量子カオスと熱化

京都大学・基礎物理学研究所　大西　明

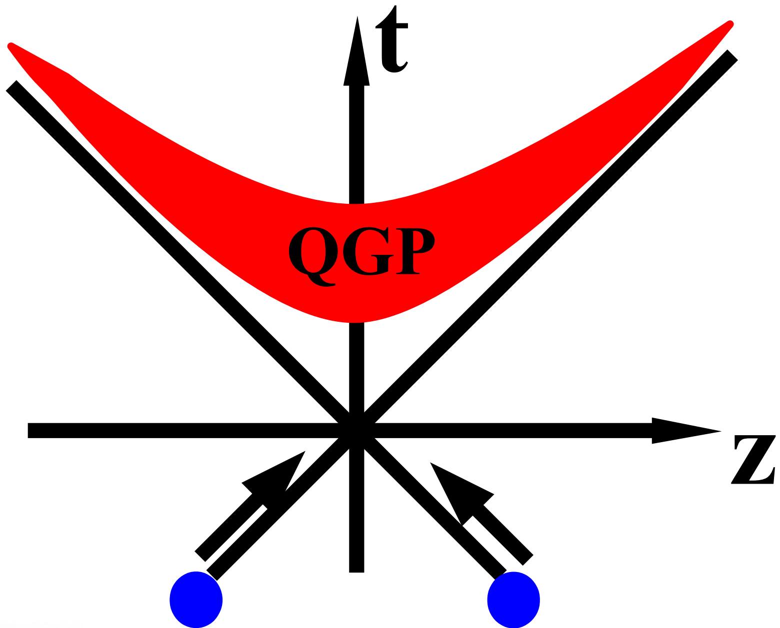
研究会「くりこみ群によるスケールの分離とスローダイナミクス」
2018 年 6 月 9 日

- Introduction
- 孤立量子系の熱化
- カオスとエントロピー生成率
- 量子系エントロピーの直接計算
- Some Reservations
- まとめ

国広さん、ご退職おめでとうございます！

RHICにおける2つの驚き

- RHIC (Relativistic Heavy-Ion Collider)
  - 2000 年から稼働している重イオン衝突型加速器
  - クォーク･グルーオン･プラズマ (QGP) 生成を (ほぼ) 確認
- ２つの驚き (1): 強結合 QGP
  - 流体力学が大きな成功、QGP はほぼ完全流体 ($\eta/s \sim 1/4\pi$) 。
    → 高エネルギーで結合定数 g は小さいはずなのに、
    小さな平均自由行程が実現



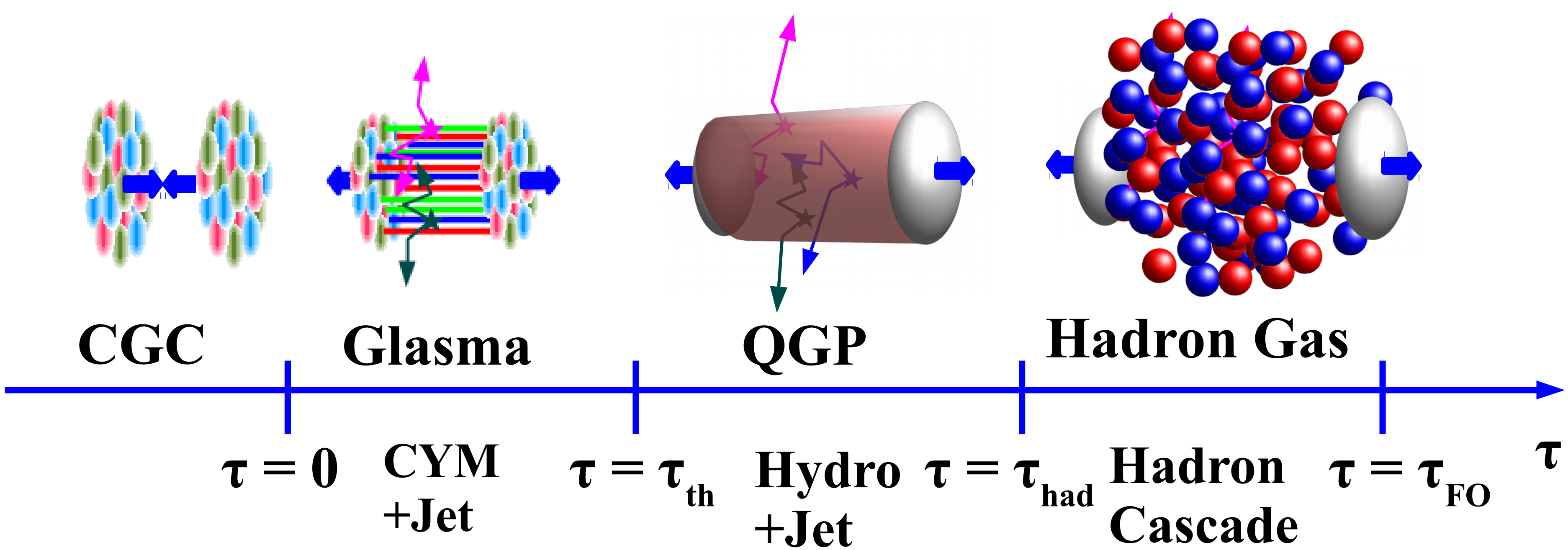
RHIC 稼働前のあるセミナーにて。

“If a miracle happens and the system thermalizes at around τ~0.5 fm/c, hydrodynamics will work to describe the evolution of QGP.”

A miracle happened !

RHICにおける2つの驚き (cont.)

- 2つの驚き (2): 早い熱平衡化
  - 摂動論的 QCD の予言 (2-5 fm/c) に比べて有意に早い時刻 (0.6-1 fm/c) で熱化が起こり、流体力学的時間発展が進む。
    → なぜ早い？
  - 高エネルギー重イオン衝突の初期条件
    = グラズマ (古典ヤンミルズ場が主要)

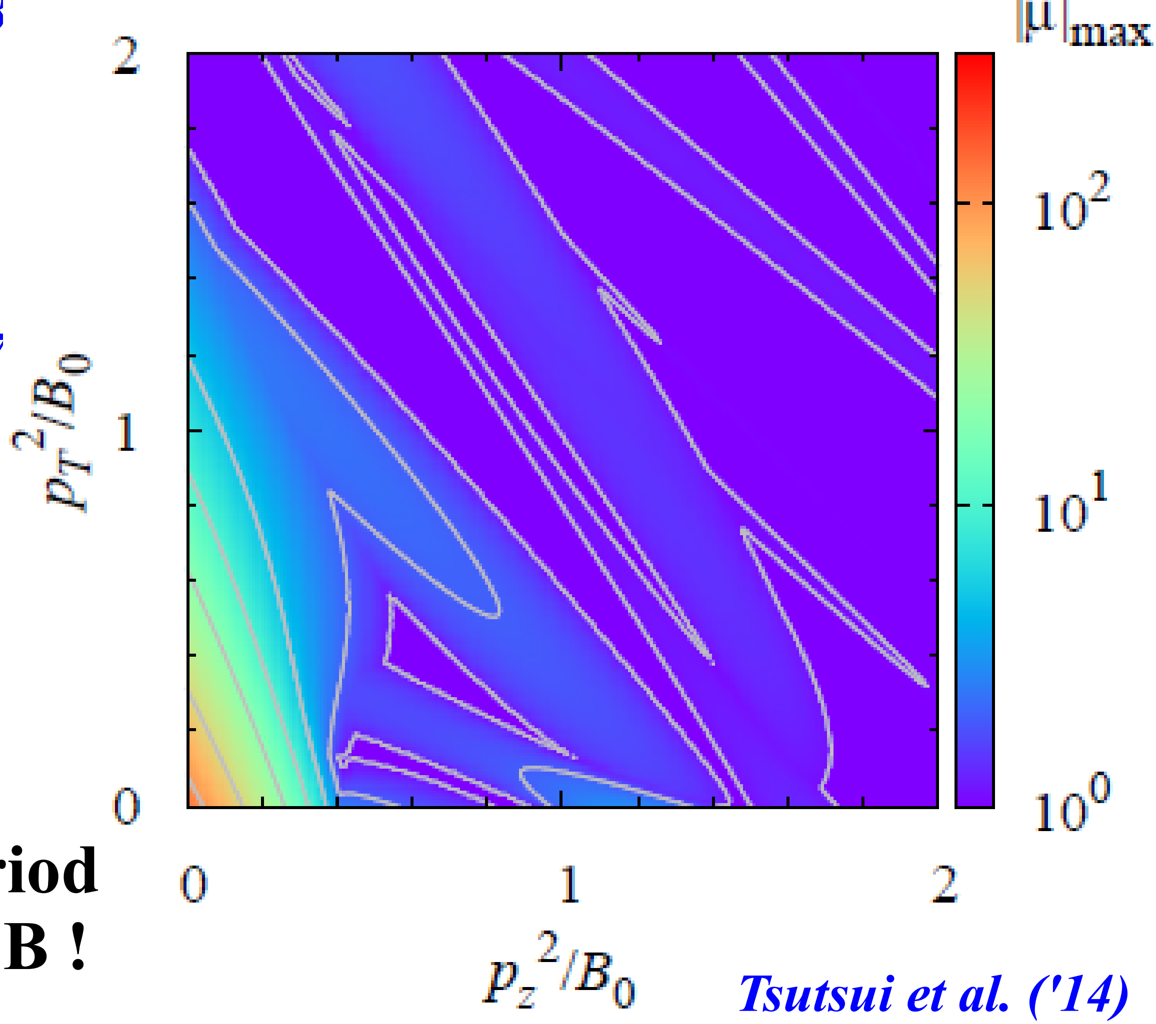


ヤンミルズ場の不安定性

- **Weibel instability**
  E.S.Weibel, PRL 2 ('59),83; S. Mrowczynski, PLB 214 ('88),587.
- **Nielsen-Olesen instability**
  N. Nielsen, P. Olesen, NPB 144 ('78), 376:
  H. Fujii, K. Itakura, NPA 809 ('08), 88
  H. Fujii, K. Itakura, A. Iwazaki,
  NPA 828 ('09), 178.
- **Parametric instability**
  J. Berges, S. Scheffler, S. Schlichting,
  D. Sexty (BSSS), PRD 85 ('12),034507
  S. Tsutsui, H. Iida, T. Kunihiro, AO,
  arXiv:1411.3809.

$$A_i^a = \tilde{A}(t)\left(\delta^{a2}\delta_{ix} + \delta^{a1}\delta_{iy}\right)$$

μ = enh. factor in one period

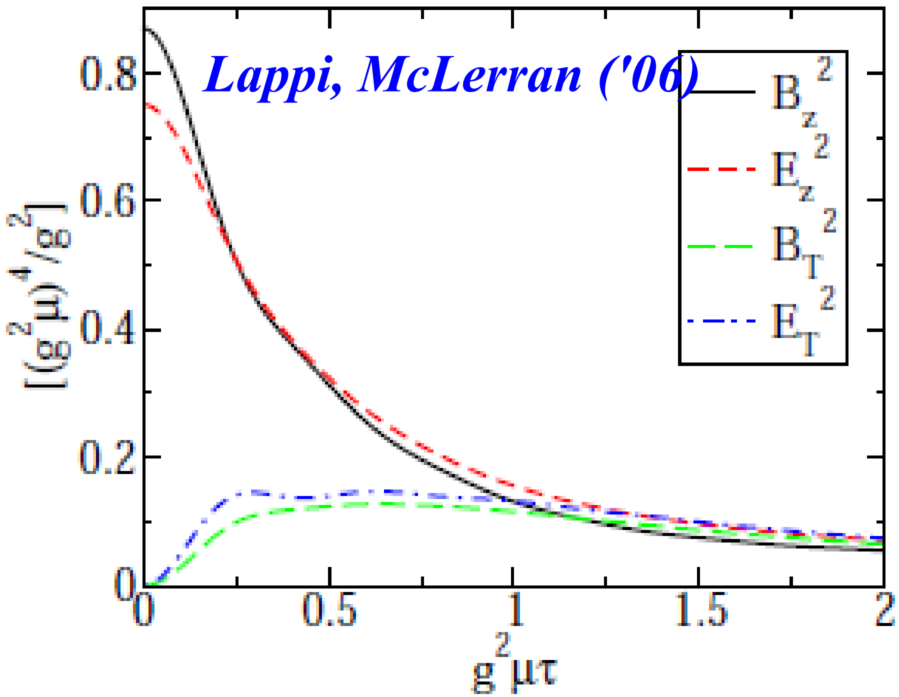


Tsutsui et al. ('14)

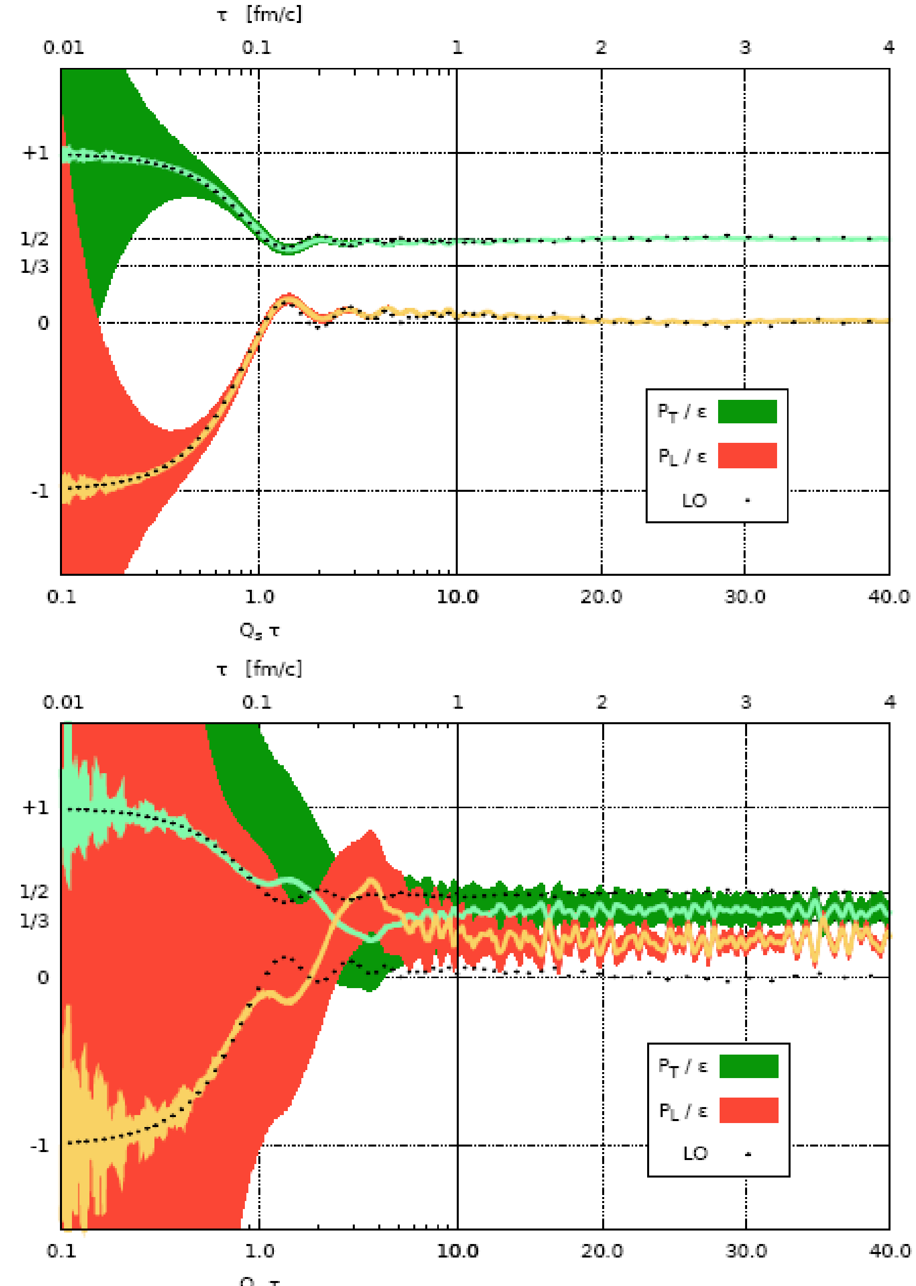
Enh. by >100 times in one period under homogeneous-periodic B !

古典ヤンミルズ場の時間発展

- **古典統計シミュレーション**
  McLerran, Venugopalan ('94), Romatschke, Venugopalan ('06), Lappi, McLerran ('06), Berges, Scheffler, Sexty ('08), Fukushima ('11), Fukushima, Gelis ('12), Epelbaum, Gelis ('13)



等方化については頻繁に議論されてきたが、エントロピーは求められていない。

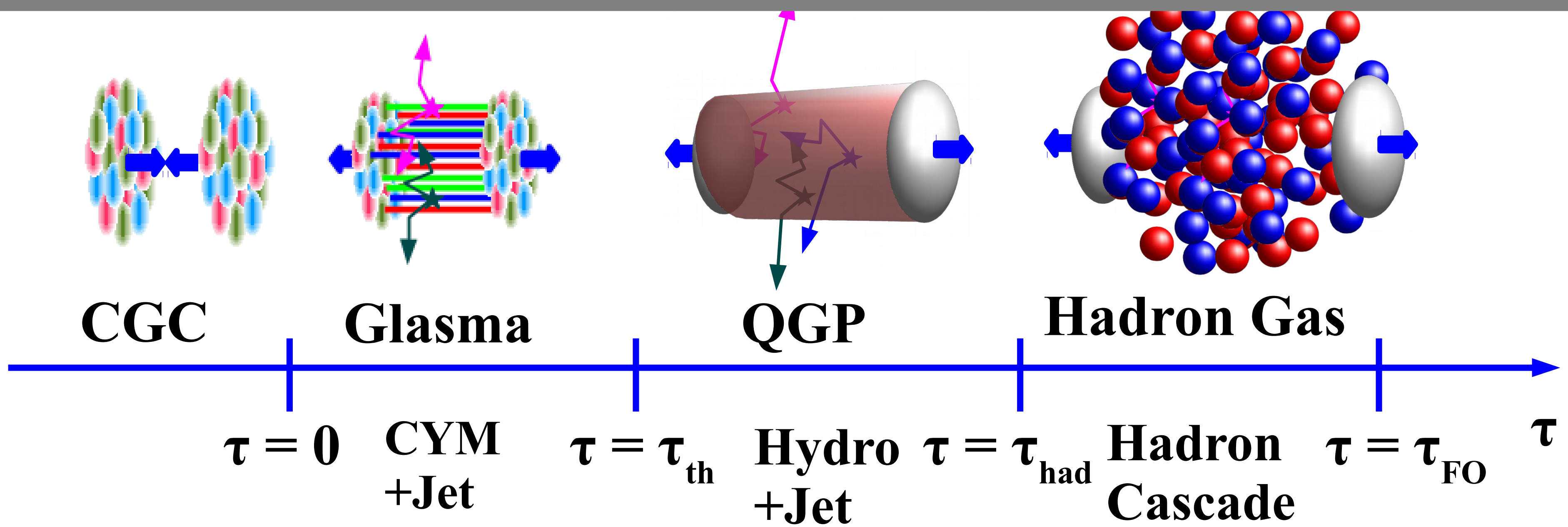


Epelbaum, Gelis ('13)

RHIC における2つの驚き (cont.)

