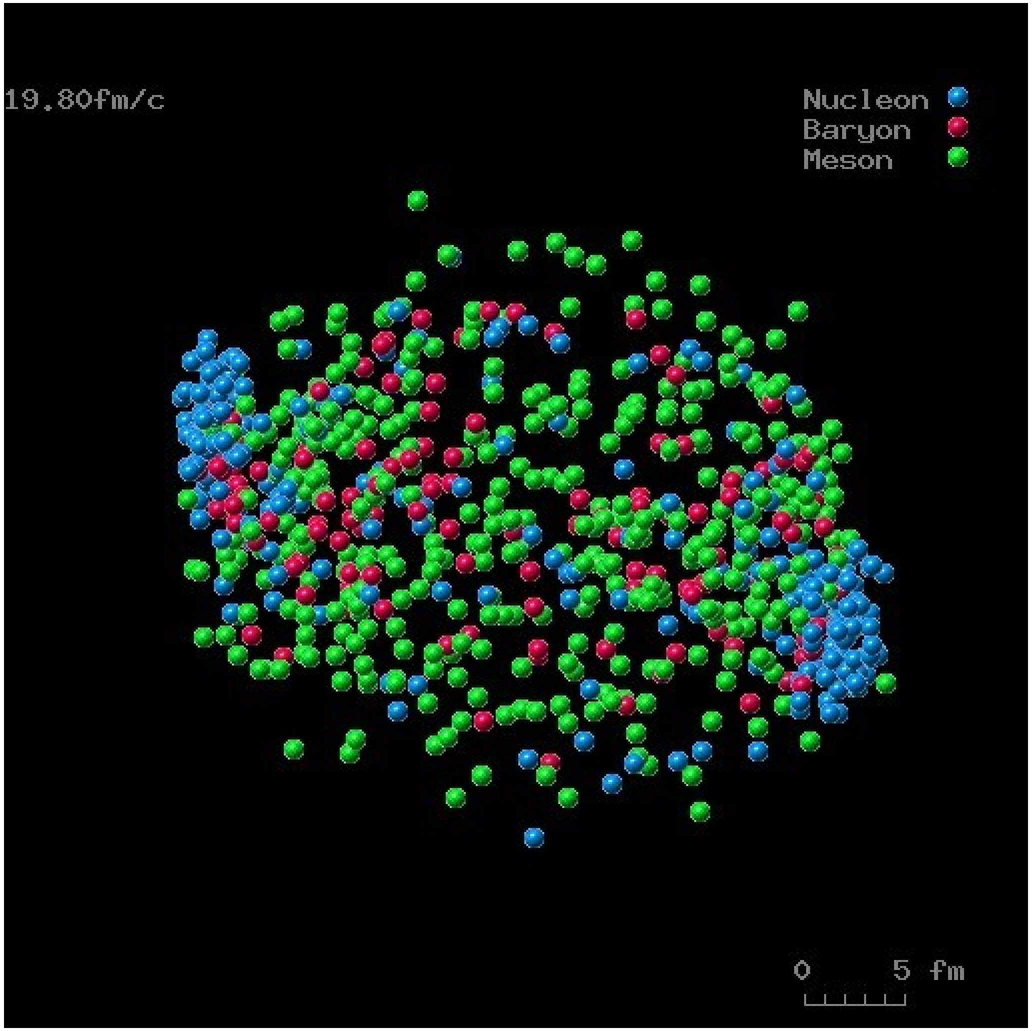


Pb+Pb, 158 A GeV



JAMming on the Web **http://www.jcprg.org/jow/**

J-PARC energy



Au+Au, 25 AGeV, b=5 fm (JOW)

QCD phase transition at BES (J-PARC) Energies ?

- ***J-PARC Energies***: $\sqrt{s_{NN}}$ = 4-40 GeV (or $\sqrt{s_{NN}}$ = 1.9-6.2 GeV)
  - E(p)=30 GeV → E(Au) ~ 12 AGeV (full strip, $\sqrt{s_{NN}}$ = 5.1 GeV for Au+Au)
  - E(p)=50 GeV → E(Au) ~ 20 AGeV ($\sqrt{s_{NN}}$= 6.4 GeV)
  - E(p)=30 GeV (50 GeV) Collider → $\sqrt{s_{NN}}$= 26 GeV (42 GeV)
- **Two Aspects of J-PARC energies**
  - Formation of highest baryon density matter
  - Various non-monotonic behaviors → Onset of deconfinement

