# *Classical Field Dynamics with Quantum Statistical Correction via Euclidean Potential*


**Akira Ohnishi (YITP, Kyoto U.)**

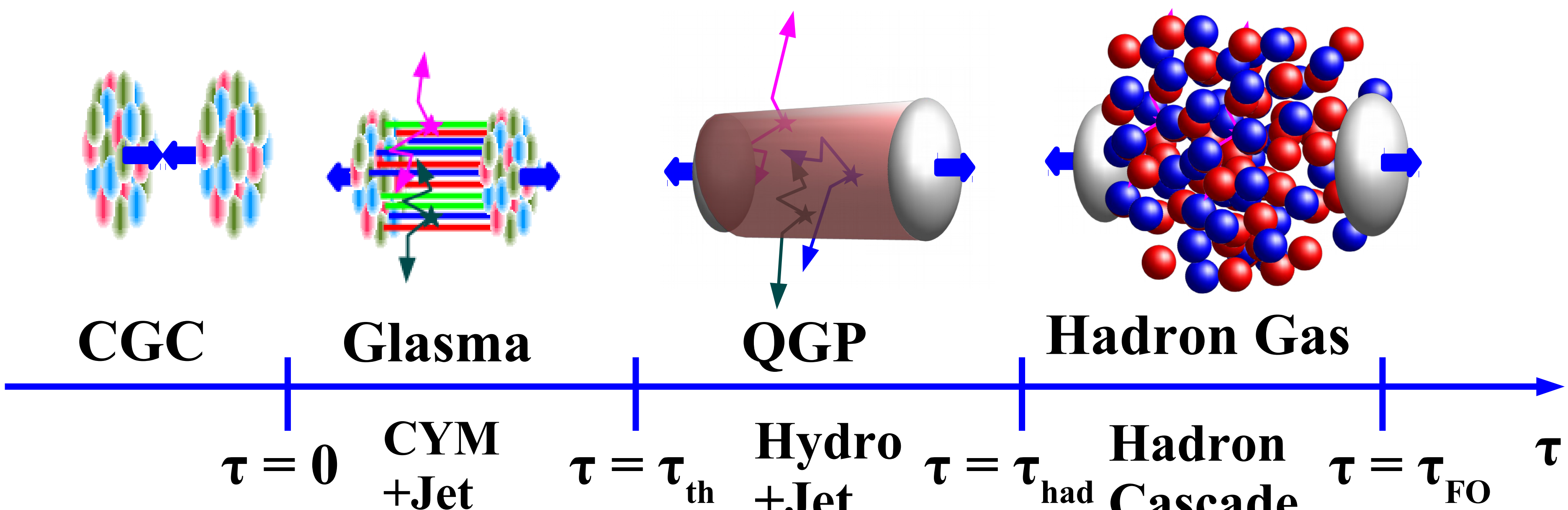
**based on the work in progress in collaboration with**

**H. Matsuda (Kyoto U.), T. Kunihiro (RCNP/YITP), and T. T. Takahashi (Gunma NCT)**


*QH Seminar, Dec. 13, 2019*

# *Two Surprises at RHIC*

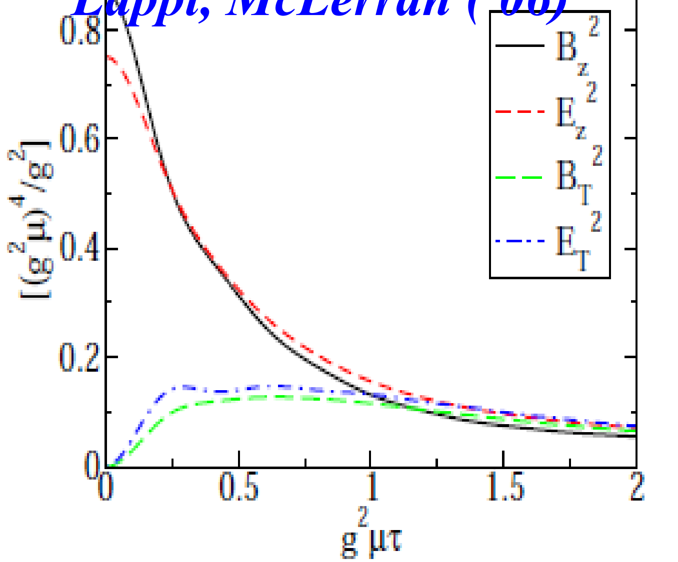
- **Surprise (1): Strongly Coupled QGP**
  - **Success of ideal hydrodynamics implies small viscosity ($\eta/s \sim 1/4\pi$)**
    **$\rightarrow$ g should be small, but mean free path is short !**
- **Surprise (2): Early Thermalization**
  - **Hydro. suggests early thermalization ($\tau_{th} \sim$ (0.6-1.0) fm/c).**
    **$\rightarrow$ Significantly short thermalization time than pQCD ($\tau_{th} \sim$ (2-5) fm/c).**
- **Possible explanation: Classical Yang-Mills field dynamics**
  - **Instability enhances classical field, which decays to gluons in early stage.**