- **2つの驚き (2): 早い熱平衡化**
  - 摂動論的 QCD の予言 (2-5 fm/c) に比べて有意に早い時刻 (0.6-1 fm/c) で熱化が起こり、流体力学的時間発展が進む。
    → なぜ早い？
  - 高エネルギー重イオン衝突の初期条件
    = グラズマ (古典ヤンミルズ場が主要)
  - 古典ヤンミルズ場の成長 (不安定性) → 粒子への崩壊 → 熱化？

むしろ古典ヤンミルズ場自体がエントロピーを作っているのでは？



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    ＋孤立量子カオス系の熱化の問題
- 孤立量子系の熱化
  - 分布関数の複雑化＋粗視化 *Kunihiro, Muller, Schafer, AO ('09)*
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    Kunihiro, Müller, AO, Schäfer, Takahashi, Yamamoto ('10)
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  Tsukiji, Iida, Kunihiro, AO, Takahashi ('15, '16)
  Tsukiji, Kunihiro, AO, Takahashi ('17)
- Some Reservations
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孤立量子系の熱化

孤立量子系の熱化

- 場の変数を正準変数と見れば 古典場もエントロピーをもつはず。
  (A, E) → (q, p) (Wigner 汎関数 , Mrowczynski, Muller)
- von Neumann entropy
  (ρ= 密度行列)
  $$S_{vN} = -\mathrm{Tr}\left[\rho\log\rho\right]$$
  - 純粋状態では ρ の固有値 =1, 0
  - エントロピーはゼロのまま
- 古典エントロピー
  (Wehrl, Boltzmann)
  ρ → f (Wigner(汎) 関数)
  $$S_{\mathrm{W}} = -\int d\Gamma f \log f$$
  $$(d\Gamma = dqdp/(2\pi))$$
  $$\frac{\partial f}{\partial t} + v\cdot\nabla f - \nabla U\cdot\nabla_p f = 0$$
  - Liouville 定理から古典軌道に沿って f は一定
    → エントロピーも一定
  - f は半正定値でない

伏見関数を使えばいいんじゃない？
国広 @ 国際モレキュール型研究会 (2008)
→ 伏見 -Wehrl エントロピー

Entropy Production before QGP

- **クォークハドロン滞在型プログラム・国際モレキュール型研究会 (2008.08.01-28)**
- **参加者 : A. Schafer (客員教授), R. Fries, B. Muller, M. Strickland, T. Schafer, M. Natsuume, Y. Nara, T. Hirano, K. Fukushima, A. Ohnishi, T. Kunihiro, (M.Ohtani)**

孤立量子系の熱化

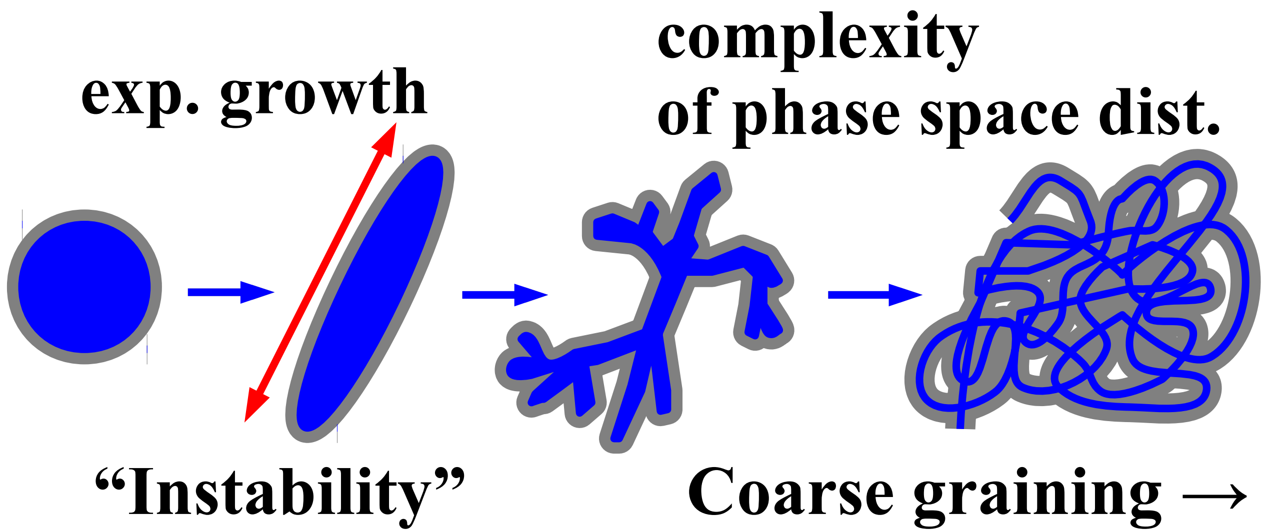
- **伏見関数 ＝ ガウス関数で粗視化した Wigner 関数**

$$f_{\rm H}(q,p) = \int d\Gamma' G(q-q',p-p') f(q',p') = \langle z \mid \rho \mid z \rangle$$

$$G = \text{Gaussian}\ ,\ |z\rangle = \text{coherent state}\ ,\ z = (\omega q + ip)/\sqrt{2\omega}$$

- **「引き伸ばし」と「折りたたみ」による位相空間分布の複雑化 (カオス系)**
  → 粗視化によりエントロピー生成 (Husimi-Wehrl entropy)
  Kunihiro, Muller, Schafer, AO ('09)

$$S_{\rm HW} = -\int d\Gamma f_H(q,p) \log f_H(q,p)$$



カオスとエントロピー生成率

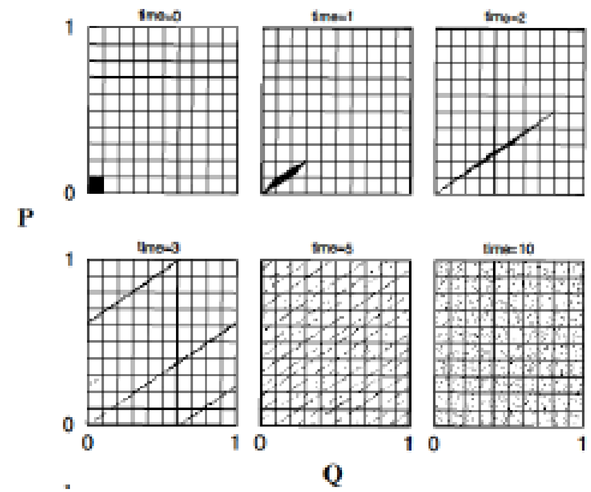
Thermalization scenario based on chaos

V. Latora and M.Baranger, PRL ('99);
M. Baranger, V. Latora and A.Rapisarda, Chaos, Soliton, Fractals (2002)

Generalized cat map(chaotic system)

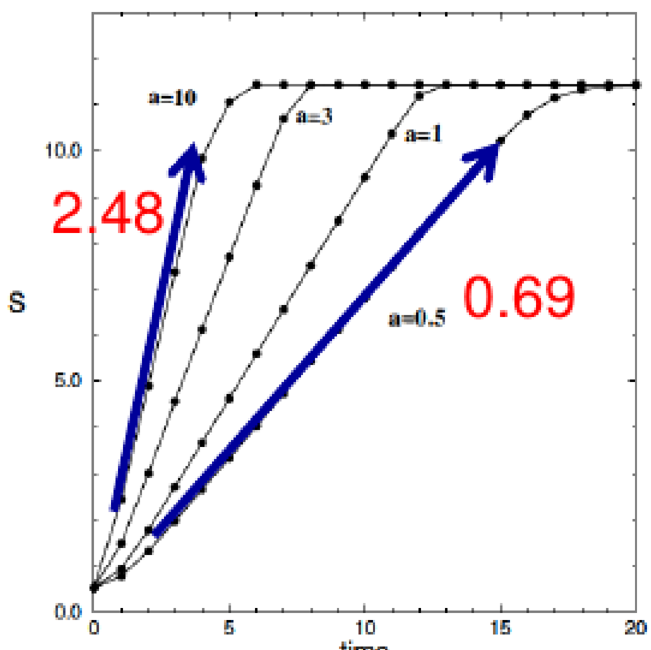
$$P = p + aq \quad (\mathrm{mod}\ 1),$$

$$Q = p + (1 + a)q \quad (\mathrm{mod}\ 1)$$



Lyapunov exponent

$$\lambda = \log \frac{1}{2}(2 + a + \sqrt{a^2 + 4a})$$



Corse-grained Boltzmann Gibbs entropy

$$S(t) = -\sum_{i:\mathrm{cell}} p_i(t) \log p_i(t)$$

$p_i(t)$:probability that the state of the system falls inside cell c_i of phase space at time t

The entropy production rate is consistent with Lyapunov exponent.

$$\boldsymbol{\lambda = 2.48, 1.57, 0.96, 0.69}$$

©Kunihiro(talk at YITP)

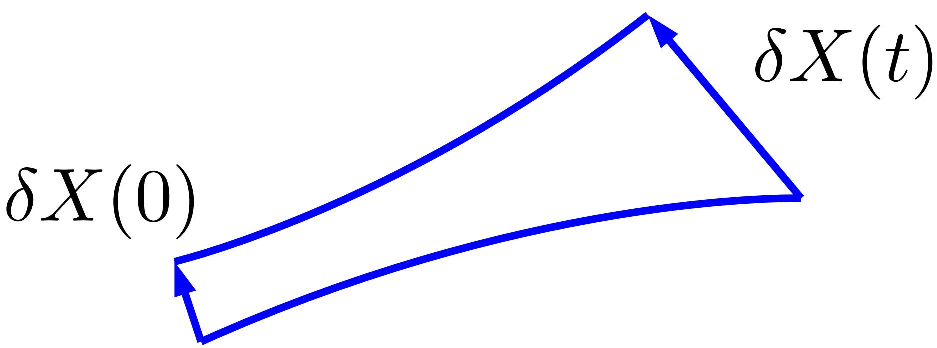
エントロピー生成率

- **古典カオス系では、**
  エントロピー生成率 = 正の Lyapunov 指数の和
  (Kolmogorov-Sinai rate)

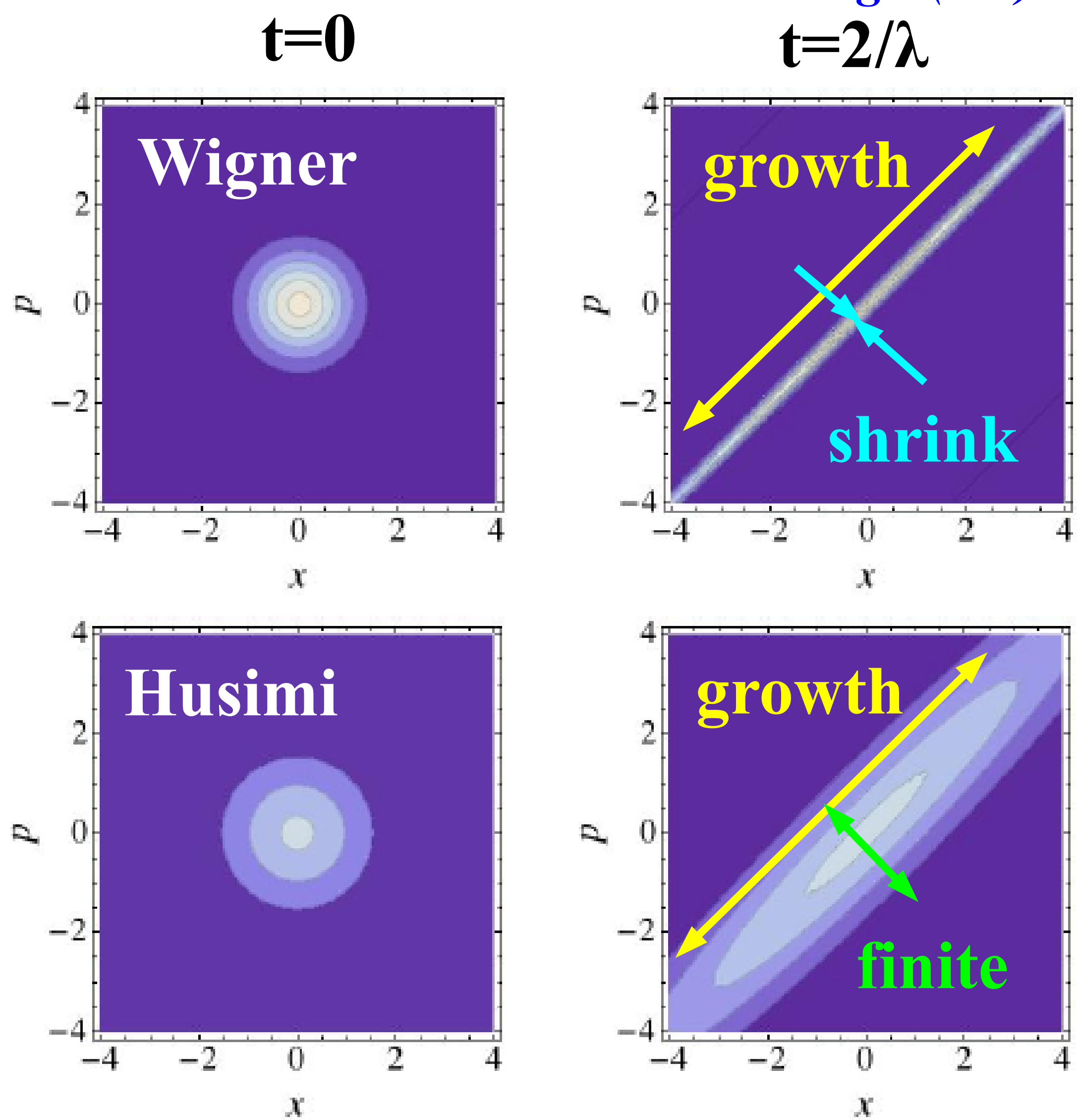
$$\frac{dS}{dt} = h_{\mathrm{KS}} = \sum_{\lambda_i > 0} \lambda_i$$

$$|\delta X_i(t)| = e^{\lambda_i t} |\delta X_i(0)|$$

λ_i : Lyapunov exponent



V. Latora and M.Baranger('99)



- **量子系でも成立**
  Kunihiro, Muller, Schafer, AO ('09)
  - **(逆)調和振動子**

ヤンミルズ場のリャプノフ指数

- 格子上の古典ヤンミルズ場

$$H = \frac{1}{2}\sum_{x,a,i}\left[\left(E_i^a(\mathbf{x})\right)^2 + \left(B_i^a(\mathbf{x})\right)^2\right]$$

$$B_i^a(x) = \partial_j A_k^a - \partial_k A_j^a - \varepsilon_{abc} A_j^b A_k^c \quad ((i,j,k) = \text{cyclic})$$

- 大自由度での Lyapunov exponent を如何に計算できるか？
  - 近い2点のズレの時間発展 → Trotter 公式を用いて積で表し対角化

Kunihiro, Müller, AO, Schäfer, Takahashi, Yamamoto ('10)

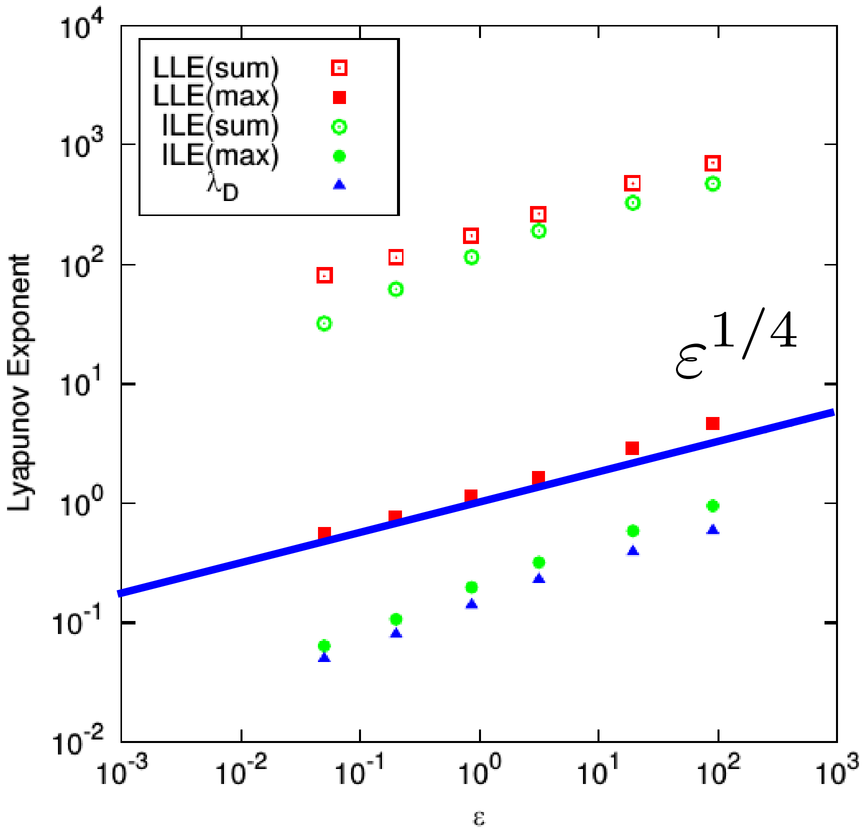
(c.f. Shimada-Nagasihima 法)

Hessian

$$\frac{d}{dt}\begin{pmatrix}\delta x \\ \delta p\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}H_{xx} & H_{xp} \\ H_{px} & H_{pp}\end{pmatrix}\begin{pmatrix}\delta x \\ \delta p\end{pmatrix} = \mathcal{H}\begin{pmatrix}\delta x \\ \delta p\end{pmatrix}$$

$$\delta X(t) \simeq \mathcal{T}\left[\prod_k (1 + \mathcal{H}(k\Delta t))\right]\delta X(0)$$

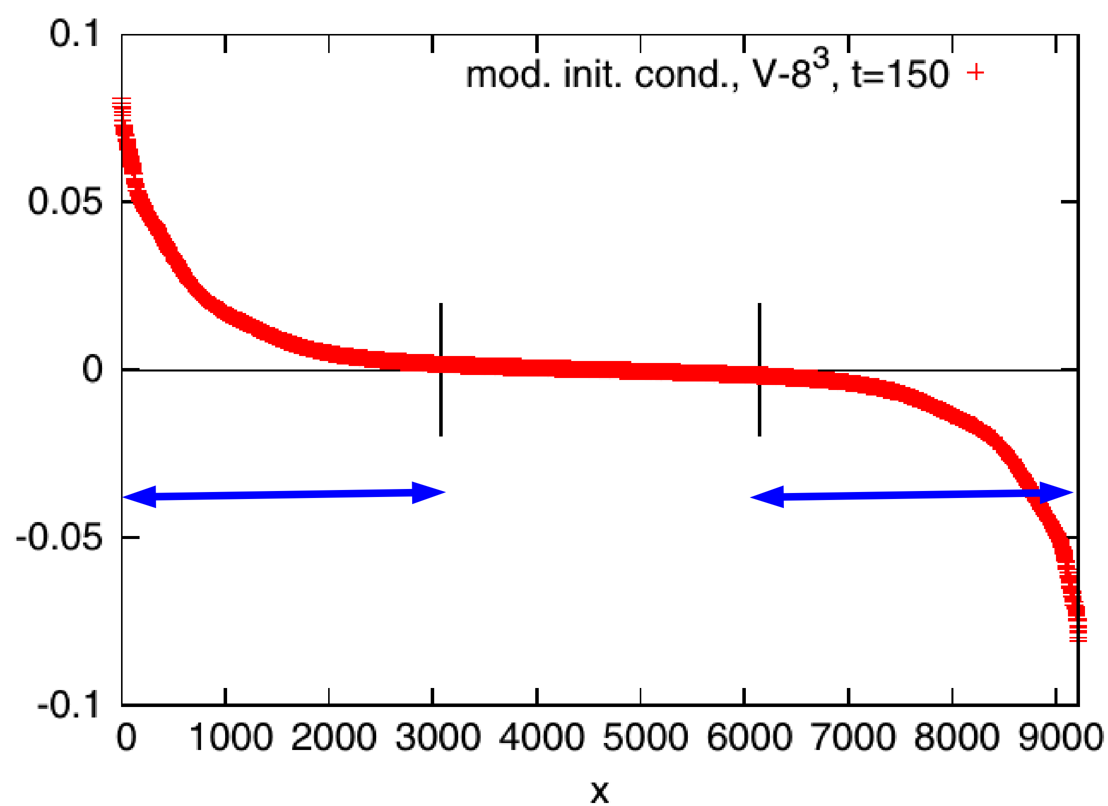
ヤンミルズ場のリャプノフ指数



exp(-20)~ exp(+20) の
Lyapunov 指数を 1000 個
正確に求め、
スケール普遍性も示す
「素晴らしい。高橋・山本、天才」*Kunihiro, Müller, AO, Schäfer, Takahashi, Yamamoto ('10)*

9000 個を超える Lyapunov 指数を求めて、正負対称。
ゲージ自由度 (1/3) でほぼ λ=0
「これは、数値計算科学自体としてもすごいんじゃない？」

Iida, Kunihiro, Müller, AO, Schäfer, Takahashi ('13)



