Replica evolution of classical field in 4+1 dimensional spacetime as a simulator of quantum field evolution

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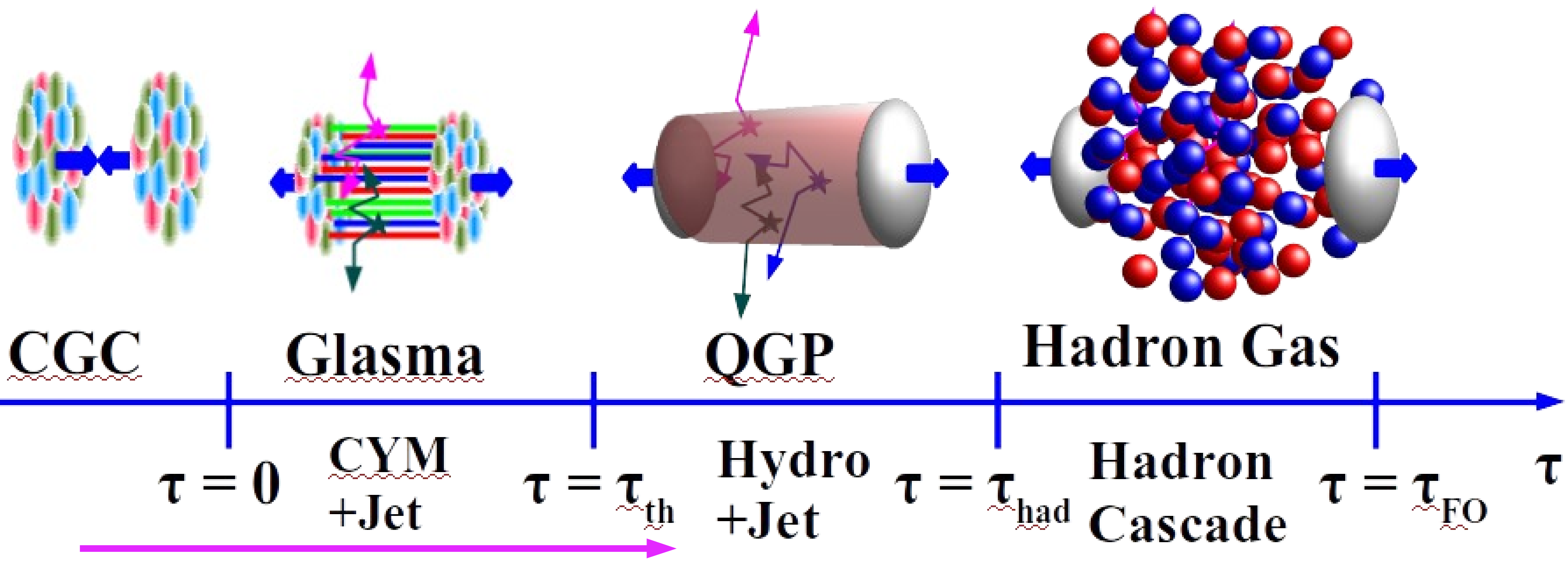
arXiv:2008.09556 [hep-lat]

10th Int. Conf. on Exact Renormalization Group 2020 (ERG2020), Nov.2-6, 2020 (Online/Kyoto).

- **Introduction**
- **Replica Evolution**
- **Application to Scalar Field Theory**
  - **Momentum dist. and Rayleigh-Jeans divergence**
  - **Time-corr. func. and thermal mass**
- **Summary**

Heavy-Ion Collisions on the Lattice ?

- Color Glass Condensate (CGC) → Glasma (initial stages) → Quark Gluon Plasma (QGP) → Hadron Gas



- **In the initial stages, we need to solve quantum field evolution under inhomogeneous & non-equilibrium background classical field.**
- **But there is a strong sign problem in real-time evolution.**

Unreachable Dream ?

Toward Classical Field with Quantum Statistics

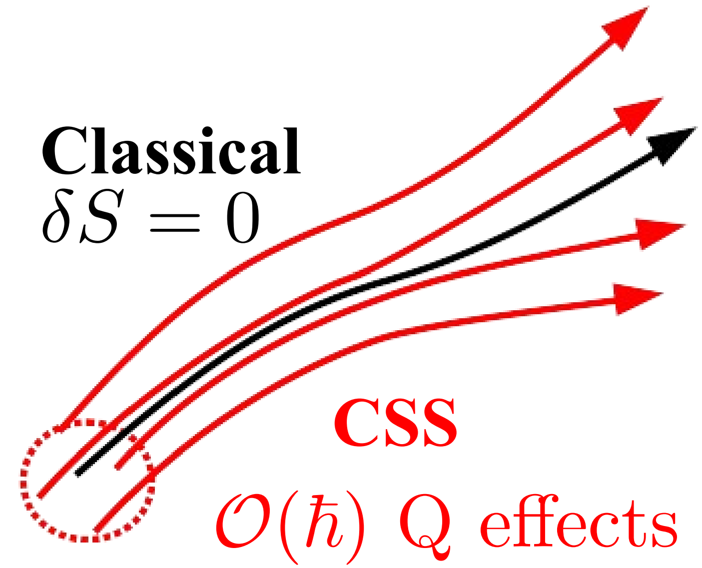
- **Classical field dynamics**
  - **Amplitude=exp(iS), Classical EOM δS=0 → No Sign problem**
  - **Applied to many non-equilibrium phenomena**
    condensate (Time dep. Gross-Pitaevski), nuclei (TD Hartree-Fock), Inflation, high-energy heavy-ion collisions (classical Yang-Mills), …
  - **Equilibrium is classical, and energy density diverges in the continuum limit (Rayleigh-Jeans Divergence)**

$$n_{\boldsymbol{k}} = T/\omega_{\boldsymbol{k}} (\text{Classical}), \quad n_{\boldsymbol{k}} = [\exp(\omega_{\boldsymbol{k}}/T) \mp 1]^{-1} (\text{Quantum})$$