# *Classical Yang-Mills Field (1)*

- **Isotropization**
  - **Pressure isotropization takes place at early times, $g^2\mu\tau \sim 1$.**
    *McLerran, Venugopalan ('94), Romatschke, Venugopalan ('06), Lappi, McLerran ('06), Berges, Scheffler, Sexty ('08), Fukushima ('11), Fukushima, Gelis ('12), Epelbaum, Gelis ('13)*

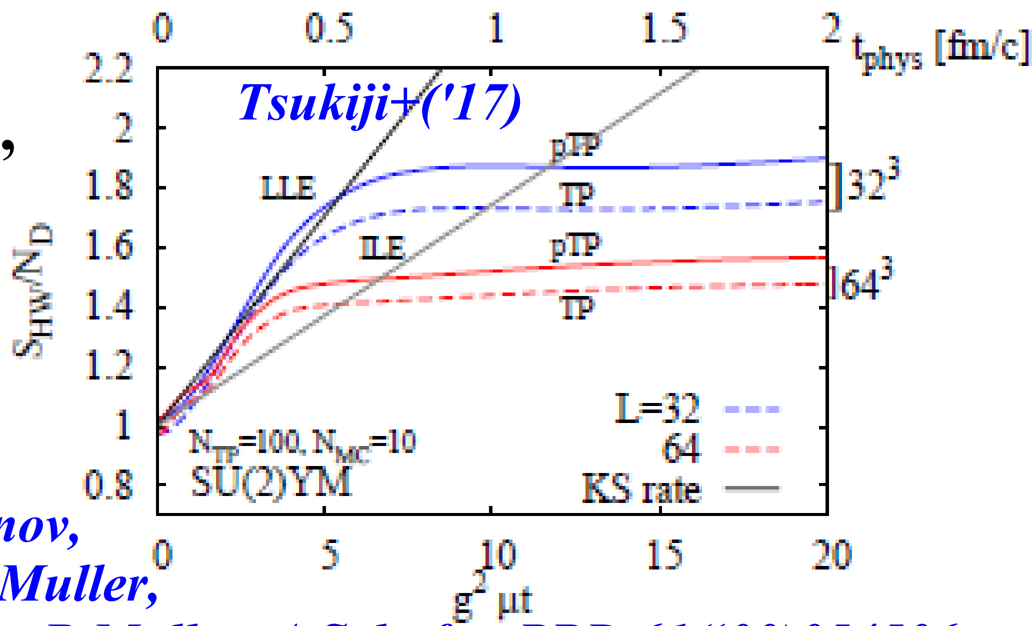
*Lappi, McLerran ('06)*



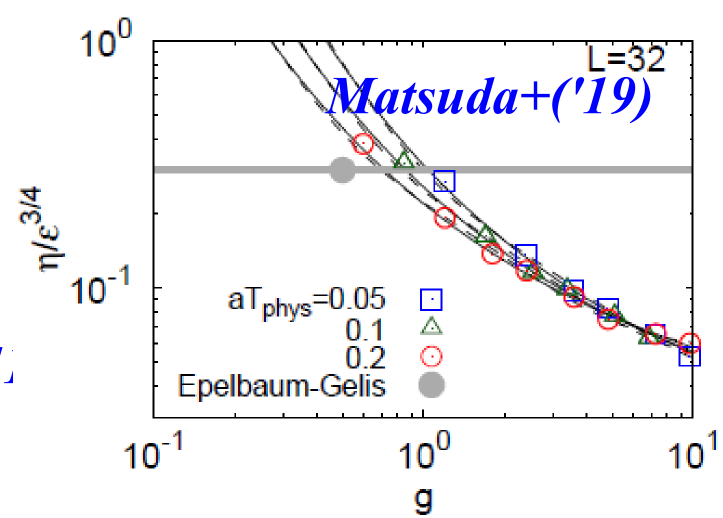

- **Instability**
  - **YM field amplitude can grow quickly.**
    **Weibel:** *E.S.Weibel, PRL 2 ('59),83; S. Mrowczynski, PLB 214 ('88)587; J. Randrup, S. Mrowczynski, PRC68 (2003) 034909;*
    **Nielsen-Olesen:** *N. Nielsen, P. Olesen, NPB 144 ('78)376; H. Fujii, K. Itakura, NPA 809 ('08), 88; H. Fujii, K. Itakura, A. Iwazaki, NPA 828 ('09), 178.*
    **Parametric instability:** *J. Berges, S. Scheffler, S. Schlichting, D. Sexty, PRD 85 ('12),034507; S. Tsutsui, H. Iida, T. Kunihiro, AO, PRD 91 ('15), 076003; S. Tsutsui, T. Kunihiro, AO, PRD 94 ('16), 016001.*

# Classical Yang-Mills Field (2)

- Entropy Production
  - Classical field itself has entropy, which grows fast and amounts to be around 40 % of that expected in HIC.
    *S.G.Matinyan, E.B.Prokhorenko, G.K.Savvidy, JETP Lett. 44('86)138; NPB298 ('88)414; B. Muller, A. Trayanov, PRL68('92)3387; T.S.Biro, C.Gong, B.Muller, A. Trayanov, IJMPC 5('94)113; J.Bolte, B.Muller, A.Schafer, PRD 61('00)054506; Kunihiro, Muller, AO, Schafer ('09); KMOS, Takahashi, Yamamoto ('10); Iida, KMOST ('13); Tsukiji, Iida, KOT ('15,'16); Tsukiji, KOT ('17).*