量子系エントロピーの直接計算

多次元量子系の *Husimi-Wehrl Entropy*

- 生成率は「推定」できたが、HW エントロピーの直接評価はなお困難
  E.g. 大西・高橋が 2009 年に挑戦するもまともな値が求まらず、惨敗

$$f_H(q,p,t) = \frac{1}{N_{\rm tp}} \sum_i G(q-\underline{q_i(t)}, p-\underline{p_i(t)})$$

テスト粒子
（古典軌道）

$$S_{\rm HW} = -\int d\Gamma\, f_H(q,p,t) \log f_H(q,p,t)$$

$$= -\frac{1}{N_{\rm tp}} \sum_i \langle \log f_H(q,p,t) \rangle_i$$

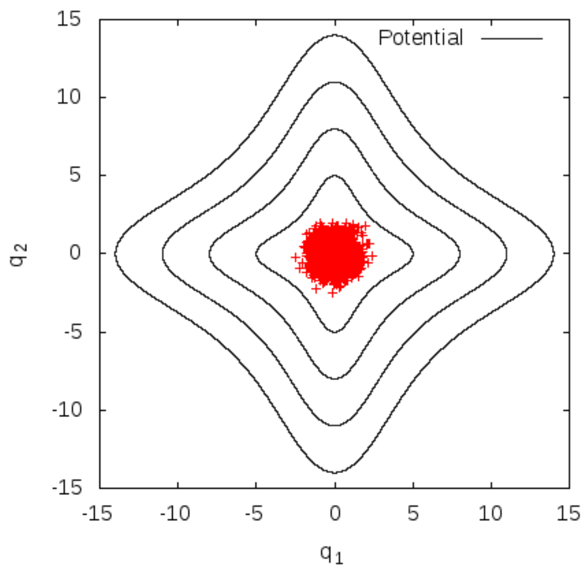
i 番目のテスト粒子の周りで MC 平均

- Monte-Carlo 法と半古典近似による手法の開発
  Tsukiji, Iida, Kunihiro, AO, Takahashi ('15,'16)
  - log f の「下限」の制御が難しい → 2 通りで挟み撃ち
  - log の内側・外側で
    同じテスト粒子群 (Test Particle 法) → log f の下限大
    異なるテスト粒子群 (Two step MC or pTP 法) → log f の下限小

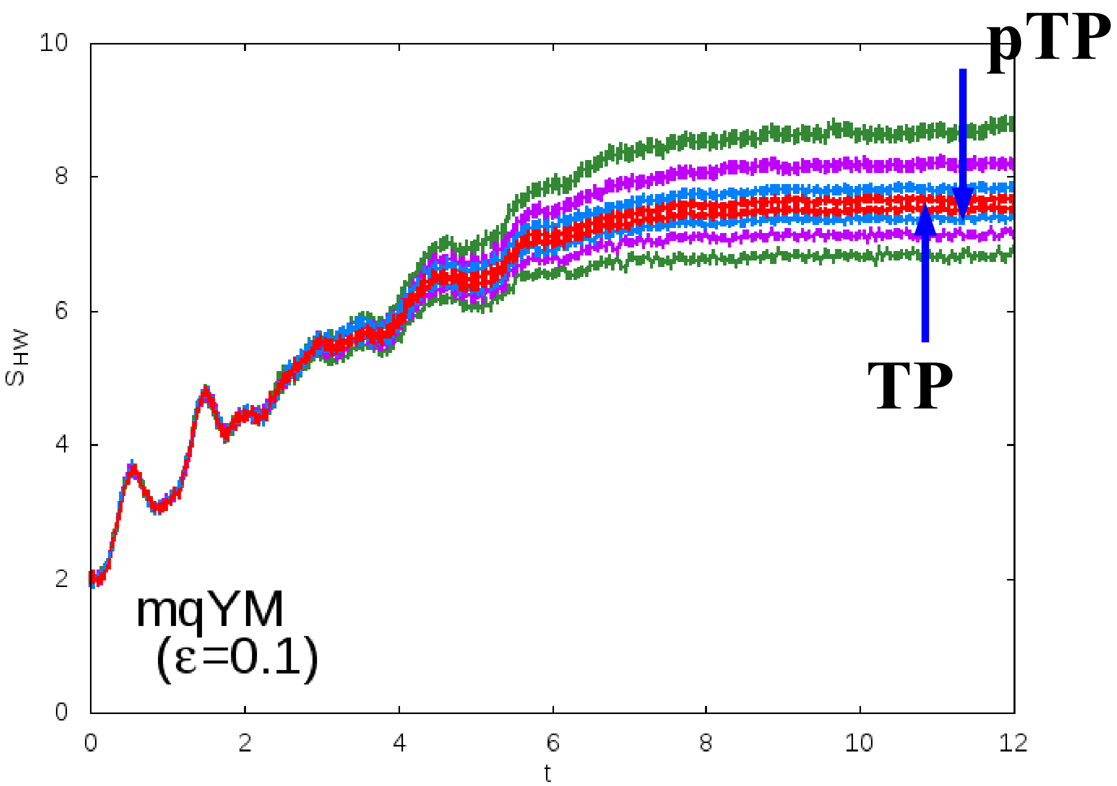
やられたなあ。（大西）

試行粒子分布の振る舞いとの比較

座標空間での振る舞い



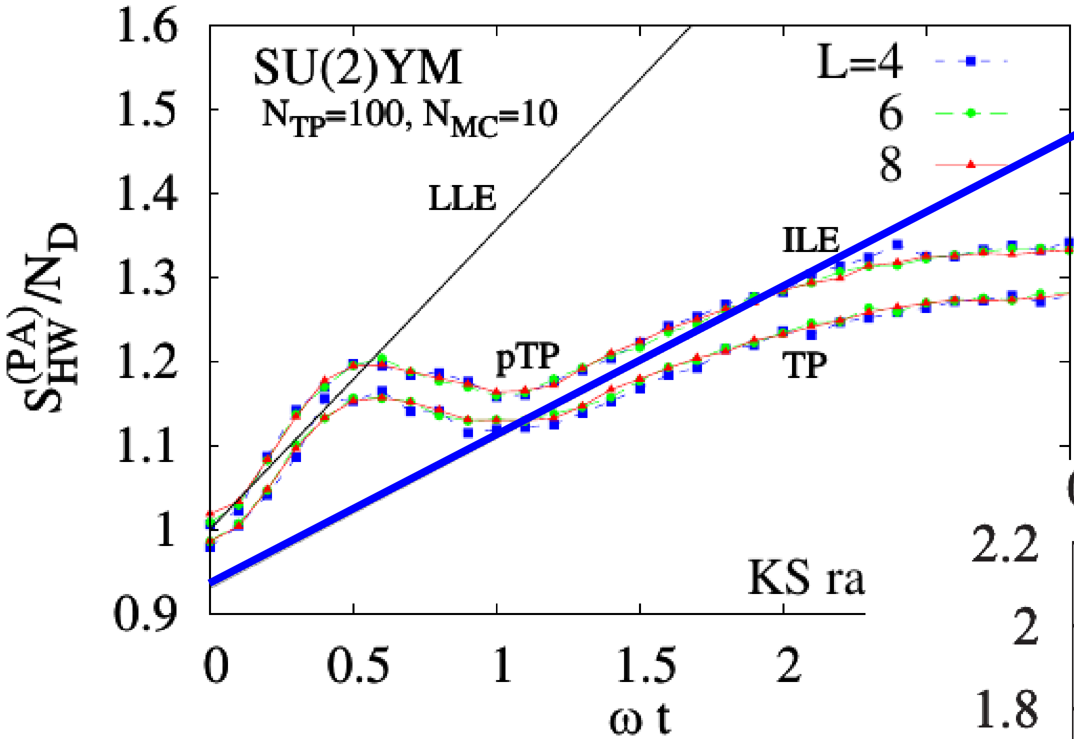
伏見バールエントロピーの時間発展



伏見関数の広がりは試行粒子分布を反映する。
エントロピーのピークは試行粒子分布がポテンシャルにぶつかる時刻に対応する。
分布が一様になるとエントロピーは飽和する。

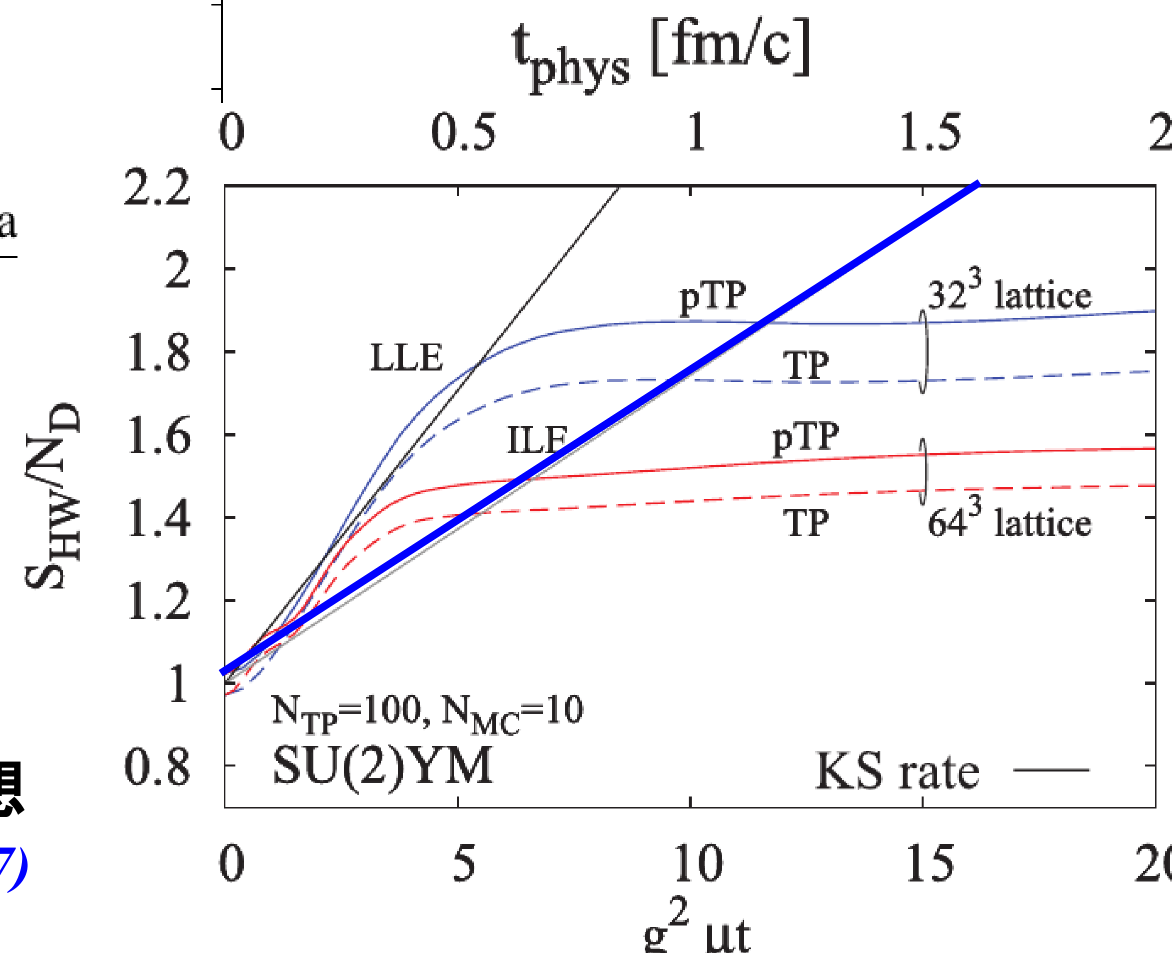
© Tsukiji

ヤンミルズ場の Husimi-Wehrl entropy



Random な初期条件
準定常状態に緩和する段階の
エントロピー生成率
＝カオス性からの予想と一致

Tsukiji, Kunihiro, Iida, AO, Takahashi ('16)



グラズマ初期条件
最初期のエントロピー生成で
ほぼ準定常状態まで到達
生成率 ＞ カオス性からの予想

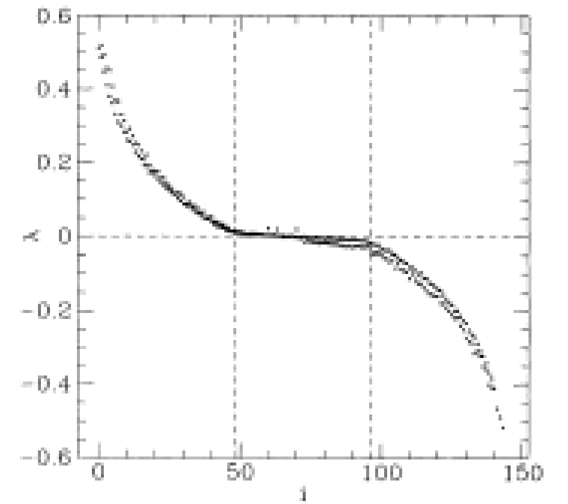
Tsukiji, Kunihiro, AO, Takahashi ('17)

Some Reservations (但し書き)

- ヤンミルズ場の Lyapunov スペクトル計算は以前にもあり。ただし 2^3 lattice.
- 「位相空間で粗視化した分布関数」を利用することは一般的ではない。
密度行列を直接用いた熱化の議論は？
→ *Matsuda, Kunihiro, AO, Takahashi, in prep.*
  - Entanglement entropy (E.g. Takayanagi-Ryu)
Decoherence entropy by Zurek
→ 純粋状態から混合状態へ。
ただし環境との相互作用が前提
  - Time-averaged observables (E.g. Deutsch)

$$\rho_{ij}(t) = \rho_{ij}(0)\exp(-i(E_i - E_j)t/\hbar)$$

→ 十分長い時間での平均が必要
- 現実的なエントロピー生成量評価には膨張効果は必須。
- Scaling を利用した最大リャプノフ指数 ~ 0.45 T (SU(3)), 0.52 T (SU(2))
上限値よりはずっと小さい。



cf) Lyapunov spectrum ($V=2^5$)
Gong, Phys.Rev.D49, 2642 (1994).

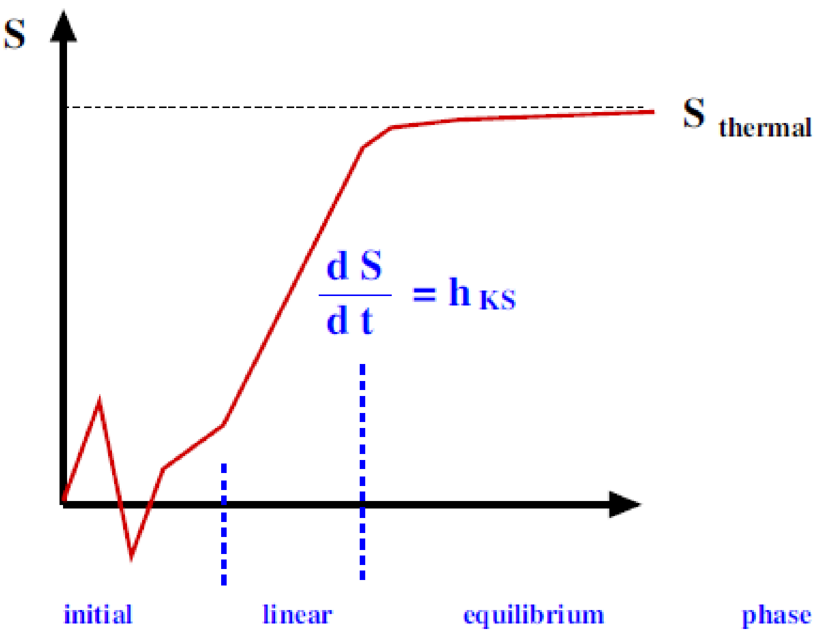
まとめ

- 孤立量子系や(第2量子化して考えた)古典場は
カオス性から分布関数が複雑化し、粗視化すれば熱化が評価できる。
- 古典ヤンミルズ場は、揺らぎが増大し、等方化し、熱化する。
高エネルギー重イオン衝突の初期段階で作られるエントロピーの一部は古典場の時間発展段階で作られたものの顕在化であろう。
- 現象を動機として物理学の基礎的問題を見出し、突破できるアイデアを出してくださった国広さんに感謝。
  - 基礎的・概念的な提案(整理)は、設定が簡単であっても論文にすべし(Kunihiro et al.('09)には後々助けられました)
  - 内側からの動機・アイデアを続ける意識の強さ
  - 大学院生・ポスドクをほめる技術(ツンデレ)
- 研究・教育・運営、ごくろうさまでした。
  - QCD物性物理学の創設、スローダイナミクスの発展。
  - 多くの大学院生・ポスドクの指導。
  - 研究室の運営、滞在型プログラムの立ち上げ。

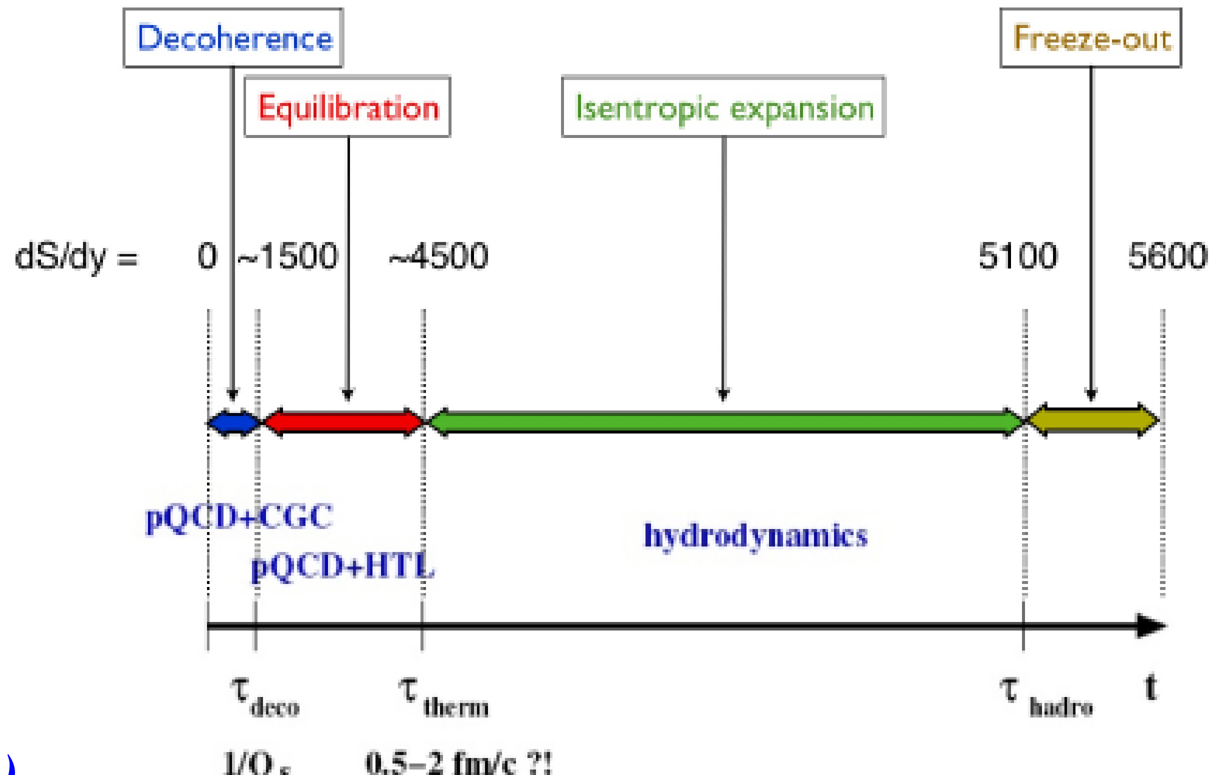
Thank you for your attention !