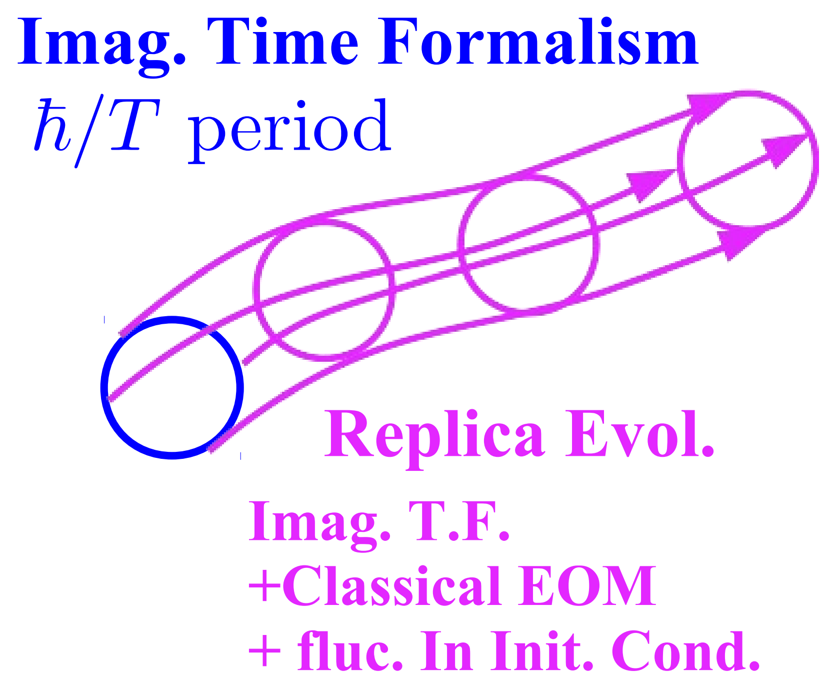
- **Toward real-time dynamics of quantum field**
  - **Closed Time Path+ 2PI effective action (Kadanoff-Baym Eq.)**
    Aarts, Berges ('02), Hatta, Nishiyama ('12) [c.f. Previous Talk by Philipp Scior]
    → CPU time is huge when background field is inhomogeneous
  - **High-momentum DOFs are integrated or implemented by particles**
    Bodeker, McLerran, Smilga ('95), Greiner, B.Muller ('97), Dumitru, Nara, (Strickland) ('05,'07) …
    → Classical field part still obeys classical statistics in equilibrium

From Classical Field to Replica Evolution

- **Classical Statistical Simulation**
  - **Classical Field equation of motion has $\mathcal{O}(\hbar^2)$ precision**
  - **Instability/Chaoticity**
  - **$\mathcal{O}(\hbar)$ effect is included in init. cond.**
  - **Classical equilibrium**



- **Imaginary Time Formalism**
  - **Enlarged "Classical" field configs. 3D → 4(=3+1)D**
  - **Quantum stat. equilibrium**



- **Replica evolution**
  - **Real time evolution of enlarged 4D "Classical" field configs.**
  - **Classical feature + Quant. Stat. Equil.**

Replica evolution

Replica Evolution (Quantum Mechanics)

- **Replica Hamiltonian**

$H_{\rm cl}(x_\tau, p_\tau)$

τ-derivative term

$\xi S[x]$

$$\mathcal{H} = \sum_{\tau=1}^{N}\left[\boxed{\frac{p_\tau^2}{2} + U(x_\tau)} + \frac{\xi^2}{2}(x_{\tau+1}-x_\tau)^2\right]$$

$$\simeq \xi \int_0^{1/T} d\bar\tau \left\{\frac{p^2(\bar\tau)}{2} + U(x(\bar\tau)) + \frac{1}{2}\left[\frac{\partial x(\bar\tau)}{\partial \bar\tau}\right]^2\right\} \quad (\xi = NT, \bar\tau = \tau/NT)$$

- **Quantum Statistical Equilibrium from Replica Evolution**
  - **Replicas = Set of N classical fields (Imag. Time formalism)**
  - **Gauss integral over p of Boltzmann weight [exp(-H/ξ)] leads to quantum statistical partition function.**

$\mathcal{Z}_Q(T)$

$$\mathcal{Z}_R(\xi) = \int \frac{\mathcal{D}x\mathcal{D}p}{2\pi} \exp\left(-\mathcal{H}/\xi\right) = \mathcal{N}\ (2\pi\xi)^{NL^3/2} \boxed{\int \mathcal{D}x \exp\left(-S[x]\right)}$$

"Classical" part. fn. of replicas ∝ "Quantum" part. fn.

Replica Evolution (Quantum Mechanics)

- **Replica Equation of Motion**

$$\frac{dx_\tau}{dt} = \frac{\partial \mathcal{H}}{\partial p_\tau} = p_\tau , \quad \frac{dp_\tau}{dt} = -\frac{\partial \mathcal{H}}{\partial x_\tau} = -\frac{\partial U(x_\tau)}{\partial x_\tau} + \underline{\xi^2 (x_{\tau+1} + x_{\tau-1} - 2x_\tau)}$$

Force from τ-derivative term

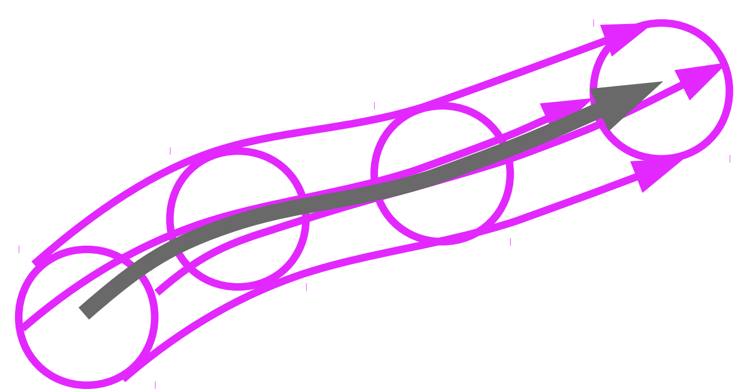
- **Replica index average**

$$\widetilde{x} \equiv \frac{1}{N} \sum_\tau x_\tau , \quad \widetilde{p} \equiv \frac{1}{N} \sum_\tau p_\tau$$

$$\frac{d\widetilde{x}}{dt} = \widetilde{p}, \qquad \frac{d\widetilde{p}}{dt} = -\frac{1}{N}\sum_\tau \frac{\partial U(x_\tau)}{\partial x_\tau} + 0 = \underline{-\frac{\partial U(\widetilde{x})}{\partial \widetilde{x}}} + \mathcal{O}((\delta x)^2)$$

Ehrenfest's theorem

Replica index average obeys classical EOM (when flucuations are small).