- Shear viscosity
  - Shear viscosity obtained by using the Green-Kubo formula roughly agrees with that estimated from classical stat. simul.
    *Matsuda, Kunihiro, AO, Takahashi, arXiv:1904.024 Matsuda,KOT, in prep; Epelbaum-Gelis*



# Problems in Classical Field

- So far, so good.
  Entropy is produced and stored in classical field
  and would be converted to particle or fluid entropy
  in the early stage.
- But equilibrium of classical field is different
  from that of quantum field.
  - Classical equipartition (E/dof=T)
  - Rayleigh–Jeans divergence and Non-renormalizability at $a \to 0$ (dof/volume $\to \infty$)
  - Thus the cutoff dependence is strong.
- Alternatives ?
  - 2PI (two particle irreducible) action w/ or w/o classical approx.
    G. Aarts, J. Berges, PRL88('02)041603 (O(N));
    Y. Hatta, A. Nishiyama, NPA873 ('12) 47 (YM).

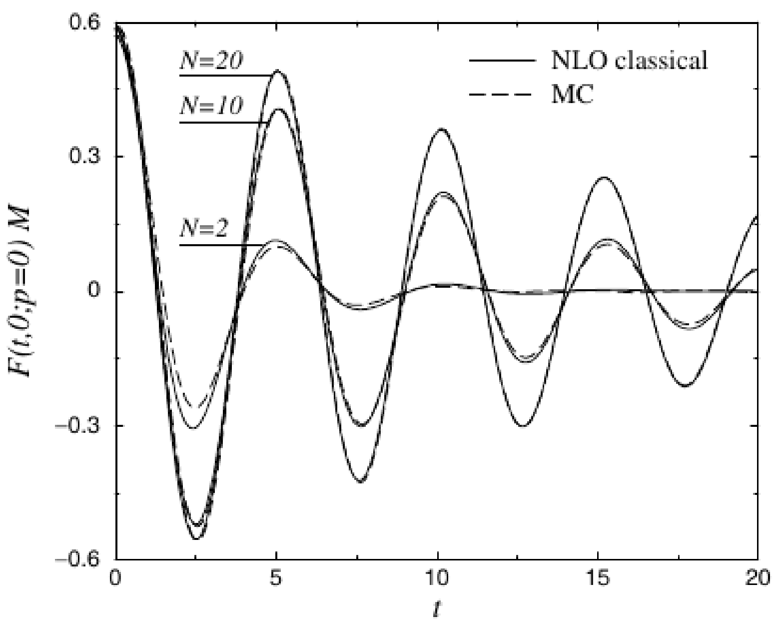
# Approaches using 2PI effective action

VOLUME 88, NUMBER 4 PHYSICAL REVIEW LETTERS 28 JANUARY 2002

## Classical Aspects of Quantum Fields Far from Equilibrium

Gert Aarts and Jürgen Berges
*Institut für Theoretische Physik, Philosophenweg 16, 69120 Heidelberg, Germany*
(Received 16 July 2001; published 15 January 2002)

We consider the time evolution of nonequilibrium quantum scalar fields in the $O(N)$ model, using the next-to-leading order $1/N$ expansion of the two-particle irreducible effective action. A comparison with exact numerical simulations in $1 + 1$ dimensions in the classical limit shows that the $1/N$ expansion gives quantitatively precise results already for moderate values of $N$. For sufficiently high initial occupation numbers the time evolution of quantum fields is shown to be accurately described by classical physics. Eventually the correspondence breaks down due to the difference between classical and quantum thermal equilibrium.




**Equation of motion for particle distribution function. Coupling with classical field is still hard.**

# *Towards a theory from the beginning to equilibrium*

- We propose a method to describe the field evolution,
  which is consistent with
  the (standard) classical field evolution when amplitude > fluc.
  and quantum field distribution in equilibrium,
  by regarding the Euclidean action as the potential
  (Euclidean potential dynamics (tentative name)).

- Contents
  - Introduction (done)
  - Euclidean potential dynamics
  - A few (very) preliminary results
  - Summary

# *Euclidean Potential Dynamics (Name is tentative)*

# Classical Field

- **Scalar theory (φ⁴) on the lattice**

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{24}\phi^4$$

$$\to H_{\mathrm{cl}} = \frac{1}{2}\sum_{\boldsymbol{x}}\pi_{\boldsymbol{x}}^2 + \sum_{\boldsymbol{x}}\left[\frac{1}{2}\left(\partial_i\phi_{\boldsymbol{x}}\right)^2 + \frac{m^2}{2}\phi_{\boldsymbol{x}}^2 + \frac{\lambda}{24}\phi_{\boldsymbol{x}}^4\right]$$

- **Classical field evolution & partition function**
  - **After the classical evolution for a long time, field configurations will be distributed according the classical partition function.**

$$\dot{\phi}_{\boldsymbol{x}} = \frac{\partial H_{\mathrm{cl}}}{\partial \pi_{\boldsymbol{x}}}\,, \quad \dot{\pi}_{\boldsymbol{x}} = -\frac{\partial H_{\mathrm{cl}}}{\partial \phi_{\boldsymbol{x}}} \quad \to \quad \mathcal{Z}_{\mathrm{cl}} = \int \mathcal{D}\pi\mathcal{D}\phi\, e^{-H_{\mathrm{cl}}/T}\ .$$