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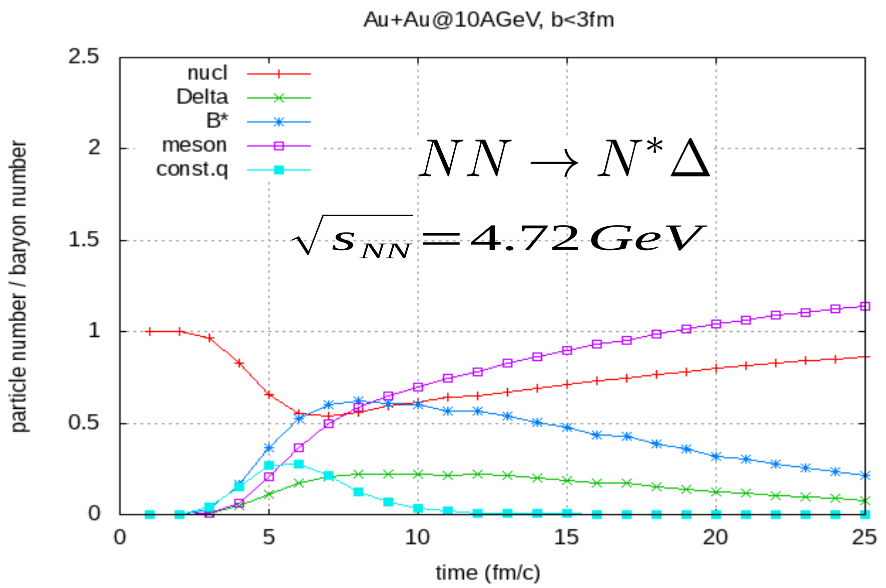
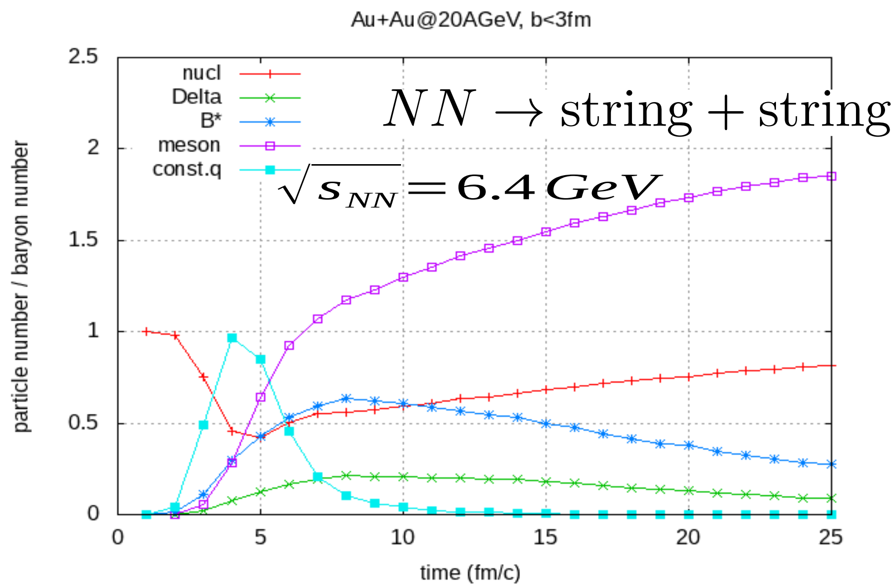
Question
Do these Non-mono. behaviors signal the onset of QCD phase transition and/or QCD critical point ?
or Do they show some properties of hadronic matter ?
→ Let's examine in hadronic transport models !

How to treat mean-field for excited matter?

Hadronic resonance dominant



constituent quark dominant due to string



Model 1 JAM/M: potential for all formed baryons

Model 2 JAM/Mq: potentials for quarks inside the pre-formed hadrons

Model 3: JAM/Mf: both formed and pre-formed baryons

Hadronic transport Approach

Purpose： Effects of hadron mean field potential on the directed flow v1

JAM hadronic cascade model : resonance and string excitation

Mean field by the framework of the Relativistic Quantum Molecular Dynamics

Nuclear cluster formation by phase space coalescence.

Statistical decay of nuclear fragment

Relativistic QMD/Simplified (RQMD/S)

RQMD based on Constraint Hamiltonian Dynamcis

Sorge, Stoecker, Greiner, Ann. Phys. 192 (1989), 266.