Entropy Production in Glasma

- **Huge entropy must be produced before QGP formation !**
  - **Thermalization time ~ (0.5-2.0) fm/c**
  - **Instability ? Rapid glasma decay ? Entropy of classical field ?**



B.Muller and A. Schaefer,
Int. J. Mod. Phys. E20, 2235 (2011)



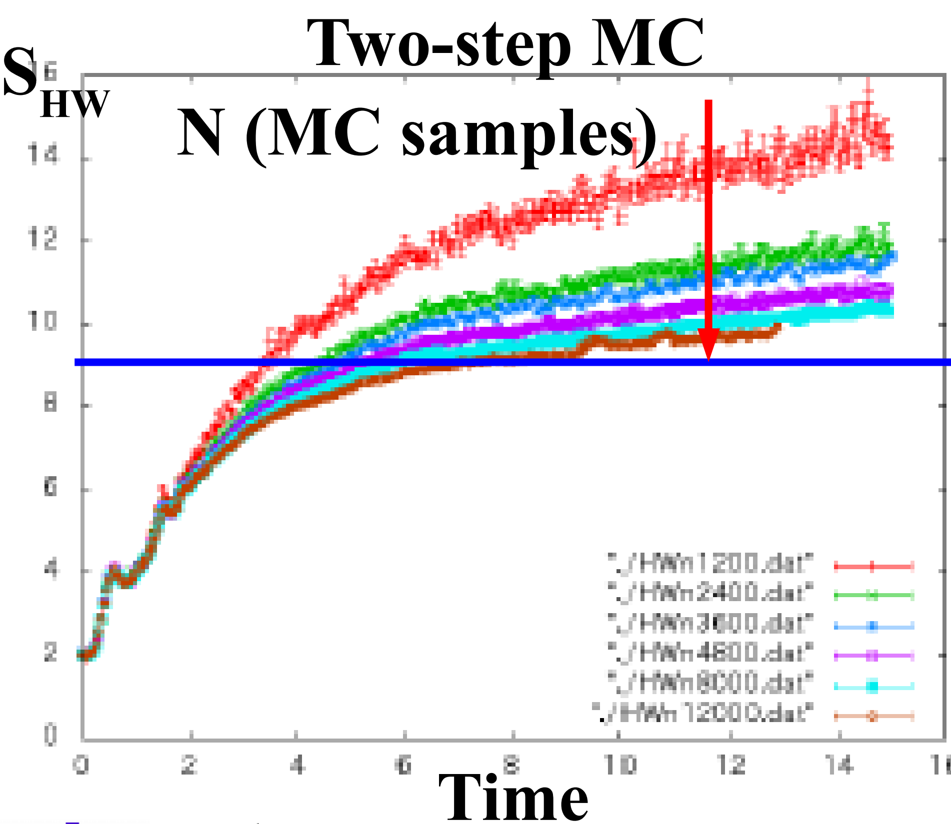
R.J. Fries et al, arXiv 0906.5293

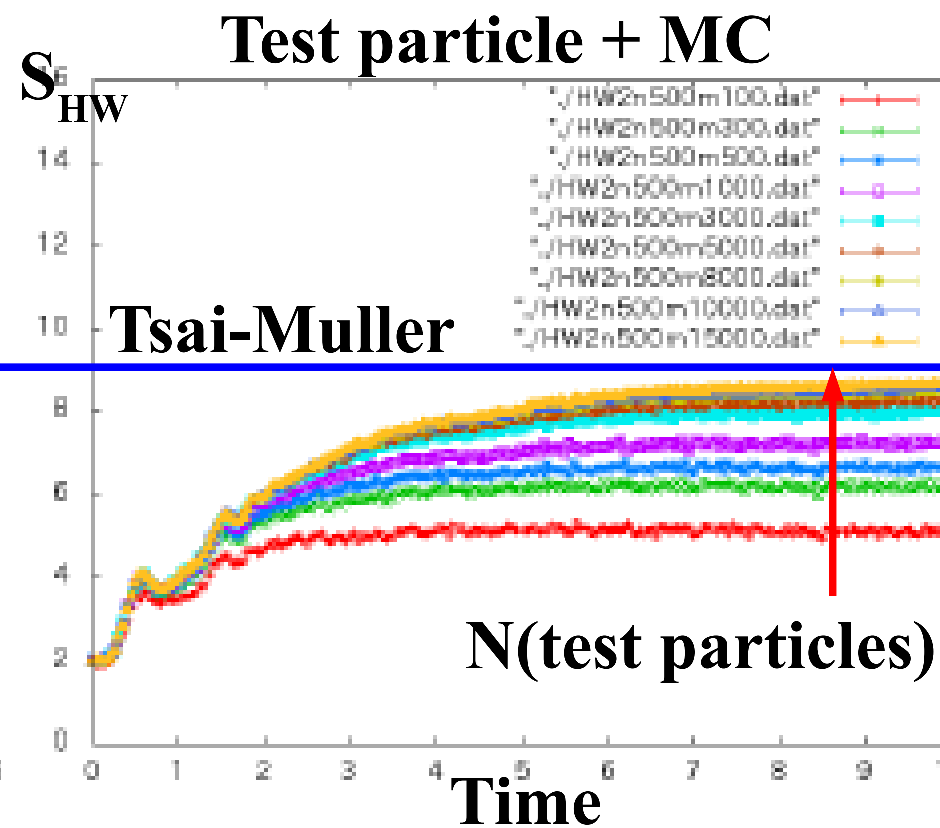
We discuss the CYM entropy and its production rate with emphasis on the chaoticity

Monte-Carlo + Semi-Classical Approx.

Tsukiji, Iida, Kunihiro, AO, Takahashi, in prep.

- **Semi-Classical + MC methods reproduce mesh integral values of S_{HW}.**
  - **Two-step MC results converge from above.**
  - **Test particle + MC results converge from below.**





まとめ

- 高エネルギー重イオン衝突物理において「早い熱平衡化」は大きな未解決問題のひとつであり、「エントロピー生成」機構解明が望まれている。
- 位相空間の複雑さにより生まれるエントロピー生成は、不安定モードの強さと数により記述され、系のカオス性と密接にかかわる。
(Kolmogorov-Sinaii entropy rate= 正の Lyapunov exponents の和)
- 多自由度系における Husimi-Wehrl entropy の評価は現在でも研究が進む課題。
  - 不安定調和振動子、Yang-Mills 量子力学系において半古典近似 +Smearing を用いた Husimi 関数の時間発展によりエントロピー生成を議論 [Kunihiro et al.('09), Tsukiji et al.(in prep.)]
- 古典ヤンミルズ理論に基づく高エネルギー重イオン衝突初期のダイナミクスが活発に議論されている。
  - 場の変数と共役運動量を正準変数として Wigner 汎関数、Husimi 汎関数を定義し、不安定性、Kolmogorov-Sinai rate 、Husimi-Wehrl entropy を議論 [Kunihiro et al.('10), Iida et al.('13, '14), Tsutsui et al.('14)]
- 乱流との関連 ....

Husimi-Wehrl Entropy in Multi-Dimensions (1)

- **Challenge: Evolution of Husimi fn. & Multi-Dim. integral**

$$S_{\rm HW} = -\int \frac{d^D q d^D p}{(2\pi\hbar)^D} f_H(q,p) \log f_H(q,p)$$

$$f_H(q,p) = \int \frac{d^D q' d^D p'}{\pi\hbar} e^{-\Delta(q-q')^2/\hbar-(p-p')^2/\Delta\hbar} f_W(q,p)$$

- **Monte-Carlo + Semi-classical approx.**
  Tsukiji, Iida, Kunihiro, AO, Takahashi, in prep.
  - **Two-step Monte-Carlo method**
    Monte-Carlo integral + Liouville theorem [$f_W(q,p,t)=f_W(q_0,p_0,t=0)$]
  - **Test particle method: Test particle evol. + Monte-Carlo integral**

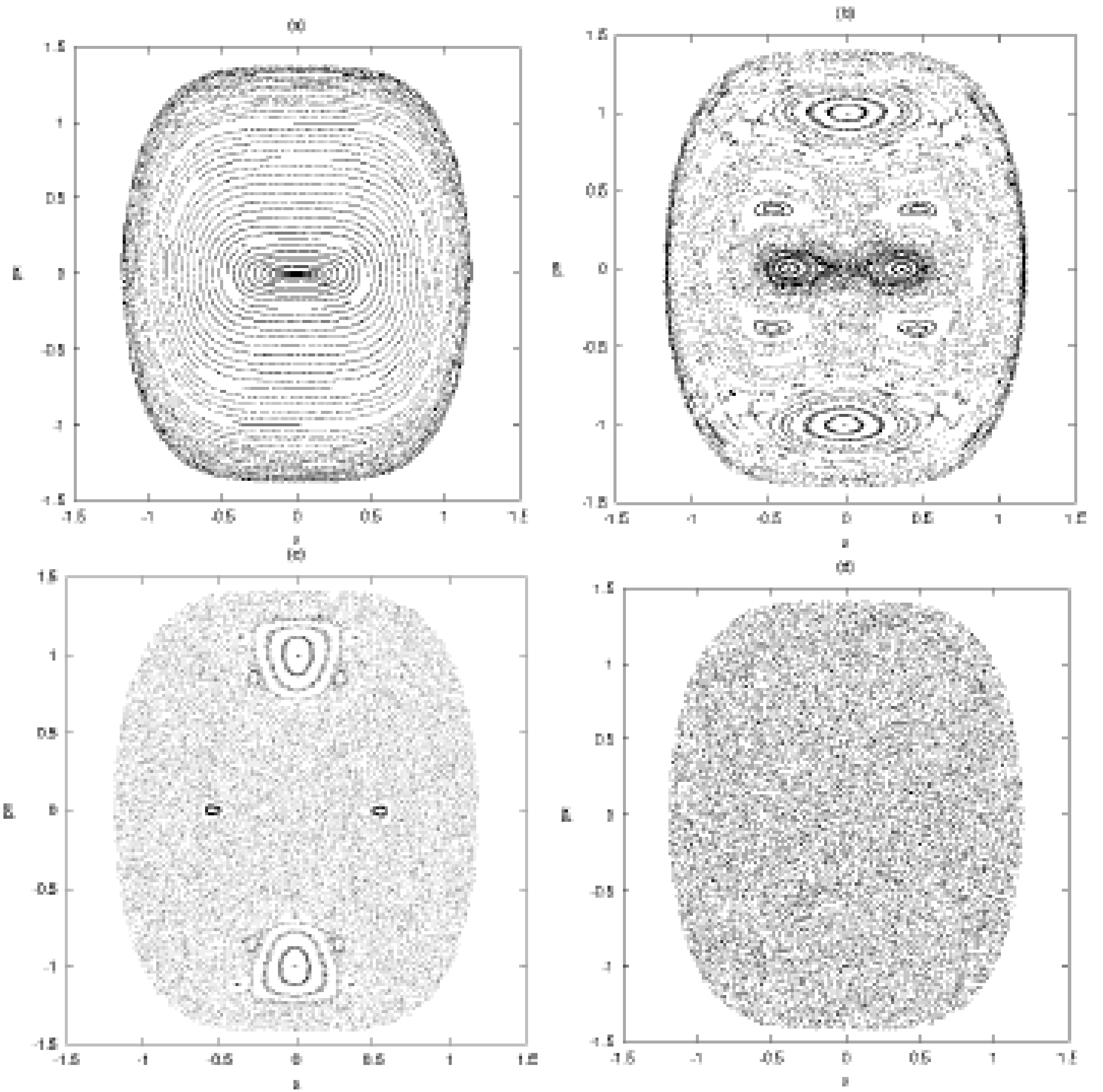
$$f_W(q,p,t) = \frac{2\pi\hbar}{N_{\rm tp}} \sum_{i=1}^{N_{\rm tp}} \delta(q-q_i(t))\,\delta(p-p_i(t)) \ ,$$

$$\frac{dq_i}{dt} = \frac{p_i}{m} \ , \quad \frac{dp_i}{dt} = -\frac{\partial U}{\partial q_i} \ .$$

Poincare Map of 2D Quartic Oscillator

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^4 + y^4) - k^2 x^2 y^2$$

k=0, 0.2, 0.4, 0.6



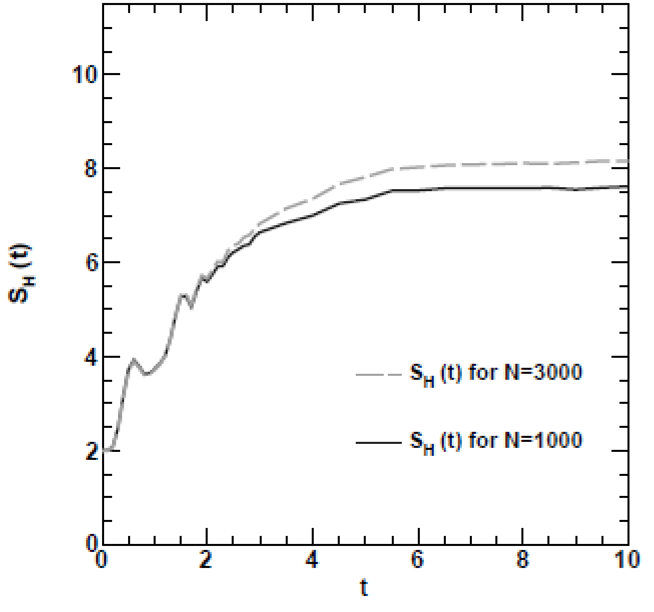
Sugita, Aiba,
Phys.Rev. A65 ('02) 036205.

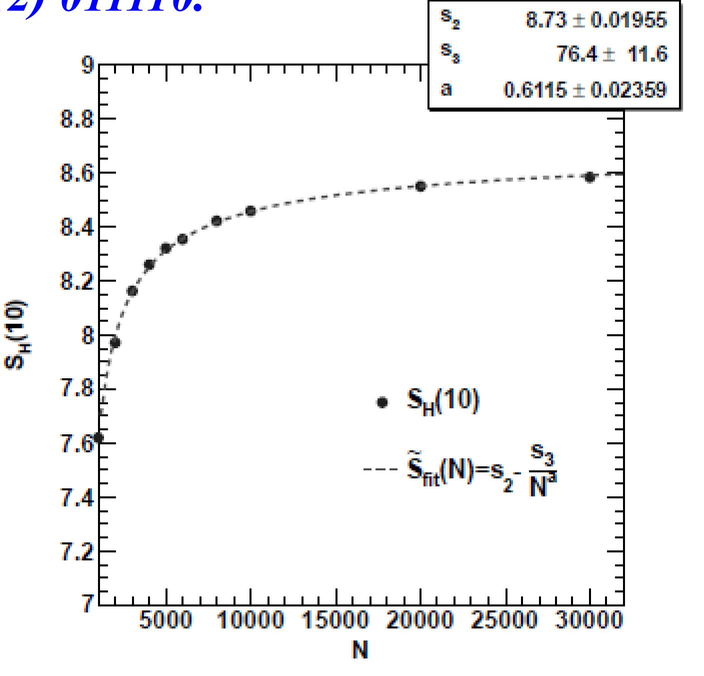
"Yang-Mills" Quantum Mechanics

- **Yang-Mills quantum mechanics**

$$H = \frac{p_1^2 + p_2^2}{2m} + \frac{1}{2} g^2 q_1^2 q_2^2$$

  - **Quartic interaction term → almost globally chaotic**
    S. G. Matinyan, G. K. Savvidy, N. G. Ter-Arutunian Savvidy, Sov. Phys. JETP 53, 421 (1981); A. Carnegie and I. C. Percival, J. Phys. A: Math. Gen. 17, 801 (1984); P. Dahlqvist and G. Russberg, Phys. Rev. Lett. 65, 2837 (1990).
  - **Husimi-Wehrl entropy in a test particle method for the Husimi fn. (w/ $\hbar^2$ corrections, EOM with a moment method)**
    H.-M. Tsai, B. Muller, Phys.Rev. E85 (2012) 011110.





Tsukiji, Iida, Kunihiro, AO, Takahashi, in prep.

- **Two-step Monte-Carlo integral**

Liouville

$$S_{\rm HW}^{\rm (tsMC)} = -\int \frac{d^DQd^DP}{(\pi\hbar)^D}\, e^{-\Delta Q^2/\hbar - P^2/\Delta\hbar} \int \frac{d^Dqd^Dp}{(2\pi\hbar)^D}\, f_{\rm W}(q,p,t)$$

$$\times \log\left[\int \frac{d^DQ'd^DP'}{(\pi\hbar)^D}\, e^{-\Delta(Q')^2/\hbar-(P')^2/\Delta\hbar} f_{\rm W}(q+Q+Q', p+P+P', t)\right]$$

$$= -\frac{1}{N_{\rm out}}\sum_{k=1}^{N_{\rm out}} \log\left[\frac{1}{N_{\rm in}}\sum_{l=1}^{N_{\rm in}} f_{\rm W}(q_k+Q_k+Q'_l, p_k+P_k+P'_l, t)\right]$$

Outside MC → S **Inside MC → f_H**

- **Test particle method: test particle evolution + MC integral**

$$S_{\rm HW}^{\rm (tp)} = -\frac{1}{N_{\rm tp}}\sum_{i=1}^{N_{\rm tp}} \int \frac{d^Dqd^Dp}{(\pi\hbar)^D}\, e^{-\Delta(q-q_i(t))^2/\hbar-(p-p_i(t))^2/\Delta\hbar} \log f_H(q,p,t)$$

$$= -\frac{1}{MN_{\rm tp}}\sum_{k=1}^{M}\sum_{i=1}^{N_{\rm tp}} \log\left[\frac{2^D}{N_{\rm tp}}\sum_{j=1}^{N_{\rm tp}} e^{-\Delta(Q_k+q_i(t)-q_j(t))^2/\hbar-(P_k+p_i(t)-p_j(t))^2/\Delta\hbar}\right]$$

Contents

- **Introduction**
- **Entropy production in quantum mechanics**
  - **Chaoticity, Lyapunov exponent, and Kolmogorov-Sinaii entropy**
  - **Coarse graining and Husimi-Wehrl entropy**
  - **HW entropy in quantum mechanics**
    T. Kunihiro, B. Müller, A. Ohnishi, A. Schäfer, PTP 121 ('09), 555.
    H. Tsukiji, H. Iida, T. Kunihiro, A. Ohnishi, T. T. Takahashi, in prep.
- **Entropy production in QCD (classical Yang-Mills field)**
  - **Wigner and Husimi functionals**
  - **KS, HW, and decoherence entropy of CYM**
    T. Kunihiro, B. Müller, AO, A. Schäfer, T.T. Takahashi, A. Yamamoto, PRD82('10),114015.
    H.Iida, T.Kunihiro, B.Müller, AO, A.Schäfer, T.T.Takahashi, PRD88('13),094006.
    H.Iida, T.Kunihiro, AO, T. T. Takahashi, arXiv:1410.7309 [hep-ph].
    S. Tsutsui, H. Iida, T. Kunihiro, AO, arXiv:1411.3809.
    H. Tsukiji, H. Iida, T. Kunihiro, A. Ohnishi, T. T. Takahashi, in progress.
- **Summary**

Classical Yang-Mills Field

- **Yang-Mills action**

$$S_{\rm YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = \frac{1}{g^2} S_{\rm CYM}(A_{cl}) + \mathcal{O}(g^0) \quad (A_{cl} = \langle gA \rangle)$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu = \frac{1}{g}\left[\partial_\mu (gA)^a_\nu - \partial_\nu (gA)^a_\mu + f^{abc} (gA)^b_\mu (gA)^c_\nu\right]$$

- **CYM Hamiltonian in temporal gauge (A_0=0)**

$$H = \frac{1}{2} \sum_{a,i,x} \left[E^a_i(x)^2 + B^a_i(x)^2 \right], \quad B^a_i(x) = \varepsilon_{ijk} F^a_{jk}(x)/2$$

$$\frac{dA^a_i(x)}{dt} = E^a_i(x), \quad \frac{dE^a_i(x)}{dt} = -\frac{\partial H}{\partial A^a_i(x)}$$

- **Wigner functional and Husimi functional**

S. Mrowczynski and B. M¨uller, PRD 50('94)7542.
T. Kunihiro, B. Muller, A. Schafer, A. Ohnishi, PTP 121('09)555.

$$f_{\rm W}[A, E] = \int ds\ e^{iEs/\hbar} \langle A - s/2 \mid \rho \mid A + s/2 \rangle$$

$$f_{\rm H}[A, E] = \int \frac{dA' dE'}{\pi \hbar} e^{-\Delta(A-A')^2/\hbar - (E-E')^2/\Delta\hbar} f_{\rm W}[A', E']$$

How to obtain Lyapunov exponents

- **Kolmogorov-Sinai entropy rate h_{KS} = Entropy production rate**

$$\frac{dS}{dt} = h_{\rm KS}, \quad h_{\rm KS} = \sum_{\lambda_i > 0} \lambda_i, \quad |\delta X_i(t)| = e^{\lambda_i t} |\delta X_i(0)|$$