  - **Classical equipartition $<\pi^2>=T$**
    **→ Rayleigh-Jeans Law E/dof ~ T**
    **→ Divergence of the energy density in the continuum limit**

$$E/V = E/L^3/a^3 \simeq T/a^3 \to \infty (a \to 0)$$

# *Quantum Field Theory at Finite T*

- **Quantum partition function is given by the Euclidean action in 3+1 dimensional lattice ($L^3$ x N).**

$$\mathcal{Z}_{\mathrm{E}} = \int \mathcal{D}\phi\, e^{-\mathcal{S}_{\mathrm{E}}} \ , \quad \mathcal{S}_{\mathrm{E}} = \xi \mathcal{K}_{\mathrm{E}}(\phi) + \frac{1}{\xi}\mathcal{V}_{\mathrm{E}}(\phi) \ ,$$

$$\mathcal{K}_{\mathrm{E}} = \sum_{\boldsymbol{x},\tau} \frac{1}{2}(\partial_\tau \phi_{\boldsymbol{x}\tau})^2 \ , \quad \mathcal{V}_{\mathrm{E}} = \sum_{\boldsymbol{x},\tau} \left[ \frac{1}{2}\left(\partial_i \phi_{\boldsymbol{x}\tau}\right)^2 + \frac{m^2}{2}\phi_{\boldsymbol{x}\tau}^2 + \frac{\lambda}{24}\phi_{\boldsymbol{x}\tau}^4 \right] \ ,$$

$$\xi = a/a_\tau = NT : \text{anisotropy.}$$

  - **Imaginary time evolution takes care of quantum fluctuations !**

- **Observables are calculated in the Monte-Carlo method.**

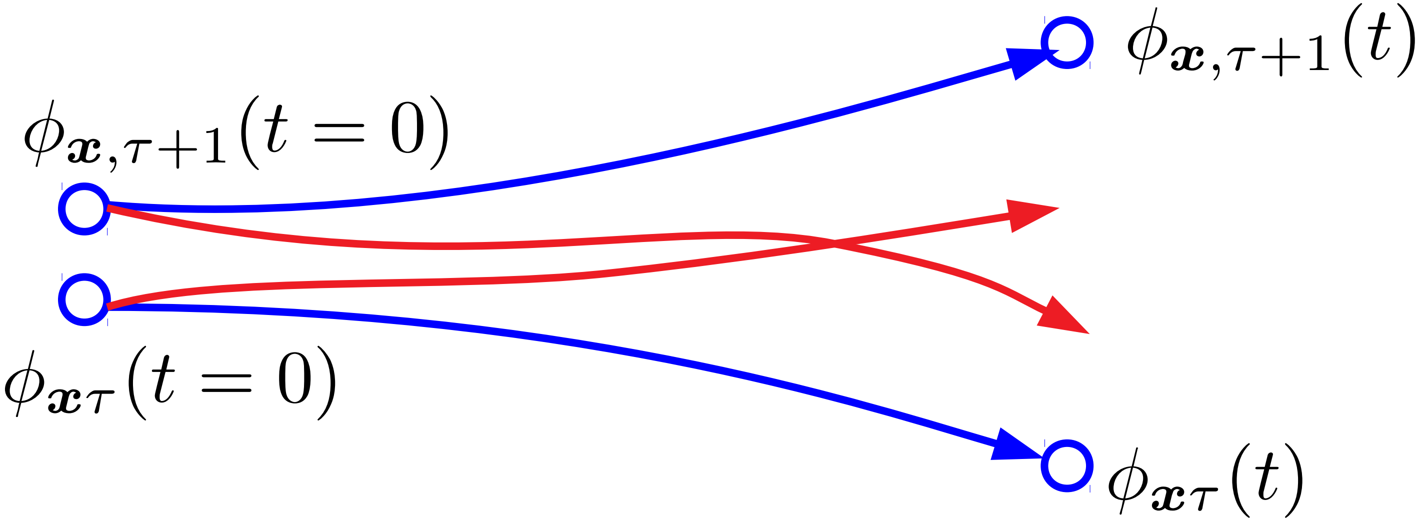
$$\langle \mathcal{O}(\phi) \rangle = \frac{1}{N_{\mathrm{conf}}} \sum_i \mathcal{O}(\phi^{(i)})$$

- **Real time evolution: Analytic continuation. Generally difficult.**

# *Real Time Evolution of MC Configurations*

- **How about considering classical real time evolution of MC configurations ?**

$$\dot{\phi}_{\boldsymbol{x}\tau} = \pi_{\boldsymbol{x}\tau} \ , \ \dot{\pi}_{\boldsymbol{x}\tau} = -(m^2 - \boldsymbol{\nabla}^2)\phi_{\boldsymbol{x}\tau} - \frac{\lambda}{6}\phi_{\boldsymbol{x}\tau}^3$$
$$+\xi^2(\phi_{\boldsymbol{x},\tau+1} + \phi_{\boldsymbol{x},\tau-1} - 2\phi_{\boldsymbol{x}\tau})$$




  - **Systems with many dof are generally chaotic**
    **→ Two adjacent configurations (e.g. τ and τ+1) depart from each other, and reach the classical equilibrium distribution.**