Harmonic Oscillator

- **Replica Hamiltonian = N free HO Hamiltonian**

$$\mathcal{H} = \sum_\tau \left[\frac{p_\tau^2}{2} + \frac{\omega^2 x_\tau^2}{2} + \underbrace{\frac{\xi^2}{2}(x_{\tau+1} - x_\tau)^2}_{\tau\text{-deriv. term}} \right] = \sum_n \left[\frac{\bar{p}_n^2}{2} + \frac{M_n^2 \bar{x}_n^2}{2} \right]$$

Fourier transf.

$$M_n^2 = \omega^2 + 4\xi^2 \sin^2(\pi n/N)$$

- **Expectation value of x^2 in Replica**

$$\langle x^2 \rangle = \frac{1}{N}\sum_\tau \langle x_\tau^2 \rangle = \frac{1}{N}\sum_n \langle \bar{x}_n^2 \rangle \overset{\exp(-\mathcal{H}/\xi)}{=} \frac{1}{N}\sum_n \frac{\xi}{M_n^2} \overset{\text{Matsubara freq. sum}}{=} \frac{\coth(\Omega/2T)}{2\omega\sqrt{1+\omega^2/4\xi^2}}$$

$$\Omega = 2\xi \operatorname{arcsinh}(\omega/2\xi)$$

$$\frac{T}{\omega^2} (N=1, \text{Classical})$$

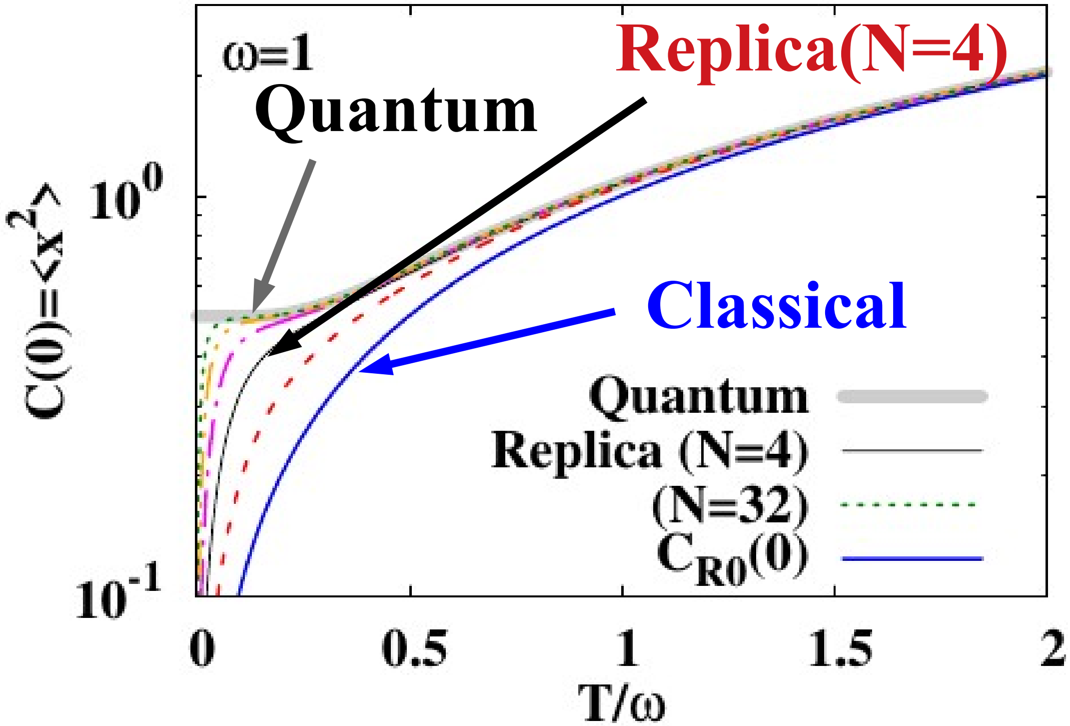
zero point **thermal**

$$\to \frac{\coth(\omega/2T)}{2\omega} = \frac{1}{\omega}\left[\frac{1}{2} + \frac{1}{e^{\omega/T}-1}\right] (N \to \infty, \text{Quantum})$$