λ_i = Lyapunov exponent

- **EOM of δX → Integral (Trotter formula)**

$$\dot{X} = \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} H_p \\ -H_x \end{pmatrix} \rightarrow \delta \dot{X} = \begin{pmatrix} H_{px} & H_{pp} \\ -H_{xx} & -H_{xp} \end{pmatrix} \delta X \equiv \tilde{H} \delta X$$

$$\left(H_{px} \equiv \partial^2 H / \partial p \partial x \quad \text{etc} \right)$$

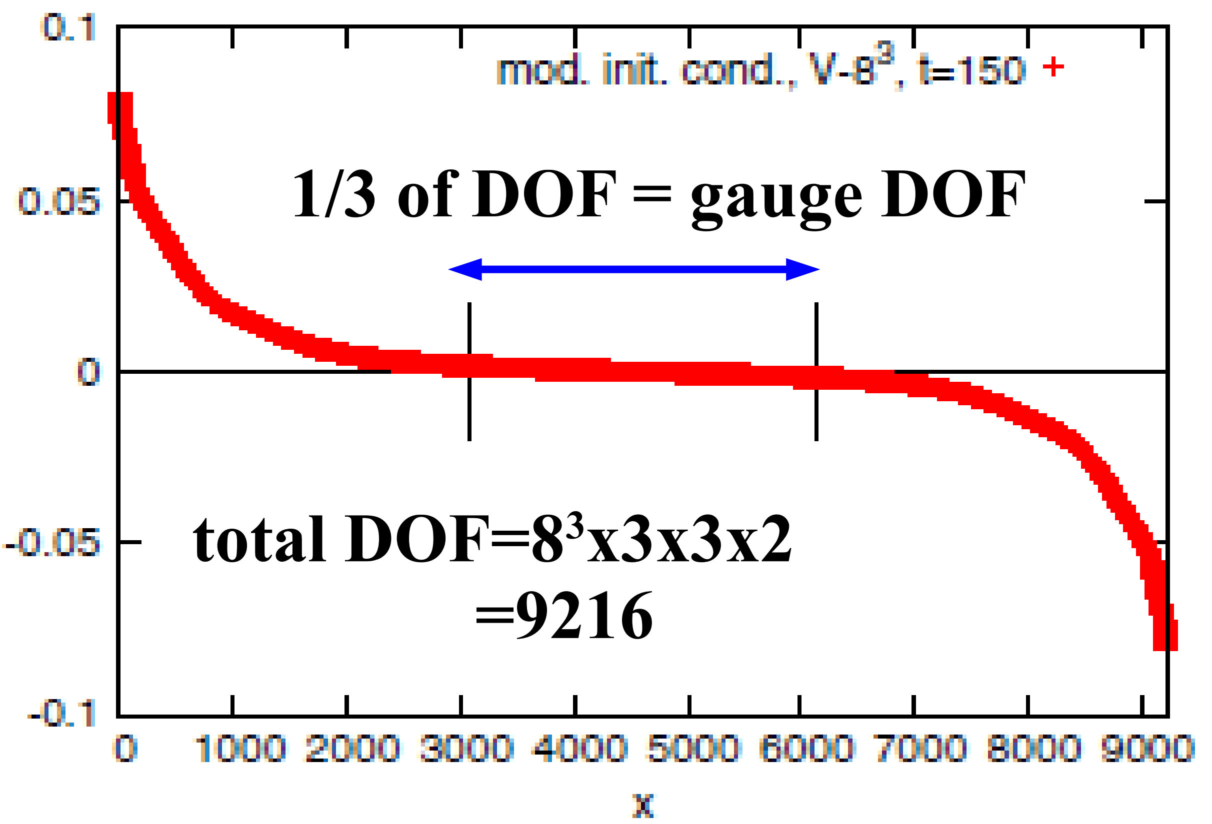
$$\delta X(t) = T \exp\left(\int_0^t dt' \tilde{H}(t') \right) \delta X(t=0) \simeq T \prod_{k=1,N} (1 + \tilde{H} \Delta t) \delta X(t=0)$$

$$= U(0,t) \delta X(t=0)$$

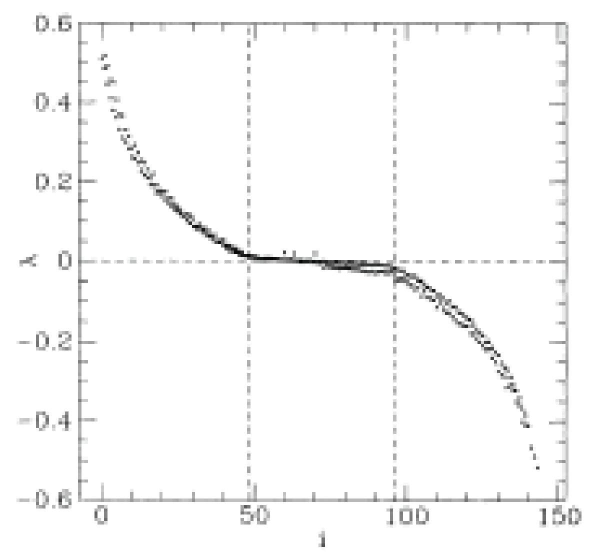
  - **Diagonalizing U and the eigen value becomes λt.**
  - **Matrix size = 3 (xyz) x (N_c^2-1) x L^3 x 2 (A,E)**

Typical Lyapunov spectrum

- **Sum of all Lyapunov exponent = 0 (Liouville theorem)**
- **1/3 Positive, 1/3 negative, and 1/3 zero (or pure imag.)**



Iida, Kunihiro, B.Müller, AO, Schäfer, Takahashi ('13)



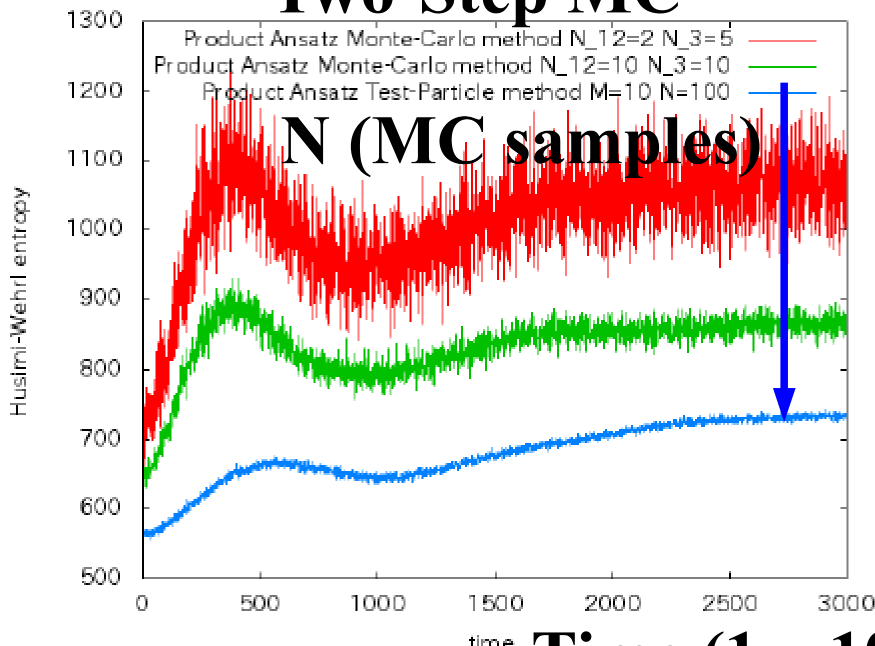
cf) Lyapunov spectrum ($V=2^3$)
Gong, Phys.Rev.D49, 2642 (1994).

Preliminary Results

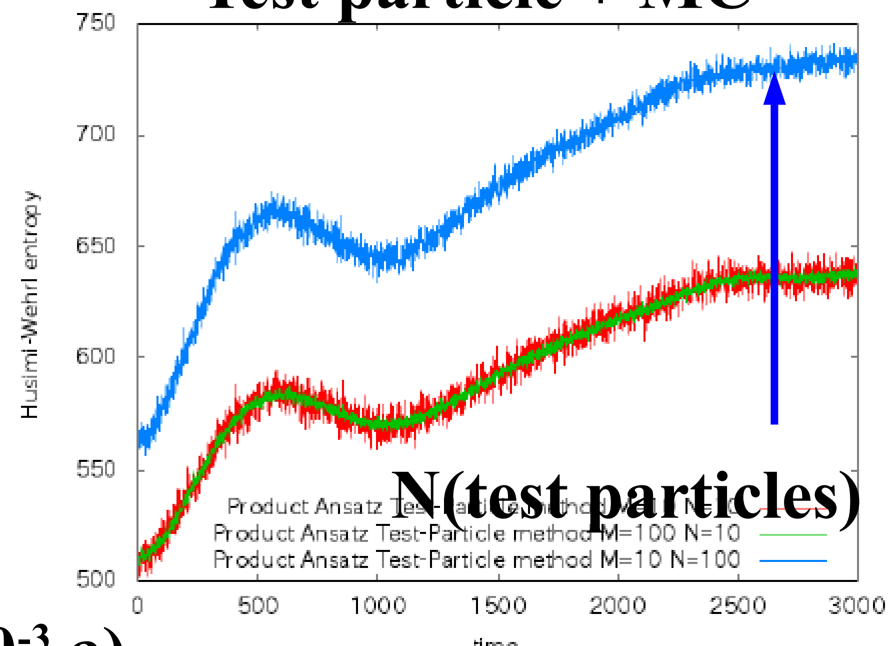
Tsukiji, Iida, Kunihiro, AO, Takahashi, in progress

- **Preliminary numerical results of SU(2) CYM on a 4^3 lattice**
  - **Initial cond. = min. wave packet (gaussian) → $S_{HW} \sim 576$**
  - **Entropy production is observed !**

Two-Step MC



Test particle + MC



Time (1 x 10^{-3} a)

Chaoticity of CYM

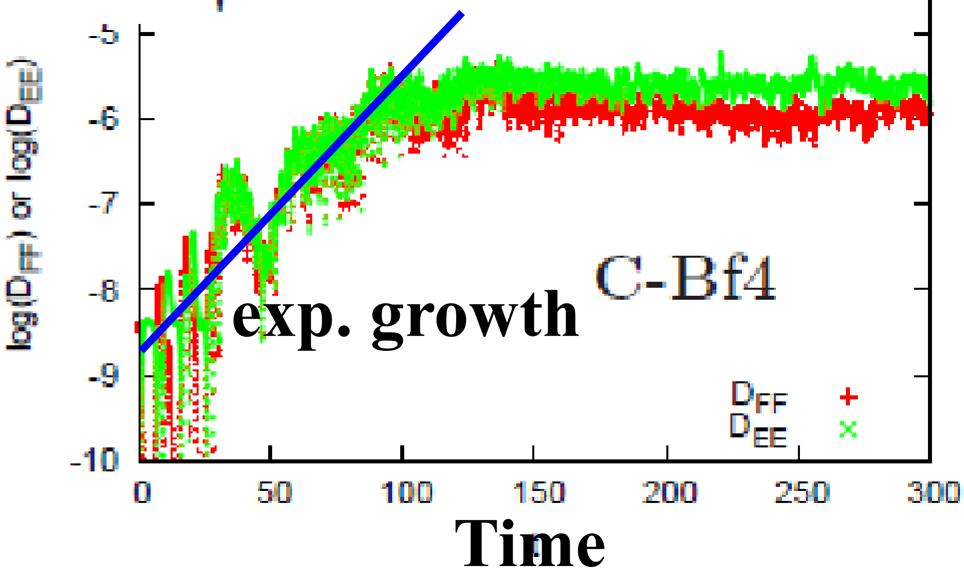
Kunihiro, Müller, AO, Schäfer, Takahashi, Yamamoto ('10)
Iida, Kunihiro, B.Müller, AO, Schäfer, Takahashi ('13)

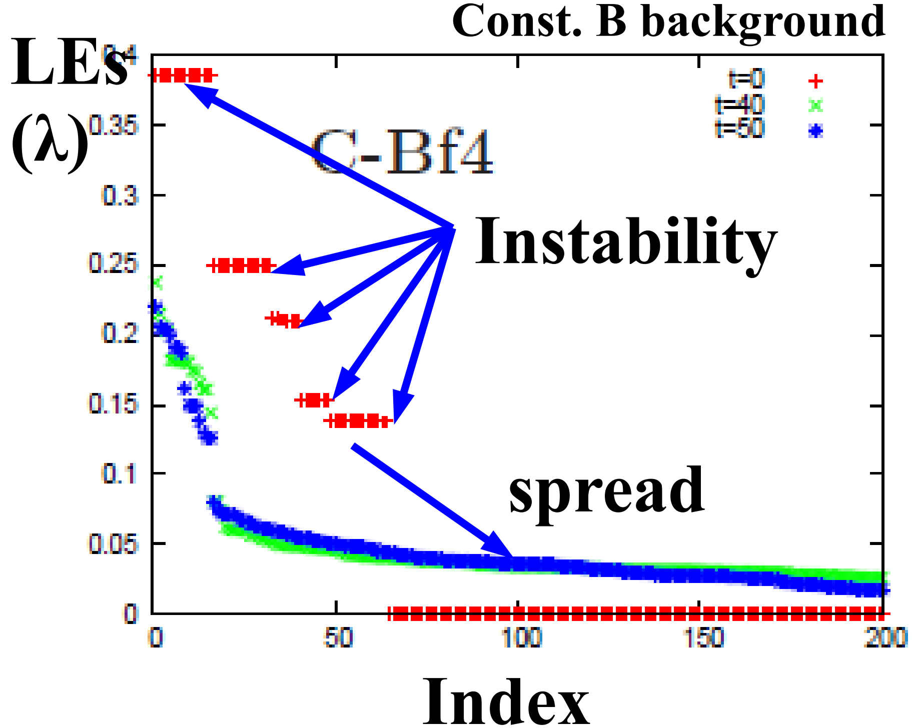
- **Chaoticity in CYM**

  T. S. Biro, S. G. Matinyan, B. Muller, Lect. Notes Phys. 56 ('94), 1; S. G. Matinyan, E. B. Prokhorenko, G. K. Savvidy, JETP Lett. 44 ('86) 138; NPB 298 ('88), 414; B. Muller, A. Trayanov, PRL 68 ('92), 3387; T. S. Biro, C. Gong, B. Muller, PRD 52 ('95),1260; C. Gong, PRD 49 ('94), 2642.

  - **Exponential growth of distance from adjacent init. cond.**
  - **Rapid spread of positive Lyapunov exponents**

$$D_{FF} = \sqrt{\sum_x \{\sum_{a,i,j} F^a_{ij}(x)^2 - \sum_{a,i,j} F'^a_{ij}(x)^2\}^2}$$





Conformal Property

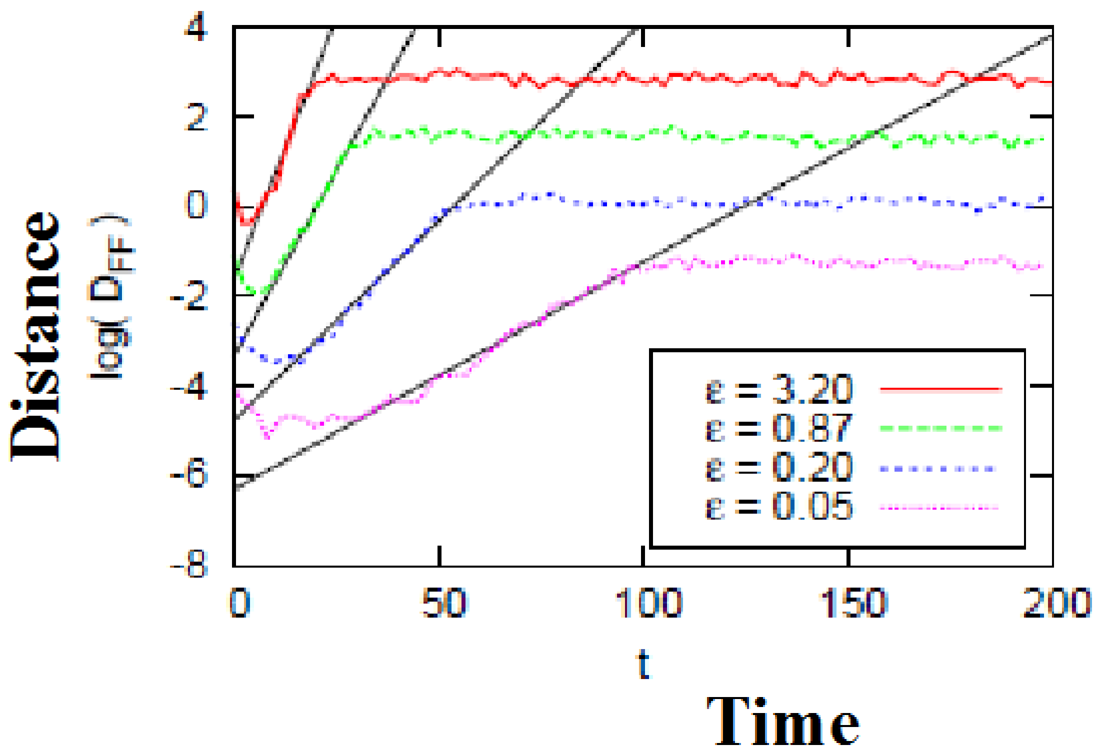
Kunihiro, Müller, AO, Schäfer, Takahashi, Yamamoto ('10)

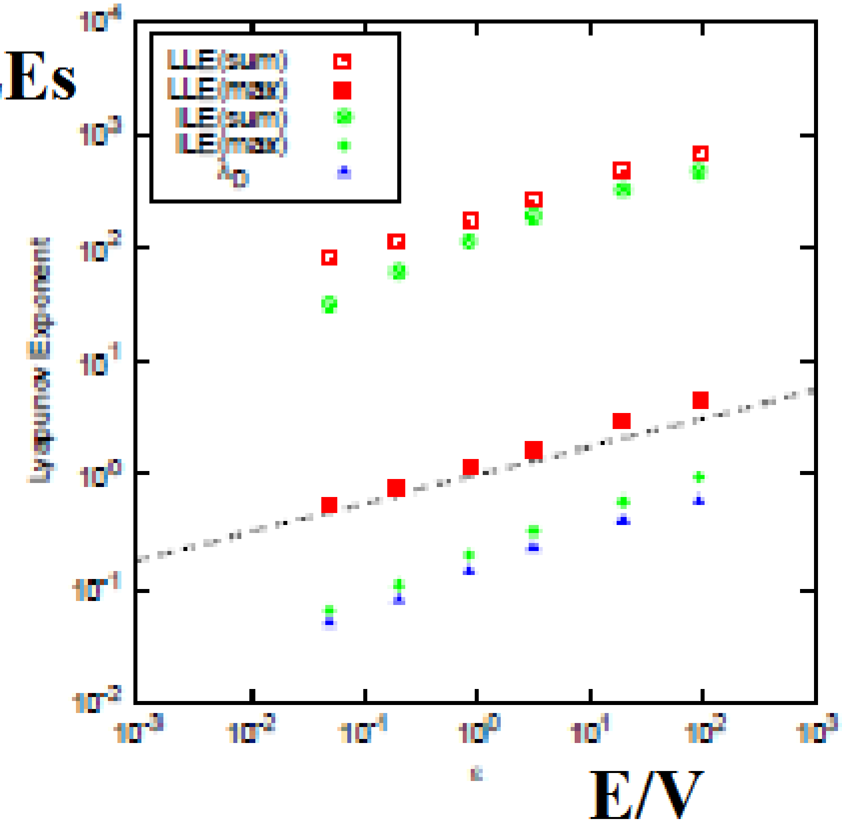
- **No conformal anomaly in CYM**
  → Any average quantity scales as $\varepsilon^{n/4}$ (ε: energy density, n: mass dim.)

$$\lambda^{\rm LLE}_{\rm sum}/L^3 \simeq 3\times\varepsilon^{1/4}\ ,\quad \lambda^{\rm LLE}_{\rm max} \simeq 1\times\varepsilon^{1/4}$$

$$\lambda^{\rm ILE}_{\rm sum}/L^3 \simeq 2\times\varepsilon^{1/4}\ ,\quad \lambda^{\rm ILE}_{\rm max} \simeq 0.2\times\varepsilon^{1/4}$$