- **Let us add the restoring force to keep the correlation of fields with different τ.**

# *Euclidean potential dynamics (tentative name)*

- **Let us utilize the Euclidean action as the potential.**

$$\mathcal{H} = \mathcal{K}_{\mathrm{M}}(\pi) + \xi\mathcal{S}_{\mathrm{E}}(\phi) = \mathcal{K}_{\mathrm{M}}(\pi) + \mathcal{V}_{\mathrm{E}}(\phi) + \xi^2\mathcal{K}_{\mathrm{E}}(\phi)\ ,$$

$$\mathcal{K}_{\mathrm{M}}(\pi) = \sum_{\boldsymbol{x},\tau} \frac{1}{2}\pi_{\boldsymbol{x}\tau}^2\ ,$$

$$\dot{\phi}_{\boldsymbol{x}\tau} = \frac{\partial\mathcal{H}}{\partial\pi_{\boldsymbol{x}\tau}}\ ,\quad \dot{\pi}_{\boldsymbol{x}\tau} = -\frac{\partial\mathcal{H}}{\partial\phi_{\boldsymbol{x}\tau}}\ .$$

- **Long time evolution leads to quantum distribution of φ.**

$$\mathcal{Z}_{\mathrm{EP}} = \int \mathcal{D}\pi\mathcal{D}\phi\, e^{-\mathcal{H}/T_{\mathrm{cl}}}$$

$$= \int \mathcal{D}\pi\, e^{-\mathcal{K}_{\mathrm{M}}(\pi)/T_{\mathrm{cl}}} \int \mathcal{D}\phi\, e^{-\xi/T_{\mathrm{cl}} \times \mathcal{S}_{\mathrm{E}}(\phi)}$$

**Tuning initial energy dist. to satisfy $\xi / T_{\mathrm{cl}}=1$**

**→ quantum partition function**

# Relation to (Usual) Classical Field Evolution

- **Classical field ~ Average over τ**

$$\phi^{\rm cl}_{\boldsymbol{x}} = \frac{1}{N}\sum_\tau \phi_{\boldsymbol{x}\tau},\ \ \pi^{\rm cl}_{\boldsymbol{x}} = \frac{1}{N}\sum_\tau \pi_{\boldsymbol{x}\tau},\ \ \frac{d\phi^{\rm cl}_{\boldsymbol{x}}}{dt} = \pi^{\rm cl}_{\boldsymbol{x}}$$

$$\begin{aligned}\frac{d\pi^{\rm cl}_{\boldsymbol{x}}}{dt} =& -\frac{1}{N}\sum_\tau \left[(m^2-\boldsymbol{\nabla}^2)\phi_{\boldsymbol{x}\tau} + \frac{\lambda}{6}\phi^3_{\boldsymbol{x}\tau}\right] \\ & + \frac{\xi^2}{N}\sum_\tau \left(\phi_{\boldsymbol{x},\tau+1} + \phi_{\boldsymbol{x},\tau-1} - 2\phi_{\boldsymbol{x}\tau}\right)(\to 0) \\ =& -(m^2-\boldsymbol{\nabla}^2)\phi^{\rm cl}_{\boldsymbol{x}} - \frac{\lambda}{6}(\phi^{\rm cl}_{\boldsymbol{x}})^3 - \frac{\lambda}{6}\sum_\tau \left[3\phi^{\rm cl}_{\boldsymbol{x}}(\delta\phi_{\boldsymbol{x}\tau})^2 + (\delta\phi_{\boldsymbol{x}\tau})^3\right]\end{aligned}$$

**If ampl. >> fluc., τ-average satisfies classical field EOM.**

$$\mathcal{H}(\phi,\pi) = NH_{\rm cl}(\phi^{\rm cl},\pi^{\rm cl}) + \delta\mathcal{H} \to \mathcal{Z} = \int \mathcal{D}\phi\,\mathcal{D}\pi\, e^{-H_{\rm cl}(\phi^{\rm cl},\pi^{\rm cl})/T - \delta\mathcal{H}/\xi}$$

# Free Field case

- **Fourier transformation → Harmonic Oscillators**

$$\mathcal{H}^{(\lambda=0)} = \sum_{\boldsymbol{k},n} \frac{1}{2} \left[\pi_{\boldsymbol{k}n}^2 + \omega_{\boldsymbol{k}n}^2 \phi_{\boldsymbol{k}n}^2\right]$$

$$\omega_{\boldsymbol{k}n} = \left[\omega_{\boldsymbol{k}}^2 + 4\xi^2 \sin^2(\omega_n/2)\right]^{1/2} \ , \ \omega_{\boldsymbol{k}} = \left[m^2 + 4\sum_{i=1}^{D} \sin^2(k_i/2)\right]^{1/2} \ ,$$

$$\begin{pmatrix} \phi_{\boldsymbol{k}n} \\ \pi_{\boldsymbol{k}n} \end{pmatrix} = \frac{1}{\sqrt{L^3 N}} \sum_{\boldsymbol{x},\tau} \left[e^{-i\boldsymbol{k}\cdot\boldsymbol{x}+i\omega_n\tau}\right] \begin{pmatrix} \phi_{\boldsymbol{x}\tau} \\ \pi_{\boldsymbol{x}\tau} \end{pmatrix} \ .$$