Equal time observables of x are reproduced at $N \to \infty$

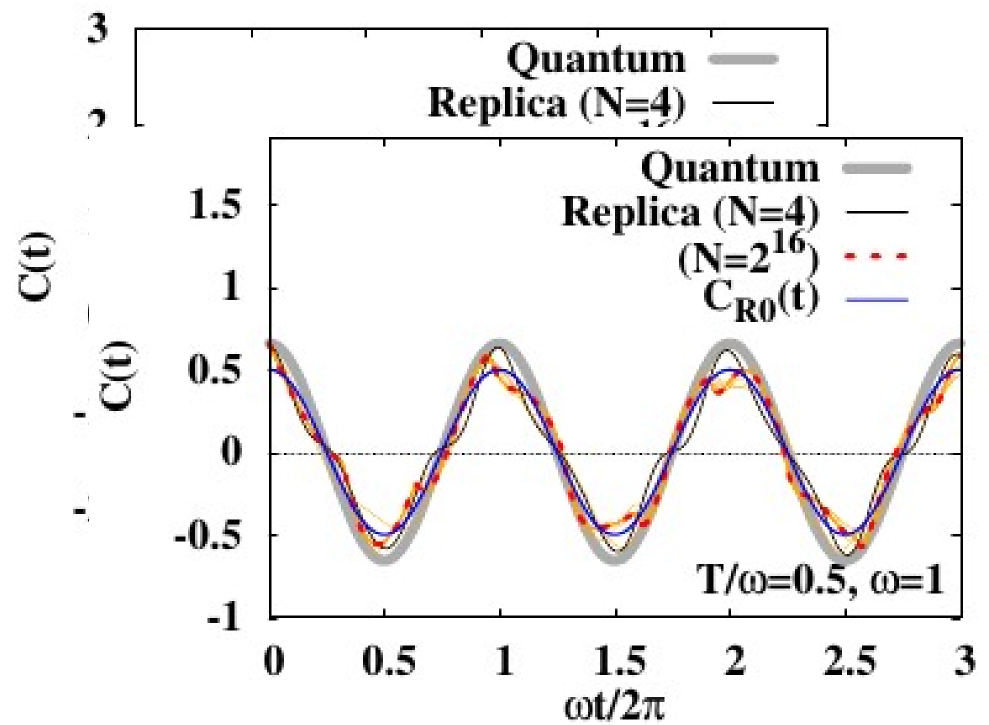
Time Correlation Function in HO

$$C(t) = \langle x(t)x(0) \rangle$$



Equal time observables
→ Exact at N → ∞

Unequal time corr. fn.
→ Not exact,
but good for T/ω > 0.5



Sounds nice. How about field theory ?

Application to Scalar Field Theory

Replica Evolution in Scalar Field Theory

- **Replica evolution in field theory**
  - **Replace variables** $(x_\tau, p_\tau) \to (\phi_{\tau\boldsymbol{x}}, \pi_{\tau\boldsymbol{x}})$
  - **Mass renormalization & Subtracting zero point contribution**
- **Example: Φ^4 theory**

τ-deriv. term

$$\mathcal{H} = \sum_{\tau,\boldsymbol{x}} \left[\frac{\pi_{\tau\boldsymbol{x}}^2}{2} + \frac{1}{2}(\boldsymbol{\nabla}\phi_{\tau\boldsymbol{x}})^2 + \frac{m_0^2}{2}\phi_{\tau\boldsymbol{x}}^2 + \frac{\lambda}{24}\phi_{\tau\boldsymbol{x}}^4 + \boxed{\frac{\xi^2}{2}(\partial_\tau \phi_{\tau\boldsymbol{x}})^2} \right]$$

$H(\phi_{\tau\boldsymbol{x}}, \pi_{\tau\boldsymbol{x}})$ $\quad$ $\xi S[\phi]$

$$m_0^2 = m^2 - \delta m^2, \partial_\tau \phi_{\tau\boldsymbol{x}} \equiv \phi_{\tau+1,\boldsymbol{x}} - \phi_{\tau\boldsymbol{x}}$$

- **Mass Counterterm (one loop)** *Aarts, Smit ('97), Kapusta, Gale (textbook)*

$$\delta m^2 = \frac{\lambda}{2}\langle\phi^2\rangle_{\mathrm{div}}$$

$$\langle\phi^2\rangle_{\mathrm{div}} = \frac{1}{L^3}\sum_{\boldsymbol{k}} \frac{1}{2\omega_{\boldsymbol{k}}\sqrt{1+(\omega_{\boldsymbol{k}}/2\xi)^2}}$$

$-\delta m^2$ $\quad$ $\lambda\langle\phi^2\rangle/2$

Numerical Calculation Setup

- **Lattice size = 32^3 x 4 (L=32, N=4)**
- **T=0.5 (ξ=NT=2); m=0, 0.5; λ=0.5, 1, 2, 4, 6, 8, 10.**
- **One loop renormalization of mass, no counterterm for λ.**
- **Initial conditions are obtained by solving the Langevin equation.**
- **Solve replica EOM until t=500 with the time step of Δt=0.025.**
- **Number of replica configurations = 1000**
  $\rightarrow$ 3-6 hours on one core of core i7 PC for a given (m, λ)

Momentum Distribution

- **Momentum distribution in replica = zero point + Bose-Einstein**

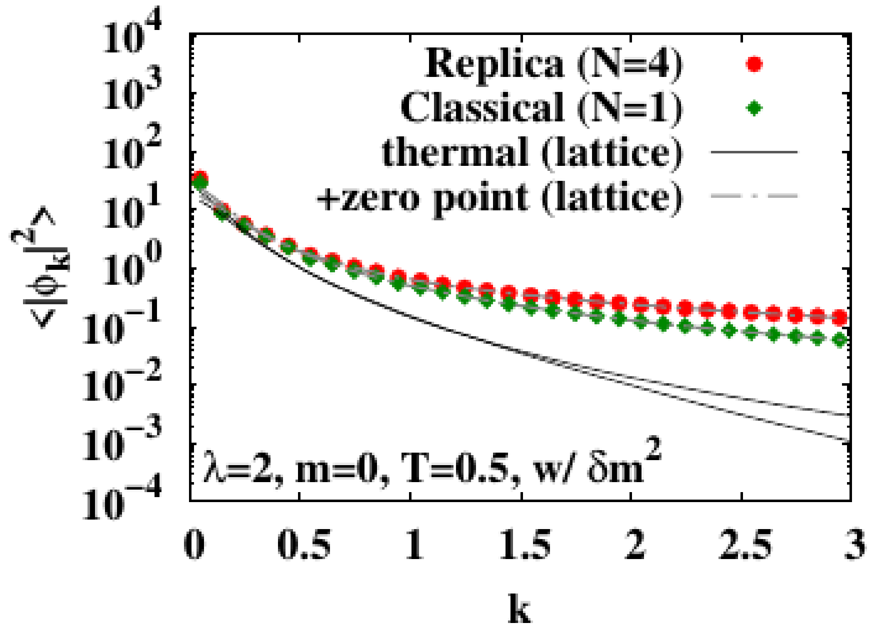
$$\langle|\phi_{\boldsymbol{k}}|^2\rangle = \frac{1}{N}\sum_n \langle\phi_{n\boldsymbol{k}}\phi^*_{n\boldsymbol{k}}\rangle = \frac{1}{\omega_{\boldsymbol{k}}\sqrt{1+(\omega_{\boldsymbol{k}}/2\xi)^2}}\left[\frac{1}{2}+\frac{1}{e^{\Omega_{\boldsymbol{k}}/T}-1}\right]$$