  - **LLE: temporally local, ILE: integral during exp. growing period**





Husimi-Wehrl entropy of CYM

Tsukiji, Iida, Kunihiro, AO, Takahashi, in progress

- **Husimi-Wehrl entropy of CYM on the lattice**

$$S_{\rm HW} = -\int \frac{d^D A d^D E}{(2\pi\hbar)^D} f_H[A,E] \log f_H[A,E]$$

$$f_H[A,E] = \int \frac{d^D A' d^D E'}{\pi\hbar} e^{-\Delta(A-A')^2/\hbar-(E-E')^2/\Delta\hbar} f_W[A,E]$$

  - **D=576 on 4^3 lattice for N_c=2 → 1152 dim. integral, average exponent ~ D (problem with large deviation !)**

- **Hartree approximation**

$$f_H[A,E] \simeq \prod_{x,i,a} f_H^{iax}(A,E)$$

$$\to S_{\rm HW} = -\sum_{x,i,a} \int \frac{dAdE}{2\pi\hbar} f_H^{iax}(A,E) \log f_H^{iax}(A,E)$$

  - **Hartree approx. gives error of 10-20 % in HW entropy for 2d quantum mech.**

Husimi Function

Kunihiro, Muller, Schafer, AO ('09)

- **A simple example with instability**
  Inverted Harmonic Oscillator

$$H = \frac{p^2}{2} - \frac{\lambda^2}{2}x^2$$

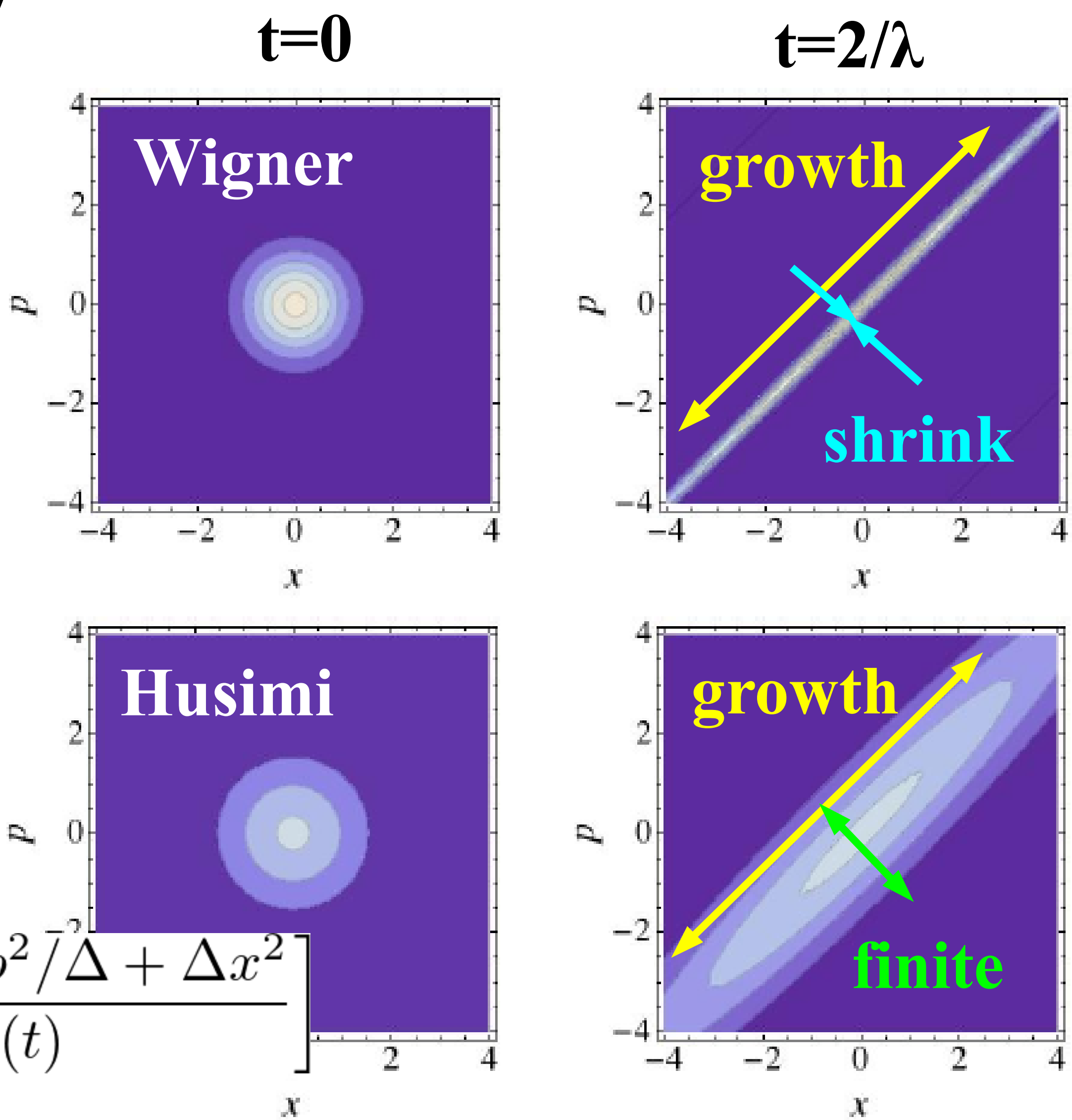
  - **exponential growth / shrink**

$$\dot{x} = p,\ \ \dot{p} = \lambda^2 x$$
$$\to p \pm \lambda x = \exp(\pm\lambda t)(p_0 \pm \lambda x_0)$$

  - **Wigner function**

$$f_{\rm W}(x,p,t) = 2\exp[-K(x,p,t)/\hbar]$$
$$K = \omega x_0^2 + p_0^2/\omega$$

  - **Husimi function**



$$f_{\rm H}(x,p,t) = \frac{2}{A(t)}\exp\left[-\frac{K(x,p,t)+p^2/\Delta+\Delta x^2}{\hbar A^2(t)}\right]$$

$$A(t) = \sqrt{2(\sigma\rho\cosh 2\lambda t + 1 + \delta\delta')}$$ **~ exp(λt)**

$$\sigma = (\lambda^2+\omega^2)/2\lambda\omega > 1, \delta = (\lambda^2-\omega^2)/2\lambda\omega, \rho = (\Delta^2+\lambda^2)/2\Delta\lambda > 1, \delta' = (\Delta^2-\lambda^2)/2\Delta\lambda$$

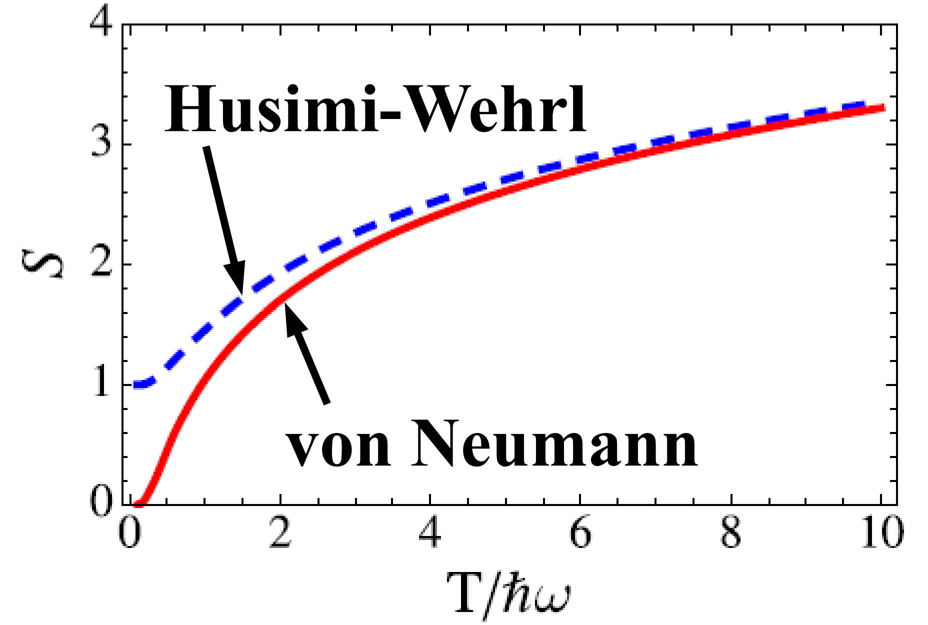
Husimi-Wehrl Entropy (1)

- **Husimi-Wehrl entropy = Wehrl entropy using Husimi function**
  Wehrl ('78), Husimi ('40), Anderson, Halliwell ('93), Kunihiro, Muller, AO, Schafer ('09).

$$S_{\mathrm{HW}} = -\int \frac{dqdp}{2\pi\hbar} f_{\mathrm{H}}(q,p) \log f_{\mathrm{H}}(q,p)$$

  - **Coarse grained entropy by minimum wave packet**

- **Harmonic oscillator in equilibrium**
  - **Min. value S_{HW}=1 (1 dim.) from smearing**
    Lieb ('78), Wehrl ('79)
  - **Husimi-Wehrl = von Neumann at high T (T/ ħω >> 1)**
    Anderson, Halliwell ('93),
    Kunihiro, Muller, AO, Schafer ('09).



Husimi-Wehrl Entropy (2)

- **Inverted Harmonic Oscillator**

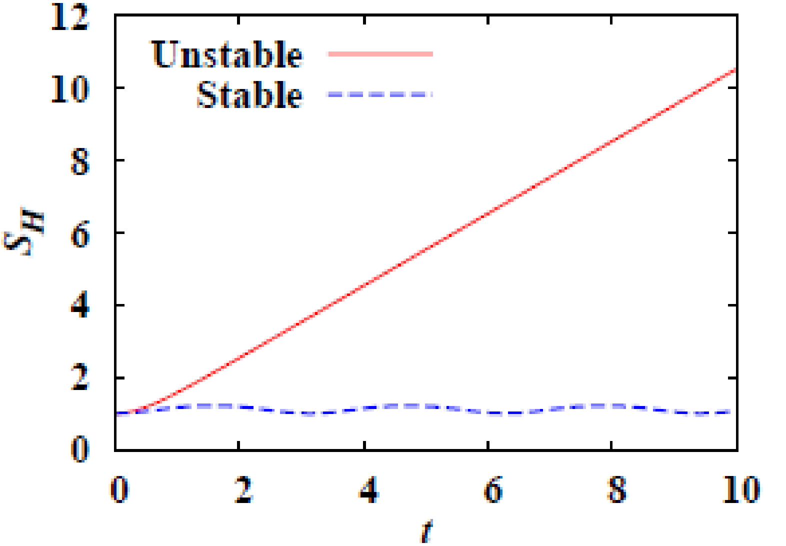
$$A(t) = \sqrt{2(\sigma\rho\cosh 2\lambda t + 1 + \delta\delta')}$$ **~ exp(λt), λ=Lyapunov exp.**

$$S_{\rm HW} = \log\frac{A(t)}{2} + 1\ , \quad \frac{dS_{\rm HW}}{dt} \to \lambda\ (t\to\infty)$$ **independent of Δ**

- **Many Harmonic & Inverted Harmonic Oscillators**

$$H = \sum_k \left(\frac{p_k^2}{2} - \frac{\lambda_k^2}{2}x_k^2\right) + \sum_i \left(\frac{p_i^2}{2} + \frac{\omega_i^2}{2}x_i^2\right)$$

$$\frac{dS_{\rm HW}}{dt} \to \sum_k \lambda_k\ (t\to\infty)$$



Classical unstable modes plays an essential role in entropy production at quantum level.

Entropy production in quantum systems

- **Entropy in quantum mech.**
  - **Time evolution is unitary, then the von Neumann entropy is const.**

$$|\,\psi(t)\rangle = \exp(-iHt/\hbar)\,|\,\psi(0)\rangle$$

$$\rho = |\,\psi\rangle\langle\psi\,| \rightarrow |\,\psi(t)\rangle\langle\psi(t)\,|$$

$$S_{\mathrm{vN}} = -\mathrm{Tr}\left[\rho\log\rho\right] \rightarrow \mathrm{const.}$$

- **Two ways of entropy production at the quantum level**
  - **Entanglement entropy**

$$\rho_{\mathrm{S}} = \mathrm{Tr}_{\mathrm{E}}\left(\rho\right) \rightarrow S_{\mathrm{S}} = -\mathrm{Tr}\left(\rho_{\mathrm{S}}\log\rho_{\mathrm{S}}\right) > 0$$

Partial trace over environment → Loss of info. → entropy production

  - **Coarse grained entropy**

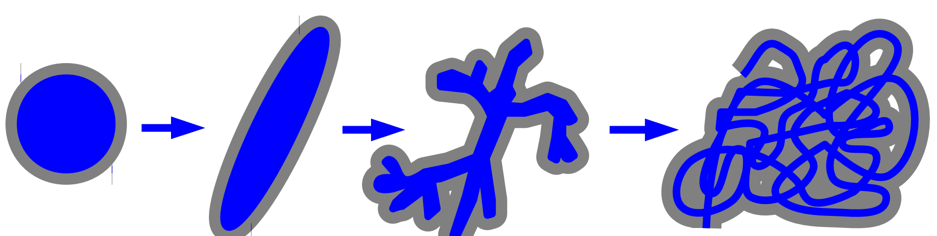
$$\rho \rightarrow \rho_z(\text{coarse grained}) \rightarrow S = -\int dz\ \rho_z \log\rho_z > 0$$

Coarse graining (粗視化) → entropy production
Yes, we can define it even in isolated systems such as HIC and early univ.!

Coarse graining in quantum mechanics

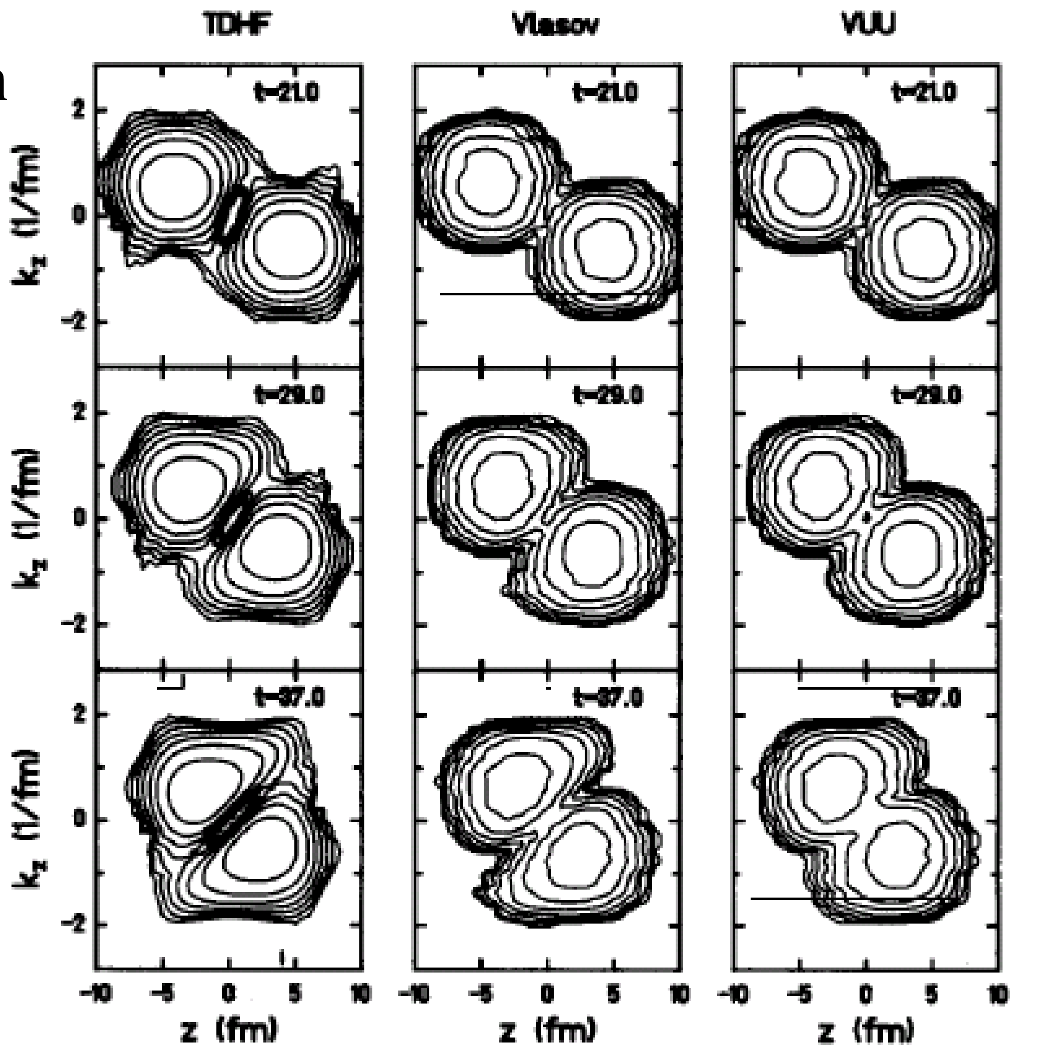
- **Wehrl entropy (Wehrl, 1978)** $$S_W = -\int \frac{dqdp}{2\pi\hbar} f(q,p) \log f(q,p)$$
- **Wigner function (Wigner, 1932)**
$$f_W(r,p) = \int ds\ e^{ips/\hbar} \langle r - s/2 \mid \rho \mid r + s/2 \rangle$$
  - **Quasi phase space dist. fn., but it can be negative.**
  - **Constant along the classical trajectory in semi-classical approx.**
    → No entropy prod.
$$\partial f_W/\partial t + v \cdot \nabla f_W - \nabla U \cdot \nabla_p f_W = 0$$
- **Husimi function (Husimi, 1940)**
$$f_H(q,p) = \int \frac{dq'dp'}{\pi\hbar} e^{-\Delta(q-q')^2/\hbar - (p-p')^2/\Delta\hbar} f_W(q',p')$$
  - **Smeared with min. wave packet.**
  - **Exp. value under a coherent state.**

$$f_H = \langle z \mid \rho \mid z \rangle\ ,\quad z = (\Delta q + ip)/\sqrt{2\hbar\Delta}$$



Comarison of Semi-classical & Quantum Evolution

- **Comparison of the evolution in**
  Time-Dependent Hartree-Fock (TDHF, ~ TDDFT)
  and
  Vlasov Eq. (semi-classical)
  for a low energy heavy-ion collision
  Ca+Ca, 40 A MeV
  Cassing, Metag, Mosel, Niita,
  Phys. Rep. 188 (1990) 363.



Separation in phase space leads
to acceleration of nuclei
(deep inter-nuclear potential)
e.g. AO, Horiuchi, Wada ('90).

Contents

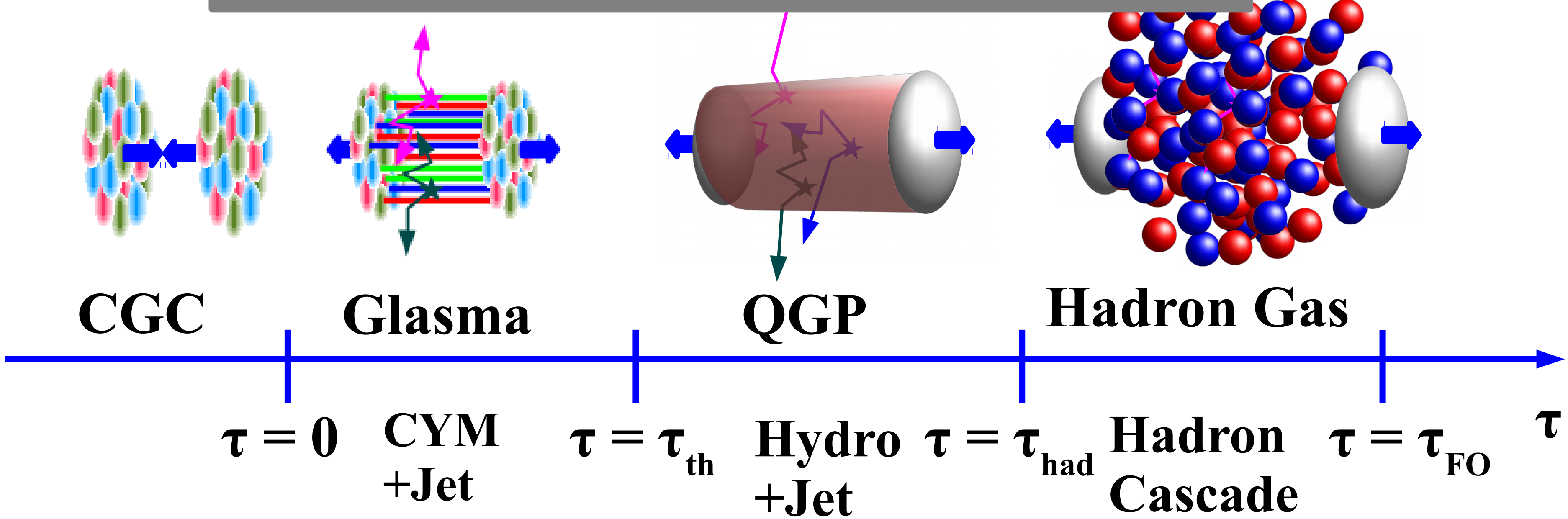
- Introduction
  - RHIC における2つの驚き
-

Husimi-Wehrl Entropy

RHIC における2つの驚き (cont.)