- **Gaussian integral, Matsubara frequency summation, …**

$$\langle E_{\boldsymbol{k}}^{(\lambda=0)} \rangle_T = -\frac{\partial}{\partial\beta} \mathcal{Z}_{\boldsymbol{k}}^{(\lambda=0)} = \frac{1}{\sqrt{1+(\omega_{\boldsymbol{k}}/2\xi)^2}} \left( \underbrace{\frac{\omega_{\boldsymbol{k}}}{2}}_{\text{Zero point E}} + \underbrace{\frac{\omega_{\boldsymbol{k}}}{e^{\Omega_{\boldsymbol{k}}/T} - 1}}_{\text{Thermal E}} \right)$$

$$\Omega_{\boldsymbol{k}} = 2\xi \, \mathrm{arcsinh}\left(\omega_{\boldsymbol{k}}/2\xi\right)$$

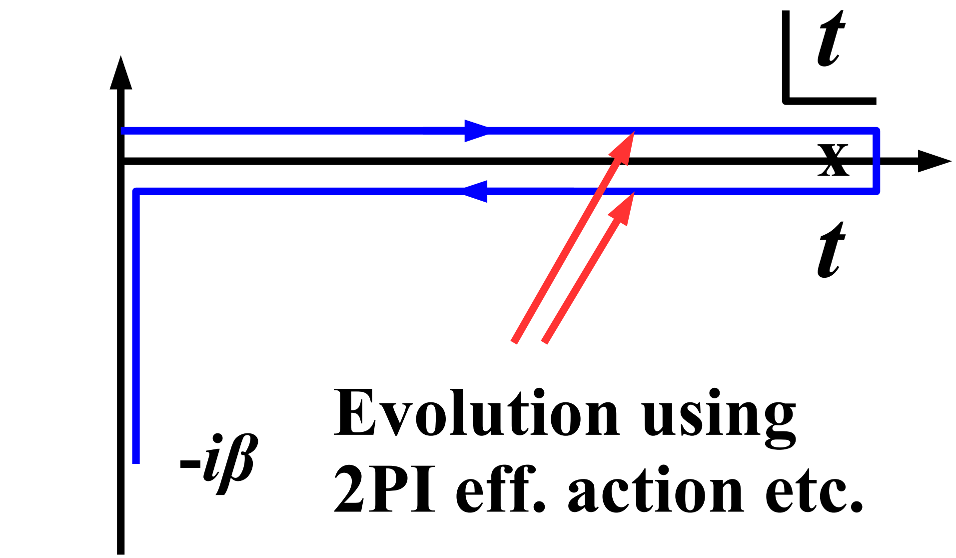
**In the large N limit (ξ = NT → ∞), continuum results are obtained.**