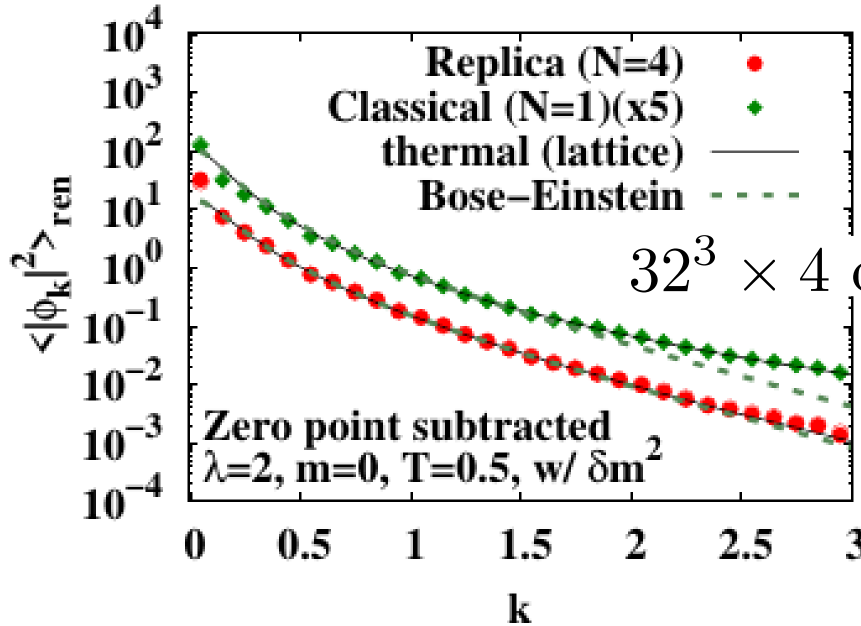
Free field, Matsubara sum

Zero point

Thermal → Bose-Einstein

  - **By subtracting the zero point part, we can avoid equipartition & Rayleigh-Jeans divergence.**





$32^3 \times 4$ or 1

Rayleigh-Jeans Divergence

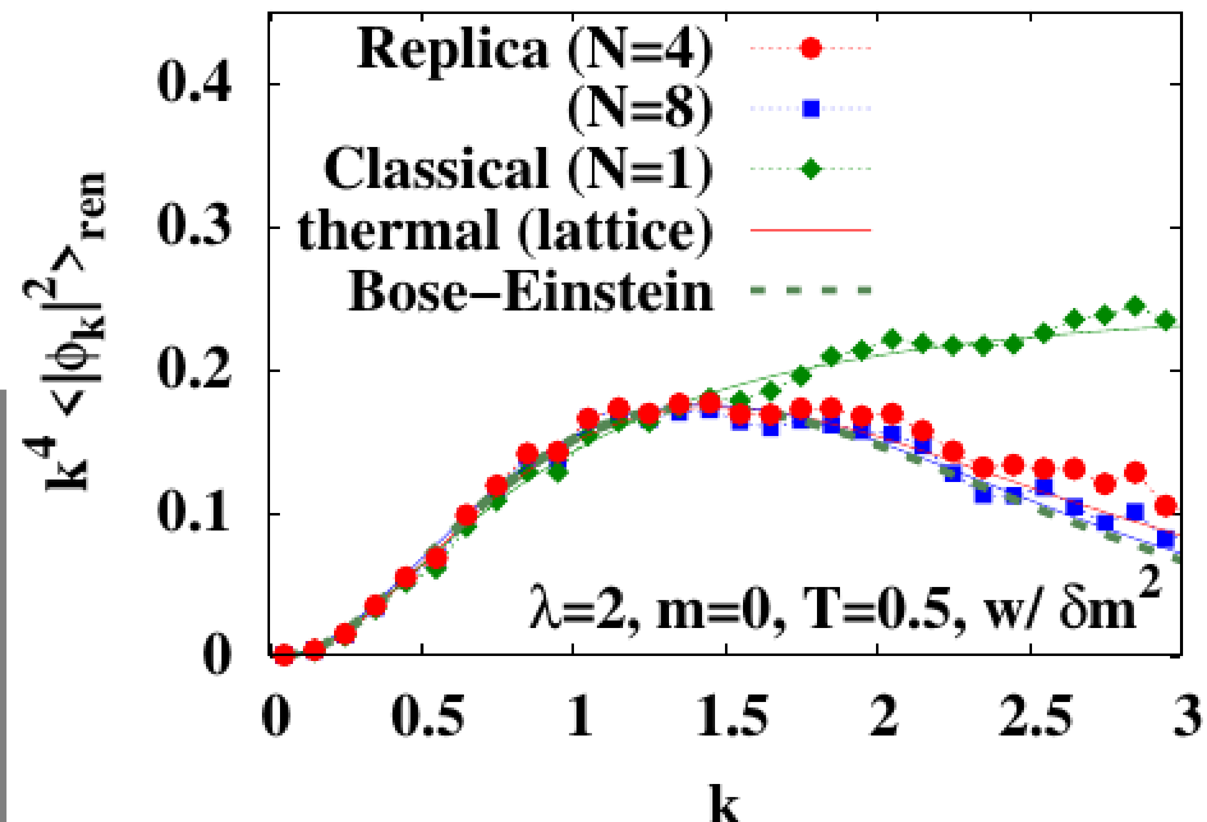
- **With N >= 2, free field energy converges in the replica method.**

$$\Omega = 2NT\,\mathrm{arcsinh}\,(\omega/2NT) \underset{\omega \gg NT}{\longrightarrow} 2NT\log(\omega/NT)$$

$$\langle|\phi_{\boldsymbol{k}}|^2\rangle_{\mathrm{ren}} \simeq \frac{2NT}{k^2}\exp(-\Omega_{\boldsymbol{k}}/T) \to 2(NT)^{2N+1}k^{-2(N+1)}$$

$$k^4\,\langle|\phi_{\boldsymbol{k}}|^2\rangle_{\mathrm{ren}} \to 2(NT)^{2N+1}k^{-2(N-1)} \quad (\mathrm{K.E.} \propto \int dk k^4\,\langle|\phi_{\boldsymbol{k}}|^2\rangle_{\mathrm{ren}})$$