RQMD/S: Tomoyuki Maruyama, et al. Prog. Theor. Phys. 96(1996),263.

Single particle energy: $p_i^0 = \sqrt{\boldsymbol{p}_i^2 + m_i^2 + 2m_i V_i}$

$$\dot{\boldsymbol{r}}_i = \frac{\boldsymbol{p}_i}{p_i^0} + \sum_j \frac{m_j}{p_j^0}\frac{\partial V_j}{\partial \boldsymbol{p}_i} \qquad \dot{\boldsymbol{p}}_i = -\sum_j \frac{m_j}{p_j^0}\frac{\partial V_j}{\partial \boldsymbol{r}_i}$$

Arguments of potential $\boldsymbol{r}_i - \boldsymbol{r}_j$ and $\boldsymbol{p}_i - \boldsymbol{p}_j$ are replaced by the distances in the two-body c.m.

Relativistic QMD/Simplified (RQMD/S)

- **RQMD = Constraint Hamiltonian Dynamics**
  (Sorge, Stocker, Greiner, Ann. of Phys. 192 (1989), 266.)
- **Constraints: φ ≈ 0 (Satisfied on the realized trajectory, by Dirac)**
  - **Variables in Covariant Dynamics = 8N phase space: (q_μ, p_μ)**
  - **Variables in EOM = 6N phase space**
    → We need 2N constraints to get EOM
- **On Mass-Shell Constraints**

$$H_i \equiv p_i^2 - m_i^2 - 2m_i V_i \approx 0$$

- **Time-Fixation in RQMD/S**

$$\chi_i \equiv \hat{a}\cdot(q_i - q_N) \approx 0 \, (i=1,\sim N-1) \quad , \quad \chi_N \equiv \hat{a}\cdot q_N - \tau \approx 0$$

$$\hat{a} = \text{Time-like unit vector in the Calculation Frame}$$

(Tomoyuki Maruyama et al., Prog. Theor. Phys. 96(1996), 263.)

RQMD/S (cont.)

- **Hamiltonian is made of constraints**

$$H=\sum_i u_i \phi_i \quad \left(\phi_i = H_i (i=1\sim N), \chi_{i-N} (i=N+1\sim 2N)\right)$$

- **Time Development** $\dfrac{df}{d\tau}=\dfrac{\partial f}{\partial \tau}+\{f,H\} \quad , \quad \{q_\mu, p_\nu\}=g_{\mu\nu}$

- **Lagrange multipliers are determined to keep constraints**
  → ***We can obtain the multipliers analytically in RQMD/S***

$$\frac{d\phi_i}{d\tau}\approx 0 \ \rightarrow \ \delta_{i,2\mathrm{N}}+\sum_j u_j \{\phi_i, \phi_j\}\approx 0$$

- **Equations of Motion**

$$H=\sum_i (p_i^2 - m_i^2 - 2m_i V_i)/2p_i^0 \quad , \quad p_i^0 = E_i = \sqrt{\vec{p}_i^2 + m_i^2 + 2m_i V_i}$$

$$\frac{d\vec{r}_i}{d\tau}\approx -\frac{\partial H}{\partial \vec{p}_i}=\frac{\vec{p}}{p_i^0}+\sum_j \frac{m_j}{p_j^0}\frac{\partial V_j}{\partial \vec{p}_i} \quad , \quad \frac{d\vec{p}_i}{d\tau}\approx\frac{\partial H}{\partial \vec{r}_i}=-\sum_j \frac{m_j}{p_j^0}\frac{\partial V_j}{\partial \vec{r}_i}$$

We can include MF in an almost covariant way in molecular dynamics

Particle "DISTANCE"

$$r_{Tij}^{\ 2} \equiv r_\mu r^\mu - \left(r_\mu P_{ij}^\mu \right)^2 / P_{ij}^2 = \vec{r}^{\,2} \qquad (in\ CM)$$

$$P_{ij} \equiv p_i + p_j \quad , \qquad r \equiv r_i - r_j$$

Particle "Momentum Difference"

$$p_{Tij}^{\ 2} \equiv p_\mu p^\mu - \left(p_\mu P_{ij}^\mu \right)^2 / P_{ij}^2 = \vec{p}^{\,2} \qquad (in\ CM)$$

$$p \equiv p_i - p_j$$

Lorentz Invariant, and Becomes Normal Distance in CM !