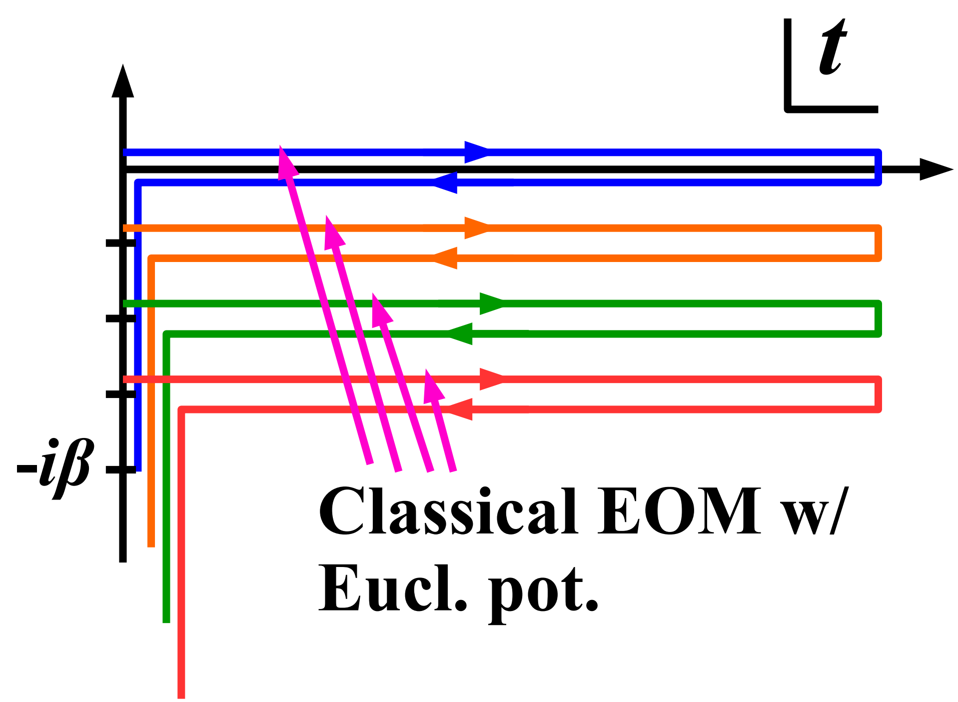
# Closed Time Path

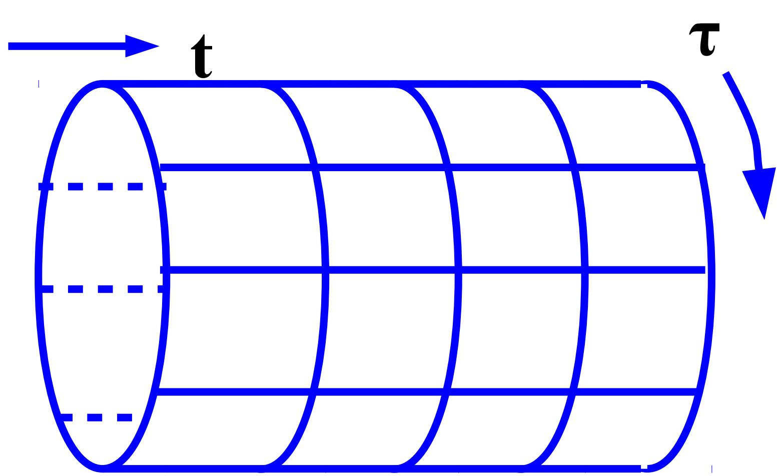
- **Keldysh closed time path**
  - **Time integration is done on the closed time path.**

$$\langle \mathcal{O}(t)\rangle = \int \mathcal{D}\phi\, \mathcal{T}_{\mathcal{C}}\left[e^{-i\int_{\mathcal{C}} dt' H(t')}\mathcal{O}(t)\right]$$

- **Euclidean potential dynamics**
  - **Integration on the imaginary time axis is assumed to be already included in preparing the initial config.**



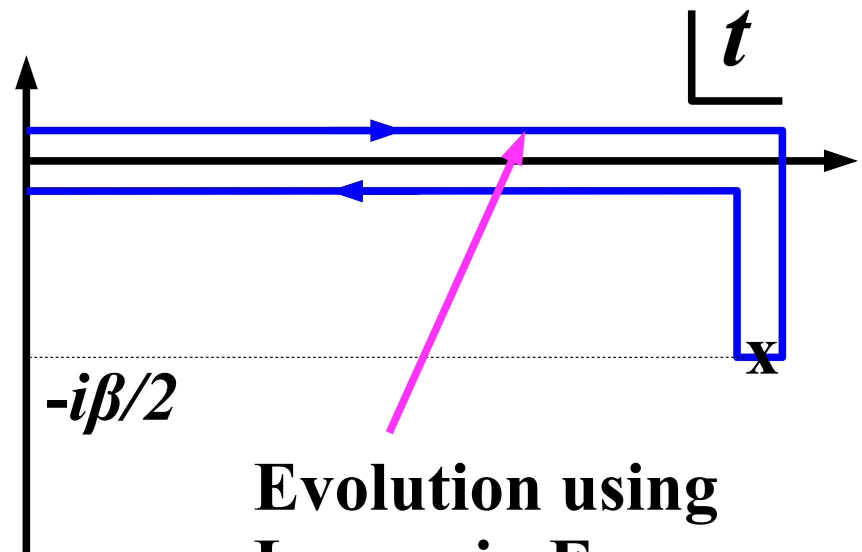



# Closed Time Path

- **Ohnishi-Randrup method**
  - **Imaginary time evolution when we make and observation.**

$$\langle \mathcal{O}(t) \rangle = \langle e^{+iHt-\beta H/2} \mathcal{O}(t) e^{-\beta H/2 - iHt} \rangle$$




**Evolution using Langevin Eq. with modified Einstein relation**

# *A few (very) preliminary results*

# Setup

- Scalar field theory ($\varphi^4$ model)
- Real time evolution of field configuration
  on 3 (spatial)+1 (imag. time) dimensional $L^3$ x N lattice.
  (L=16, N=1 (classical), 2, 4, 8)
- Initial condition
  Langevin equation at temperature of $T_{cl} = \xi = NT$.