- **Convergence cond.**
  2(N-1) > 1 → N > 1.5

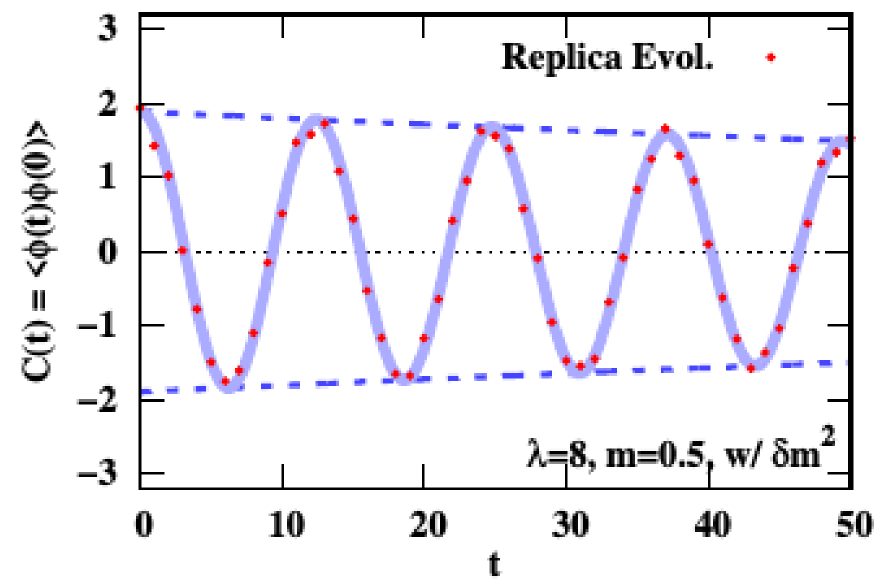


We can remove divergence of energy in the replica method (N>=2) with mass counterterm.

Time-correlation function

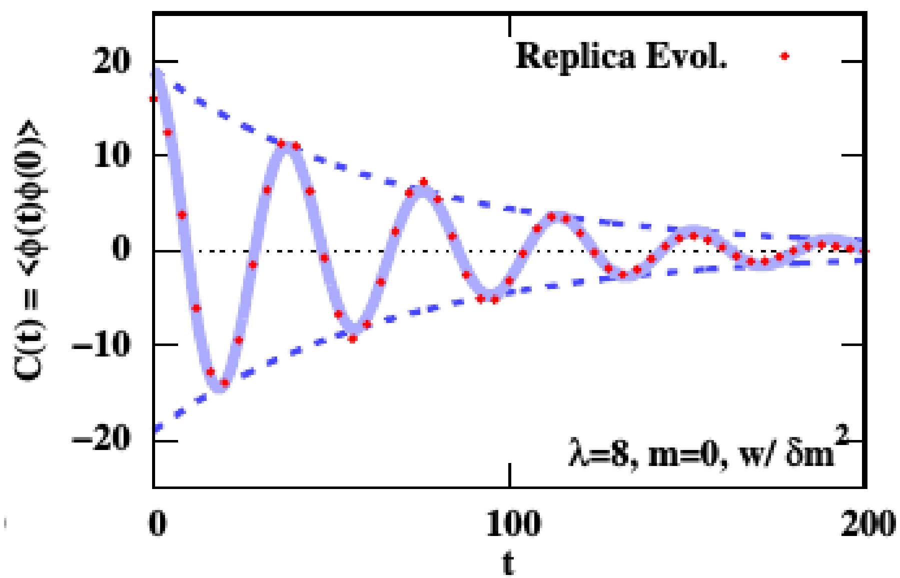
- **Time-correlation function of free field (zero momentum)**

$$
\begin{aligned}
C(t) &= \frac{1}{L^3} \sum_{\boldsymbol{x},\boldsymbol{y}} \langle \phi_{\boldsymbol{x}}(t) \phi_{\boldsymbol{y}}(0) \rangle \\
&= \frac{1}{NL^3} \sum_{\tau,\boldsymbol{x},\boldsymbol{y}} \langle \phi_{\tau\boldsymbol{x}}(t) \phi_{\tau\boldsymbol{y}}(0) \rangle \\
&= \sum_n \frac{T}{M_n^2} \cos M_n t \\
&\quad (M_n^2 = \omega^2 + 4\xi^2 \sin^2(n\pi/N))
\end{aligned}
$$



From the dominant frequency of C(t), we obtain thermal mass.

- **TCF of interacting field**
  - **Interaction induces thermal mass**
  - **Coupling of different momentum modes induces damping.**