- **2つの驚き (2): 早い熱平衡化**
  - **摂動論的 QCD の予言 (2-5 fm/c) に比べて有意に早い時刻 (0.6-1 fm/c) で熱化が起こり、流体力学的時間発展が進む。**

なぜ早い熱化のが起こるのか？
→ 古典ヤンミルズ場が原因ではないか。



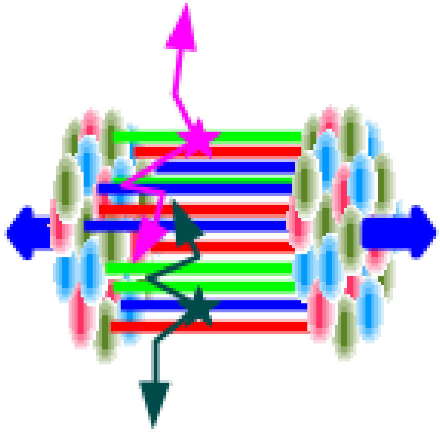
古典ヤンミルズ場

- ヤンミルズ場

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} \;,\; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

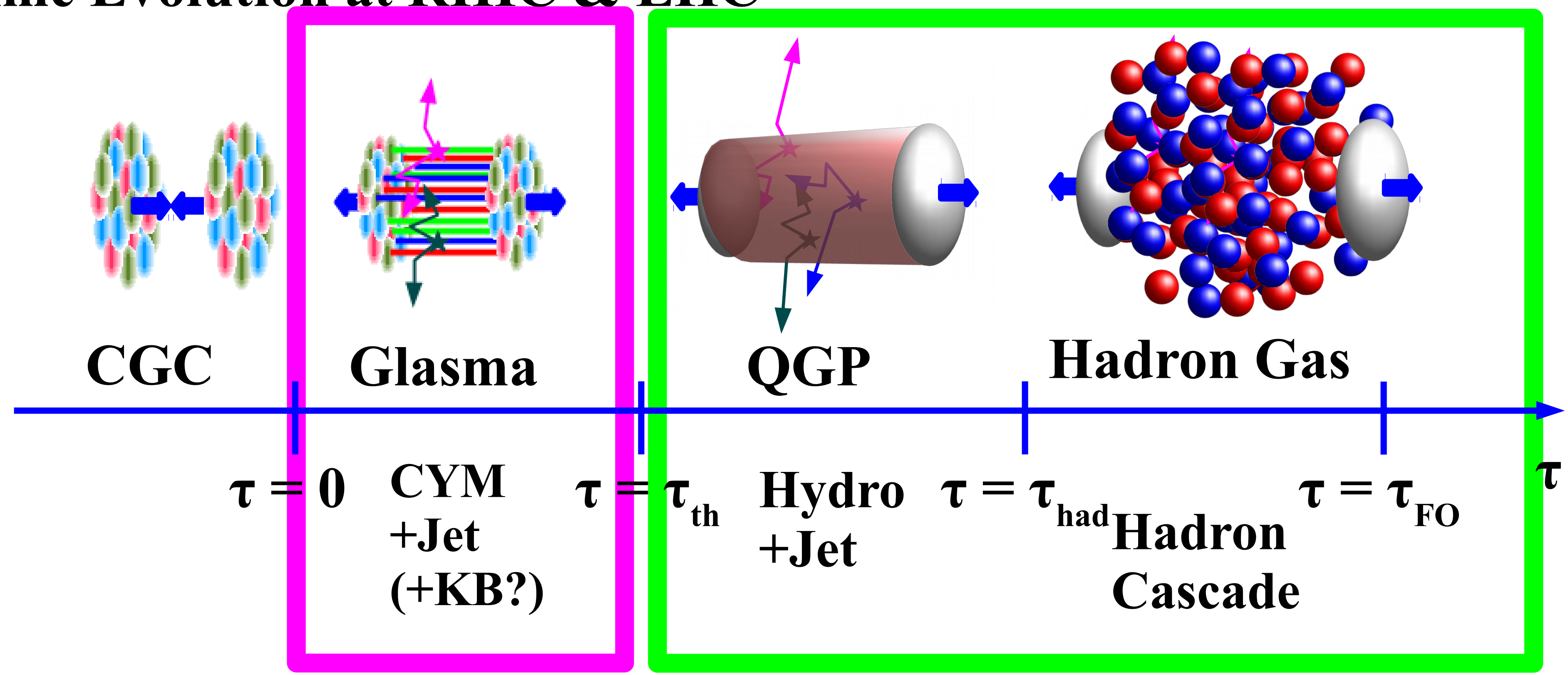
$$S = \frac{1}{g^2} S[A_{\rm cl}] + S_{\rm free}[a, A_{\rm cl}]$$

-



Thermalization in High-Energy Heavy-Ion Collisions

- **Time Evolution at RHIC & LHC**



Theor. Challenges

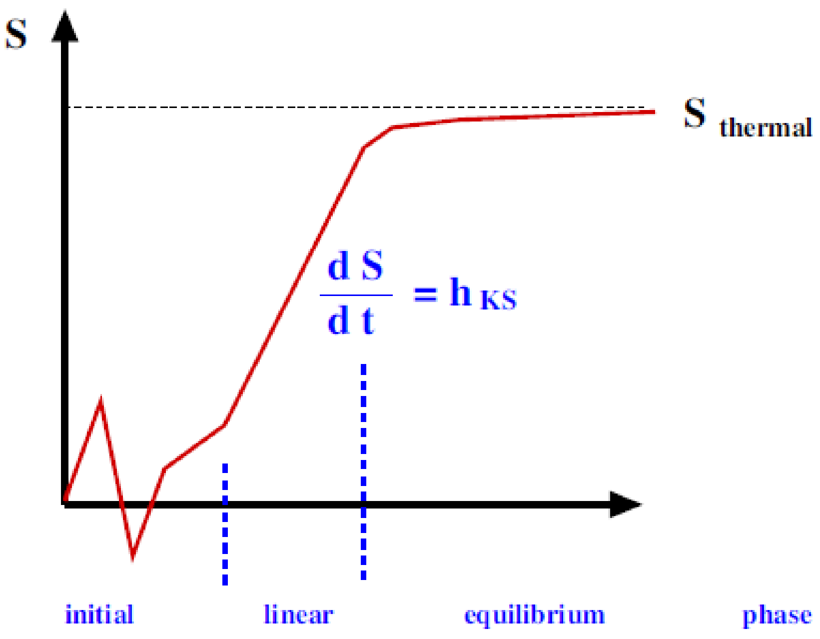
- **Thermalization under dynamical classical field**
- **Theoretically interesting and Phenomenologically important. dN/ dη, init. cond. of hydro.**

Phen. Challenges

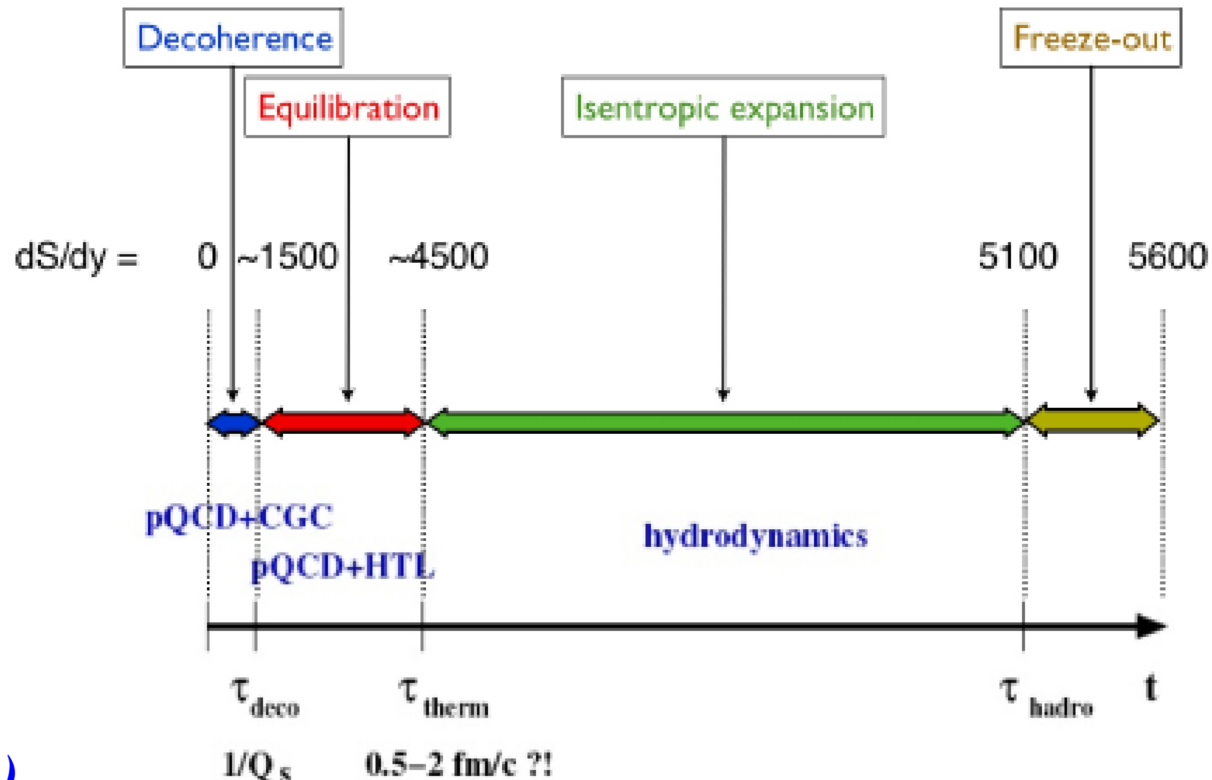
flow, jet, hard probes
→ hydro., transport coef., E-loss, hadron prop., phase diagram, ...

Entropy Production in Glasma

- **Huge entropy must be produced before QGP formation !**
  - **Thermalization time ~ (0.5-2.0) fm/c**
  - **Instability ? Rapid glasma decay ? Entropy of classical field ?**



B.Muller and A. Schaefer,
Int. J. Mod. Phys. E20, 2235 (2011)



R.J. Fries et al, arXiv 0906.5293

We discuss the CYM entropy and its production rate with emphasis on the chaoticity

Contents

- Introduction
- Entropy production in quantum mechanics
  - Chaoticity, Lyapunov exponent, and Kolmogorov-Sinaii entropy
  - Coarse graining and Husimi-Wehrl entropy
  - KS and HW entropy in quantum mechanics

    T. Kunihiro, B. Müller, A. Ohnishi, A. Schäfer, PTP 121 ('09), 555.
    H. Tsukiji, H. Iida, T. Kunihiro, A. Ohnishi, T. T. Takahashi, in prep.
- Entropy production in QCD (classical Yang-Mills field)
  - Wigner and Husimi functionals
  - KS, HW, and decoherence entropy of CYM

    T. Kunihiro, B. Müller, AO, A. Schäfer, T.T. Takahashi, A. Yamamoto, PRD82('10),114015.
    H.Iida, T.Kunihiro, B.Müller, AO, A.Schäfer, T.T.Takahashi, PRD88('13),094006.
    H.Iida, T.Kunihiro, AO, T. T. Takahashi, arXiv:1410.7309 [hep-ph].
    S. Tsutsui, H. Iida, T. Kunihiro, AO, arXiv:1411.3809.
    H. Tsukiji, H. Iida, T. Kunihiro, A. Ohnishi, T. T. Takahashi, in progress.
- Summary

QGP での本当の turbulence については
浅川さん、福嶋さんへ

Decoherence Entropy

Muller, Schafer ('03,'06), Fries, Muller, Schafer ('09), Iida, Kunihiro, AO, Takahashi ('14)

- **Coherent State**

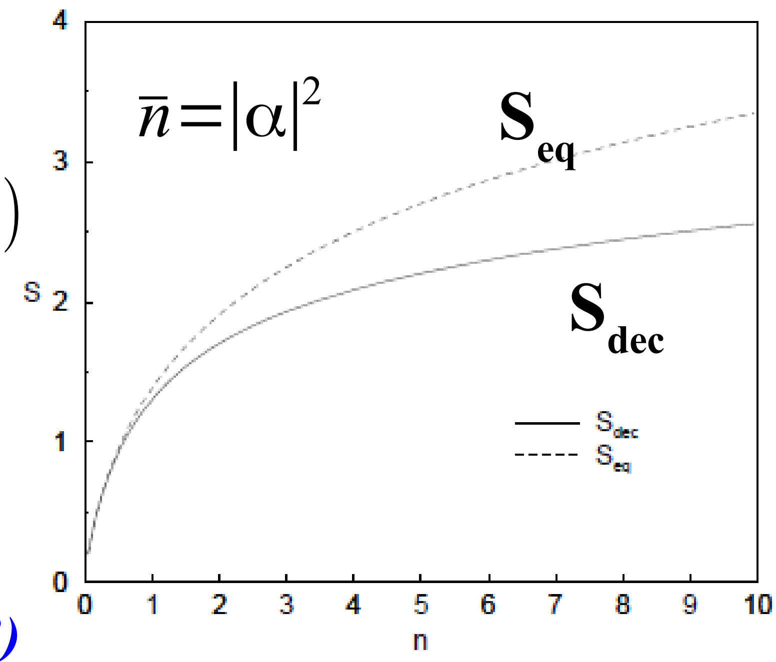
$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

$$|\alpha\rangle = N\exp(\alpha\hat{a}^{+})|0\rangle = \exp(-|\alpha|^2/2)\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

- **n-quanta states are coherently superposed in a coherent state.**
- **When this coherence is broken, entropy is generated (decoherence entropy)**

$$P_n = \frac{|\alpha|^{2n}}{n!}\exp(-|\alpha|^2) \quad (\text{Poisson dist.})$$

$$\rightarrow \quad S_{\text{dec}} = -\sum_{n=0}^{\infty} P_n \log P_n > 0$$



Muller, Schafer ('03)

CYM as a Coherent State

Muller, Schafer ('03,'06), Fries, Muller, Schafer ('09), Iida, Kunihiro, AO, Takahashi ('14)

- **What kind of state does the CYM correspond to ?**
  → Natural guess = Coherent State

$$|\,\mathrm{CYM}> \simeq \prod_{\boldsymbol{k},a,i} |\alpha_{\boldsymbol{k}ai}>$$

- **Decoherence entropy from CYM**

$$S_{\mathrm{dec}} = -\sum_{\boldsymbol{k},a,i} \sum_{n} P_n(\alpha_{\boldsymbol{k}ai}) \log P_n(\alpha_{\boldsymbol{k}ai})$$

$$\alpha_{\boldsymbol{k}ai} = \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} \left[\omega_{\boldsymbol{k}} A_{ai}(\boldsymbol{k},t) + i\, E_{ai}(\boldsymbol{k},t) \right], \quad \omega_{\boldsymbol{k}} = \sqrt{\sin^2 k_x + \sin^2 k_y + \sin^2 k_z}$$

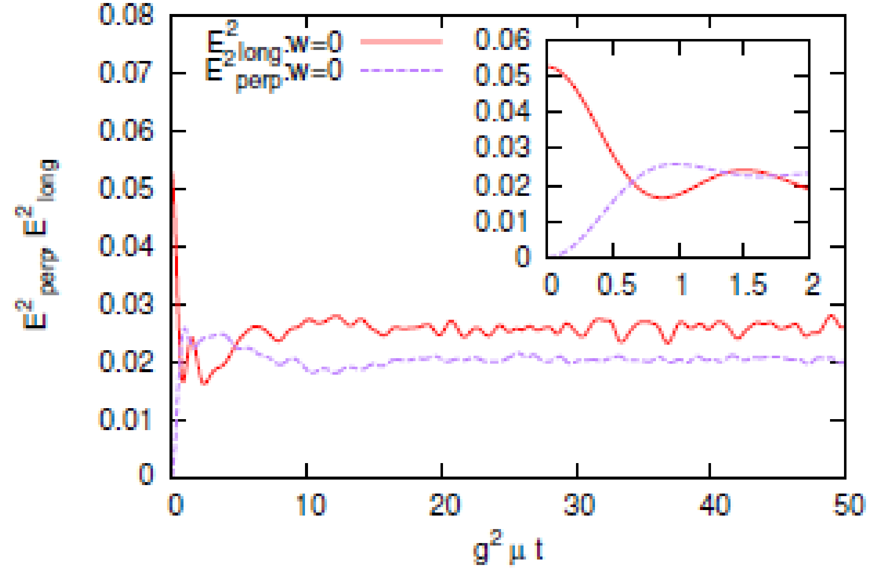
- **Is the above assignment unique ?**
  - **Coherent state in each "coherent domain"** *Fries, Muller, Schafer ('09)*
  - **Deviation from Poisson dist. with coupled oscillator**
    Glauber ('66), Gelis, Venugopalan ('06)

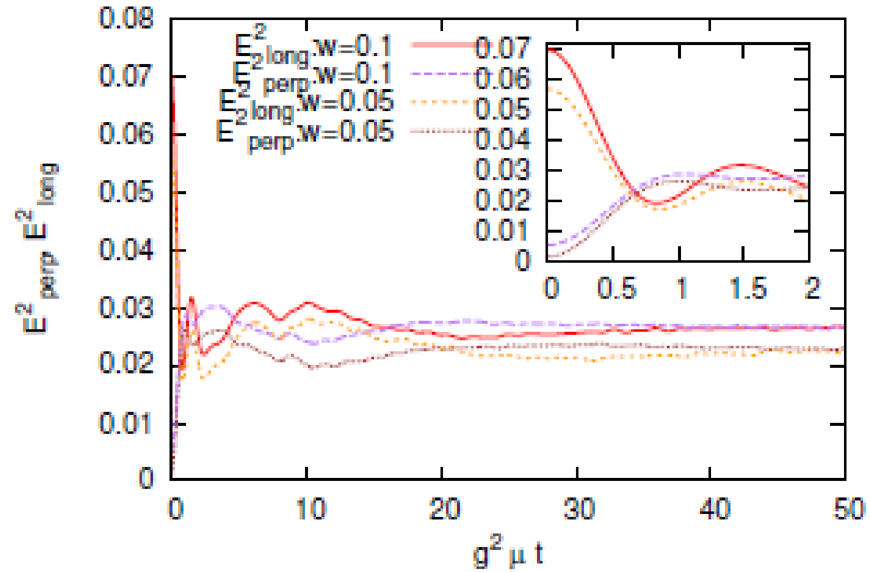
Initial Condition and Time Evolution

- **"Glasma-like" init. cond.**
  - **MV model (boost inv.)**
    + Longitudinal fluctuations

    → Bx,y, Ex,y, Bη, Eη
    McLerran, Venugopalan ('94), Romatschke, Venugopalan ('06), Fukushima, Gelis ('12)
  - **Non-expanding geometry is assumed, Substitute Bη and Eη in MV model into Bz and Ez at t=0.**
- **Time-evolution**
  - **Short time behavior of E^2 does not depend on the fluctuation strength. (and similar to expanding geo. results.)**
    E.g. Lappi, McLerran ('06)
  - **Long-time behavior:**
    Earlier "isotropization" in perp. and long. directions of E^2.