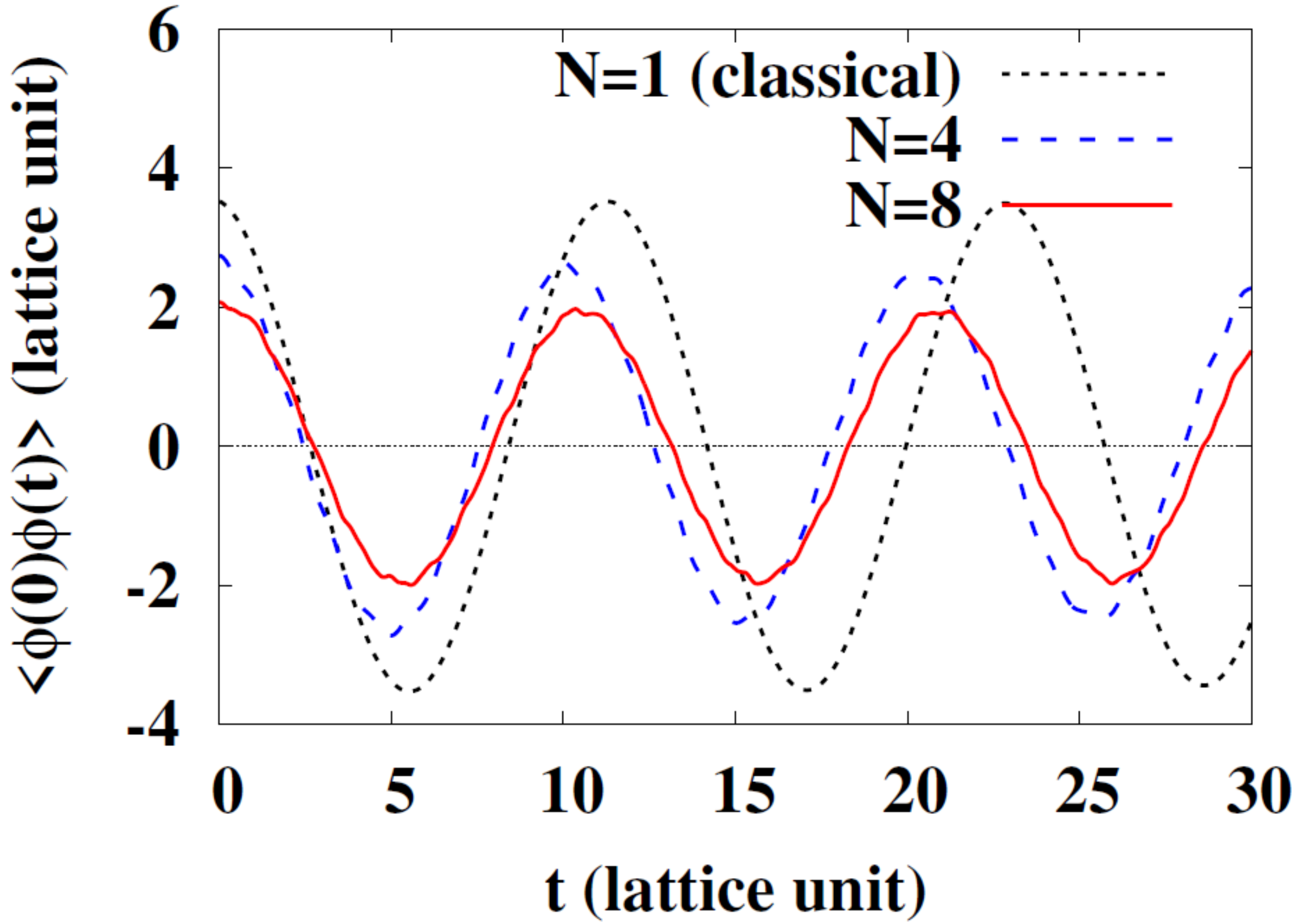
$$\langle \pi_{\boldsymbol{x}\tau} \rangle = NT, \ \langle \pi_{\boldsymbol{x}}^{\mathrm{cl}} \rangle = T$$

  (Simulating temp. T as τ-average of temp. NT fields)
- Observable (as an example)
  Two point function of different times.

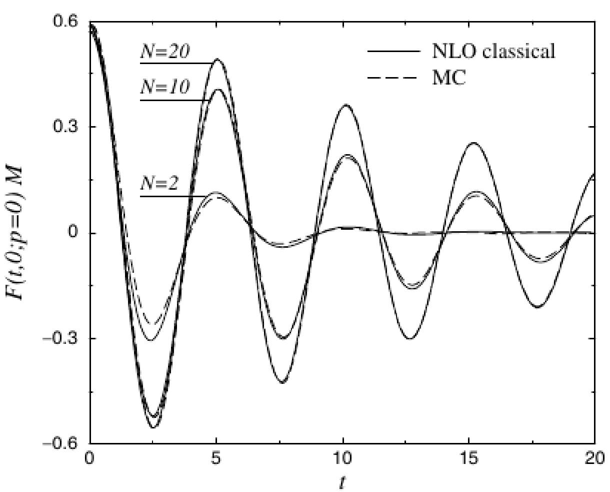
$$F(t) = \langle \phi(\boldsymbol{k}=0, t=0)\phi(\boldsymbol{k}=0, t) \rangle = \frac{1}{L^3 N} \sum_{\tau} \sum_{\boldsymbol{x},\boldsymbol{x}'} \phi_{\boldsymbol{x}\tau}(t=0)\phi_{\boldsymbol{x}'\tau}(t)$$

# Two Point Function

- **Two point function in scalar theory (m=0, λ=3, $16^3$ x N lattice, N=1,4,8)**
  - **Evolution with quantum statistical correction → Smaller amplitude, faster oscillation**




*Aarts, Berges ('02)*



# *Summary*

- **Classical field evolution with quantum statistical correction is investigated by using the Euclidean action of qauntum fields at finite temperature as the potential in the Hamiltonian.**
  - **EOM of τ-average ~ Classical Field EOM**
  - **Equilibrium distribution of φ = quantum field theory at finite T (Boltzmann dist. in 3+1D space is robust even if we add "collision term" in the dynamics)**
  - **Molecular dynamics part of hybrid Monte-Carlo sampling**
- **Two point function is shown as an example of application.**
  - **Unequal time two point function shows faster oscillation and smaller amplitude ($\omega_n$ effects ?).**
    **→ To be examined !**

# *Future works*

- There are many things to examine.
  - Scalar theory, entropy production
  - O(N) theory, comparison with 2PI results
  - Yang-Mills field, isotropization and entropy production
  - Relation with the previous works
    *S.-i. Sasa, PRL 112 ('14), 100602;*
    *T. Hayata, Y. Hidaka, T. Noumi M. Hongo, PRD 92 ('15), 065008.*
    → "local" equilibrium ?
  - and many.