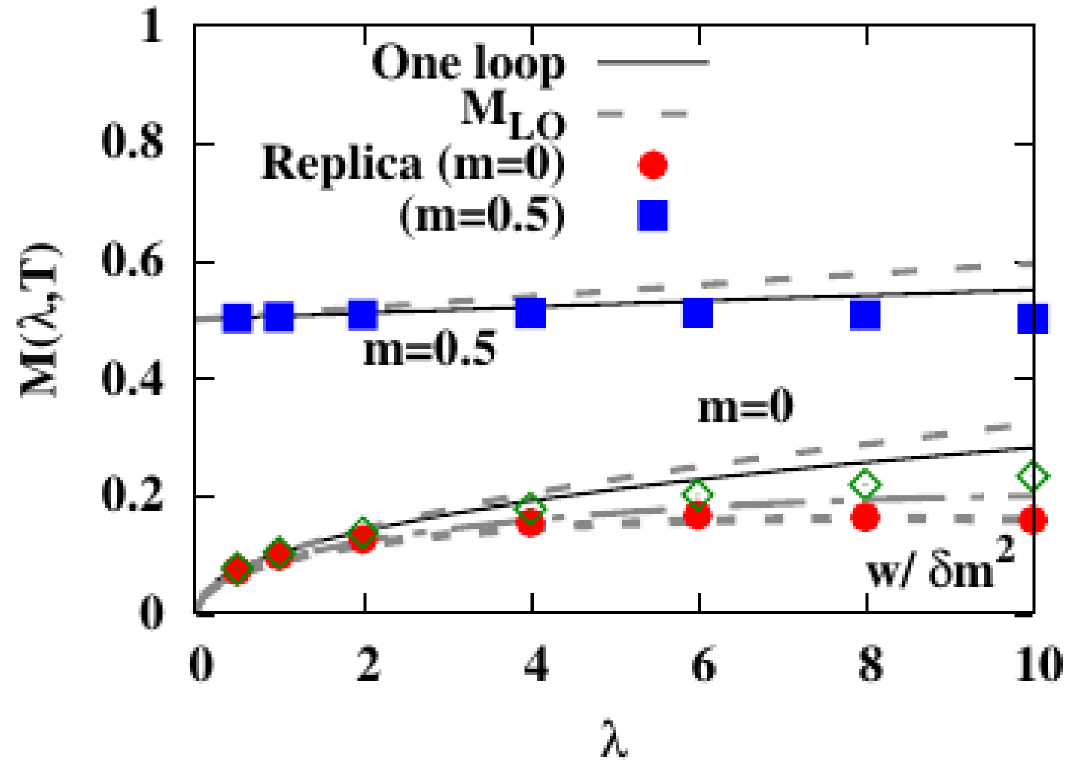


Thermal Mass

- **Thermal Mass**
  - **Leading Order (one-loop)** $M^2_{\rm LO} = m^2 + \lambda T^2/24.$
  - **Resummed One-Loop** $M^2_{\rm resum} = \dfrac{\lambda T^2}{24}\left[1 - \dfrac{3}{\pi}\sqrt{\dfrac{\lambda}{24}}\right]$
  - **Two-Loop**

$$M^2_{2-\rm loop} = \frac{\lambda T^2}{24}\left\{1 - \frac{3}{\pi}\sqrt{\frac{\lambda}{24}} + \frac{\lambda}{(4\pi)^2}\left[\frac{3}{2}\log\left(\frac{T^2}{4\pi\mu^2}\right) + 2\log\left(\frac{\lambda}{24}\right) + \alpha\right]\right\},$$

Kapusta, Gale (textbook)
Parwani ('92, '93)



Thermal mass in Replica Evolution ~ Two-loop results

Summary

- **Replica evolution is classical field dynamics, which reaches quantum statistical equilibrium.**
  AO, H. Matsuda, T. Kunihiro, T. T. Takahashi, arXiv:2008.09556
  - **Configurations of N classical fields (replicas) interacting via τ-derivative terms (kinetic E. in imag. time formalism) at temperature ξ=NT reach quantum statistical distribution of at temperature T.**
    (Chaoticity is assumed.)
  - **Since 4D "classical" field statistics = 3D quantum field statistics, it is not unreasonable to expect**
    4+1D "classical" field evolution ~ 3+1D quantum field evolution
  - **Replica-index (~ imag. time) average provides classical field.**
    (Fictitious time in the molecular dynamics part of the hybrid Monte-Carlo works as a real time.)
  - **Subtracting zero point motion part from $\langle\Phi^2\rangle$**
    $\rightarrow$ mass renormalization and removing Rayleigh-Jeans divergence
  - **Thermal mass ~ 2-loop perturbation results.**

To do list

- **There are many subjects to be investigated**
  - **Comparison with previously proposed frameworks.**
    Hard mode effects [Bodeker, McLerran, Smilga ('95), Greiner, B.Muller ('97)], Field-particle sim. [Dumitru, Nara ('05), Dumitru, Nara, Strickland ('07)], 2PI [Aarts, Berges ('02), Hatta, Nishiyama ('12)], …
  - **Formal discussions, e.g. relation to Boltzmann Eq.,**
    A.Muller, Son ('04).
  - **Shear viscosity, Thermalization of classical field, Entropy production, ...**

Thank you for your attention !