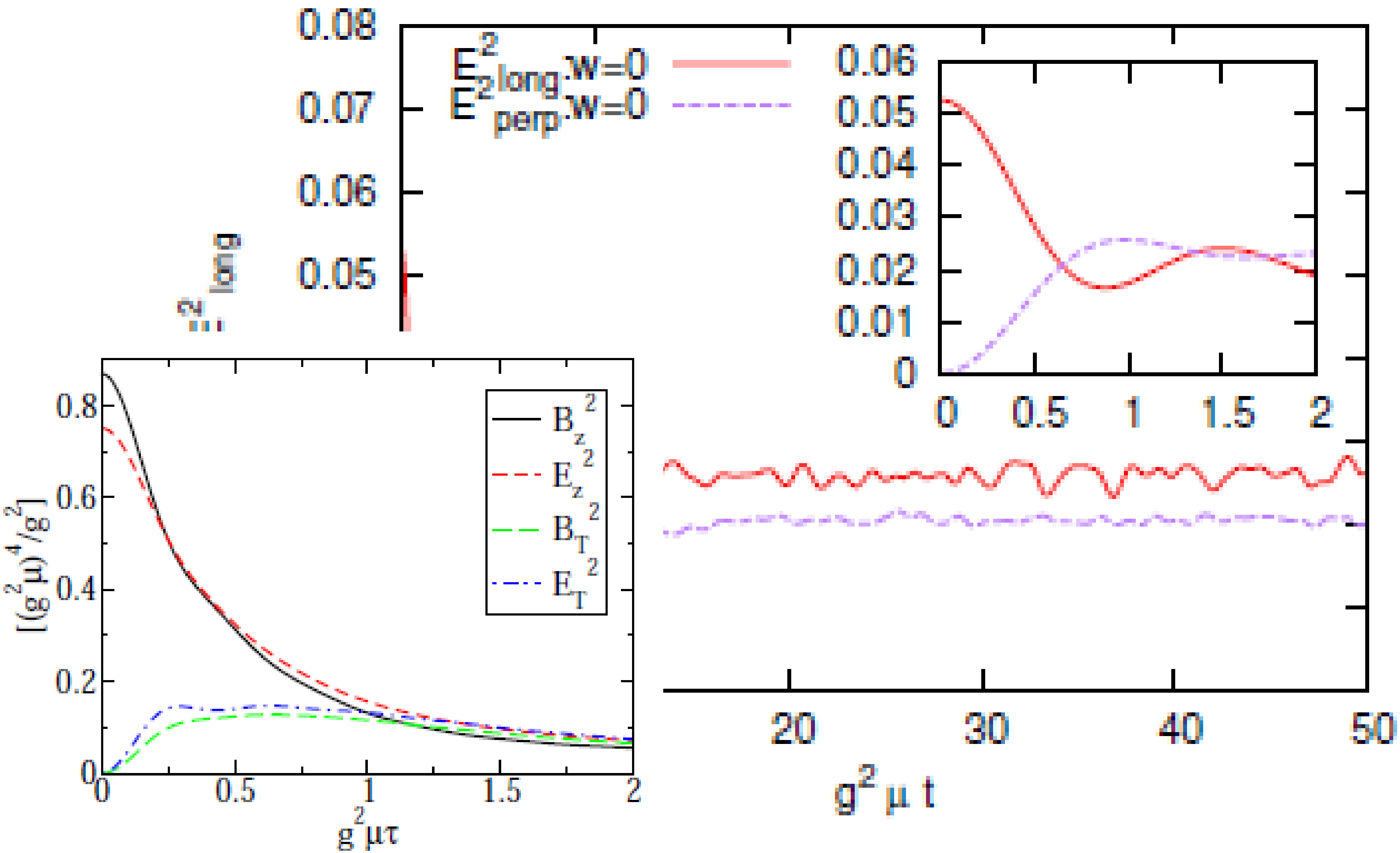
20^3 lattice





Iida, Kunihiro, AO, Takahashi ('14)

Initial Condition and Time Evolution



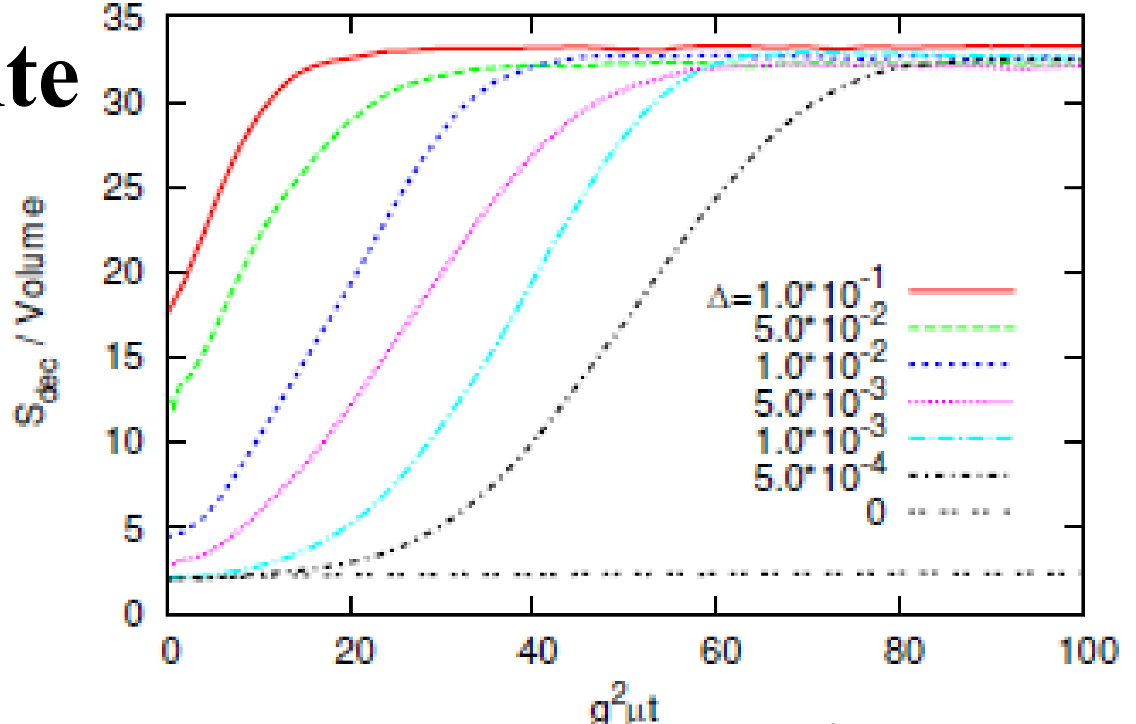
Lappi, McLerran ('06)

Iida, Kunihiro, AO, Takahashi ('14)

Decoherence Entropy of CYM

- **How about the decoherence entropy ?**
  - **< δE²>/<E²> ~ 0.1 (Δ=0.05) and 0.3 (Δ=0.1)**
  - **S_{dec} ~ 2.3 (Δ=0) and 33 (Δ=0.05, 0.1)**
  - **Entropy from initial state fluc. and chaoticity**
  - **No long. fluc. results in 2D (pz=0 mode) entropy, while 3D entropy is realized with finite long. fluc. (non-zero Δ).**

S_{dec}/site



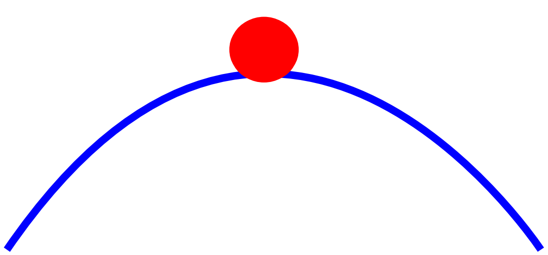
Time

Decoherence Entropy Production Rate

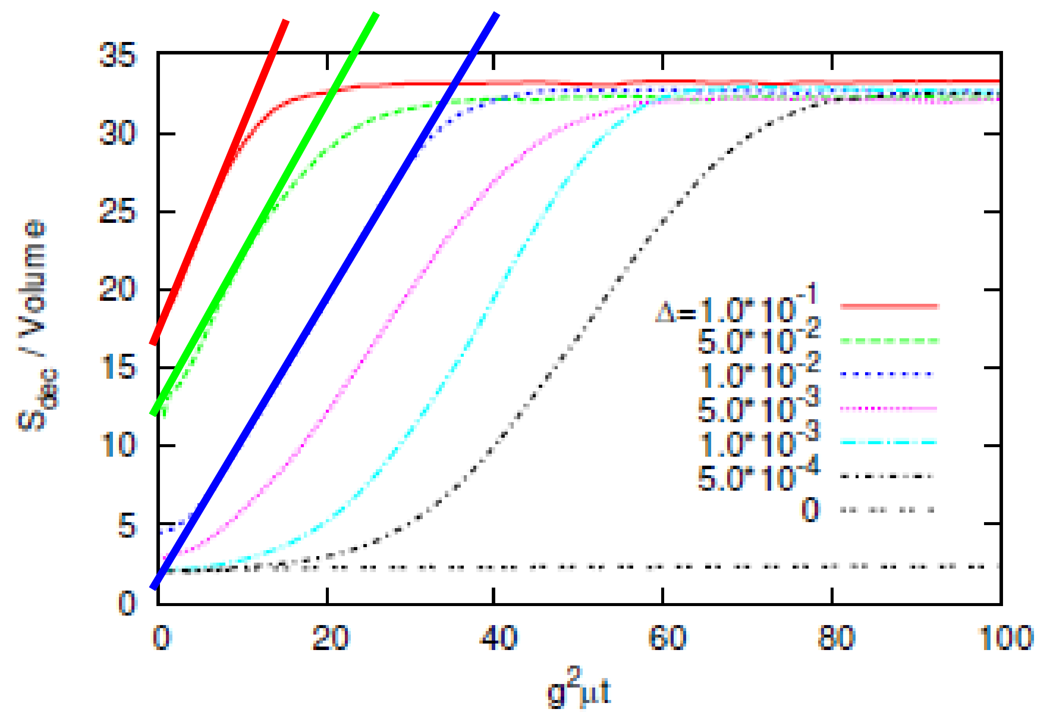
- **Decoherence entropy growth rate should be compared with KS entropy**
  - **$dS_{dec}/dt \sim 0.88$ (Δ=0.01), 1.05 (Δ=0.05), 1.36 (Δ=0.1)**
  - **KS entropy estimate: $S_{KS} \sim c_{KS}\, \varepsilon^{1/4}$, $c_{KS} \sim 2$ (conformal chaotic value)**
  - **Energy density: ε = 0.17 (Δ=0.01), 0.18 (Δ=0.05), 0.21 (Δ=0.1)**
    $\rightarrow c_{KS} = dS^{dec}/dt/\varepsilon^{1/4}$ = 1.4 (Δ=0.01), 1.6 (Δ=0.05), 2.0 (Δ=0.1)

$$\frac{1}{S_{\mathrm{KS}}}\frac{dS_{\mathrm{dec}}}{dt} \sim (0.7-1.0)$$

- **KS entropy**
  = Potentially realized
  growth rate

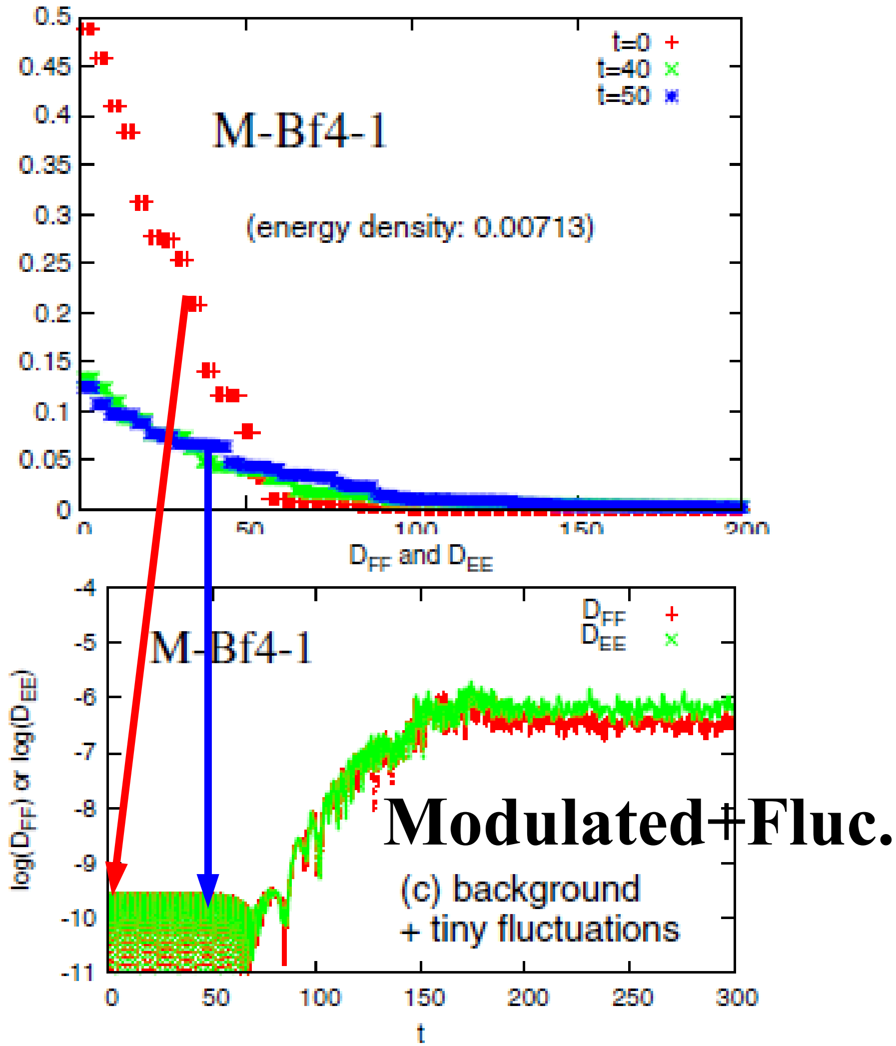


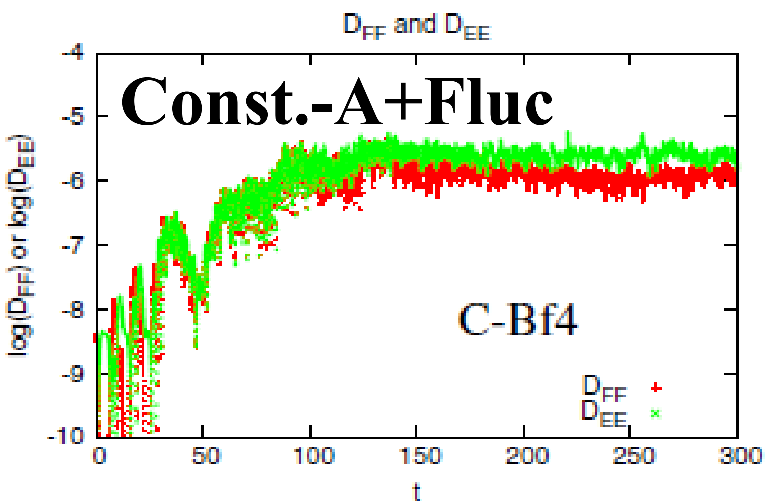
Δ=0: unstable
but stationary



KS entropy in CYM from glasma-like init. cond.

- **Instability under strong color-magnetic field**
  Nielsen, Olesen ('78), Fujii, Itakura ('08), Berges, Scheffler, Schlichting, Sexty ('12)
  - **No chaotic behavior is observed with sine waves and constant-A w/o fluctuations.**
  - **Small fluctuations activate instability and chaoticity.**
  - **Chaoticity emerges after instability spreads to many modes.**





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- *Chaotic behavior in classical Yang-Mills dynamics*, **T. Kunihiro, B. Müller, A. Ohnishi, A. Schäfer, T. T. Takahashi, A. Yamamoto, Phys. Rev. D 82 (2010), 114015 [arXiv:1008.1156].**
- *Entropy production in classical Yang-Mills theory from Glasma initial conditions*, **H. Iida, T. Kunihiro, B. Müller, A. Ohnishi, A. Schäfer, T. T. Takahashi, Phys. Rev. D 88 (2013), 094006 [arXiv:1304.1807].**
- *Time evolution of gluon coherent state and its von Neumann entropy in heavy-ion collisions*, **H. Iida, T. Kunihiro, A. Ohnishi, T. T. Takahashi, arXiv:1410.7309 [hep-ph].**
- *Parametric Instability of Classical Yang-Mills Fields under Color Magnetic Background*, **S. Tsutsui, H. Iida, T. Kunihiro, A. Ohnishi, arXiv:1411.3809.**

Chaoticity, Lyapunov exponent, and KS entropy

- **Entropy in classical dynamics = Wehrl entropy**

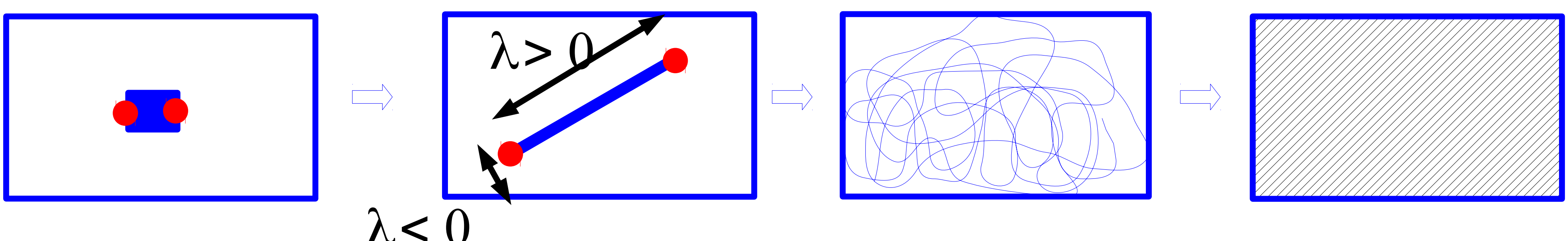
$$S = -\int d\Gamma\, H \log H$$

($d\Gamma$= dx dp = phase space, H= phase space dist. fn., e.g. Husimi fn.)

- **Lyapunov exponent and Kolmogorov-Sinaii entropy**

$$\delta X_i(t) = \delta X_i(t_0) \exp[\lambda_i (t - t_0)] \quad (X = (x, p)),$$
$$dS/dt = S_{\mathrm{KS}} \equiv \sum_{i, \lambda_i > 0} \lambda_i$$

  - **δ X = difference of two trajectories from adjacent initial conditions**
    λ = initial state sensitivity (Lyapunov exponent, measure of chaoticity)
  - **When λ > 0, exponentially growing number of phase space cells are visited**
    → phase space dist. fn. becomes smooth after proper coarse graining
    → entropy production (Kolmogorov-Sinaii entropy)



Classical Yang-Mills dynamics on the lattice

- **Lattice CYM Hamiltonian in temporal gauge (A_0=0) in the lattice unit**

$$H = \frac{1}{2} \sum_{x,a,i} \left[E_i^a(x)^2 + B_i^a(x)^2 \right]$$

$$F_{ij}^a(x) = \partial_i A_j^a(x) - \partial_j A_i^a(x) + \sum_{b,c} f^{abc} \boxed{A_i^b(x) A_j^c(x)} = \varepsilon_{ijk} B_k^a(x)$$

Non-linear & coupling

- **Non-compact (A, E) form !**
  - **Demerit: Gauge invariance is not fully satisfied at finite lattice spacing.**
  - **Merit: Easy to consider the coherent state, and conformality is manifest.**
- **Initial conditions ($E_i^a(x)$=0 is assumed here.)**
  - **Random initial condition: $A_i^a(x)$ = random in [-η, η],**
  - **Modulated init. cond.:** $A_i^a(\vec{r}) = \delta_{i2}\left[\epsilon_1 \sin\left(\frac{2x\pi}{N_x}\right) + \epsilon_2 \sin\left(\frac{2nx\pi}{N_x}\right)\sin\left(\frac{2mz\pi}{N_z}\right)\right]$
  - **Constant-A init. cond.** $A_i^a(x) = \sqrt{B/g}\left(\delta_{i2}\delta^{a3} + \delta_{i3}\delta^{a2}\right)$ *Berges et al.('12)*

magnetic field ~ z direction ($\varepsilon_1 >> \varepsilon_2$), w and w/o fluc.