AO

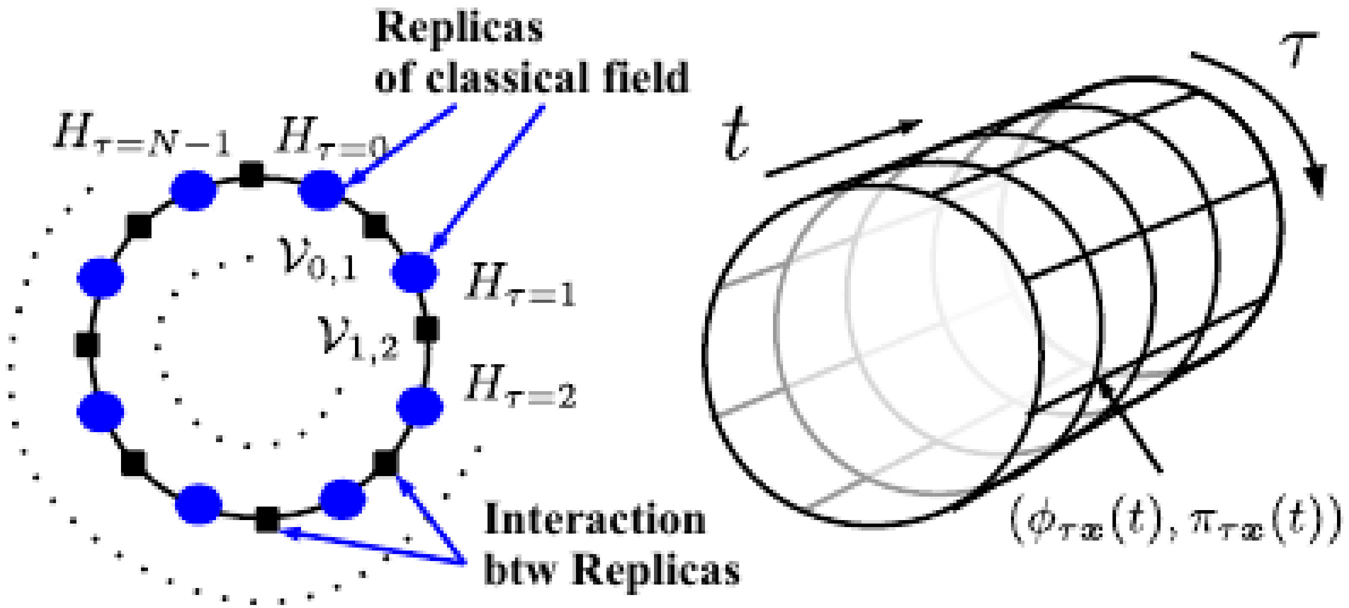
Hidefumi Matsuda

Teiji Kunihiro

Toru T. Takahashi

"Replica evolution of classical field in 4+1 dimensional spacetime toward real time dynamics of quantum field",
A. Ohnishi, H. Matsuda, T. Kunihiro, T. T. Takahashi, arXiv:2008.09556 [hep-lat].

Replica Evolution



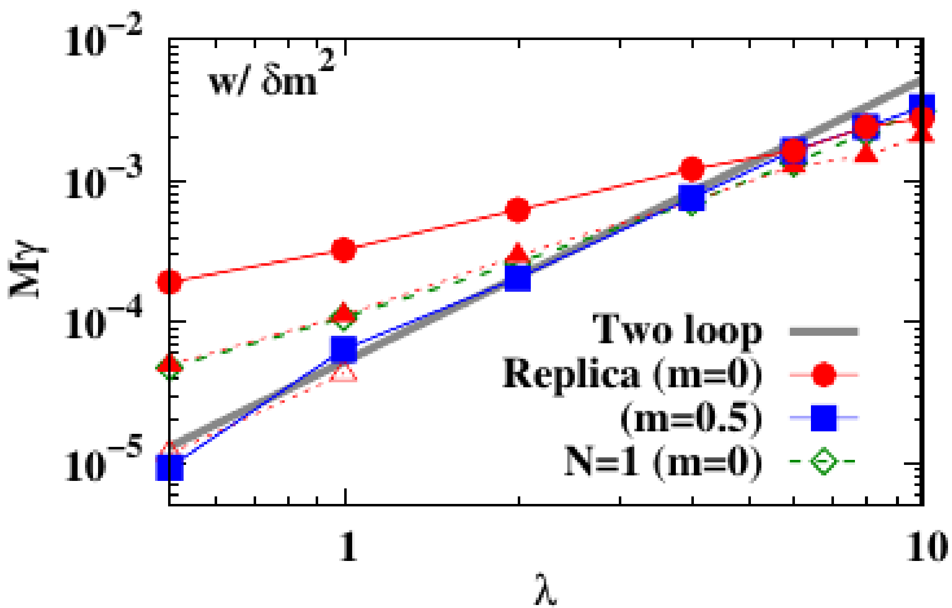
$$\mathcal{H} = \sum_\tau H_\tau + \sum_\tau \mathcal{V}_{\tau,\tau+1} = \frac{1}{2}\sum_{\tau,\boldsymbol{x}} \pi^2_{\tau,\boldsymbol{x}} + \xi S[\phi]$$

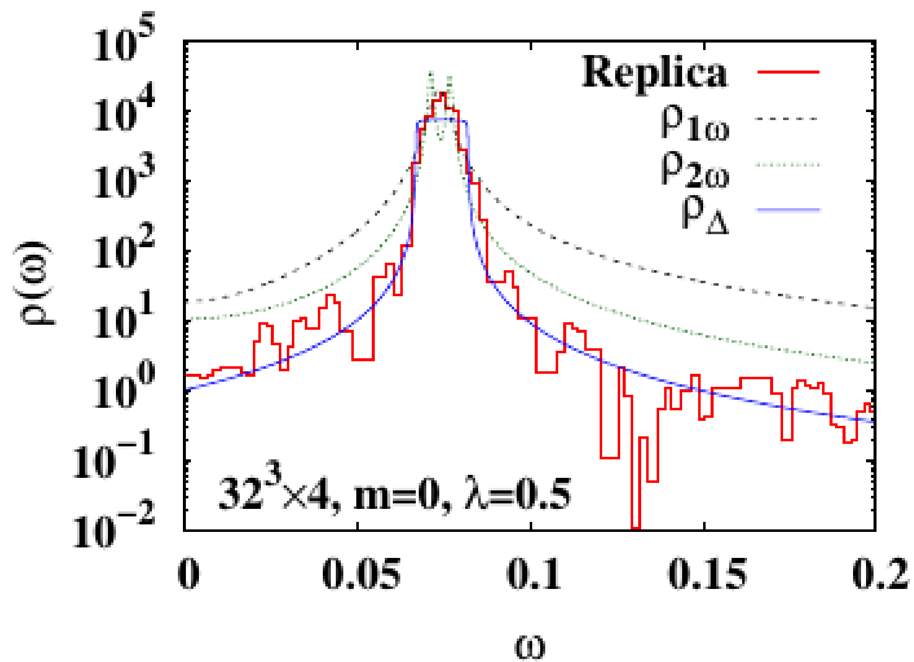
$$\mathcal{Z}_R = \int \mathcal{D}\pi\,\mathcal{D}\phi\,\exp(-\mathcal{H}/\xi) \propto \int \mathcal{D}\phi\,\exp(-S[\phi])$$

Replica Evolution
= Classical Dynamics
with Quantum Statistics in Equilibrium

Damping Rate

- **Apparent damping rate in replica evolution is larger than 2-loop results at small coupling. Why ?**
  - **Classical results (N=1) better agrees with 2-loop results.** *Aarts ('01)*
  - **Power spectrum shows wide spread of the mass, but falls off quickly. Fragmentation of zero-momentum single particle mode ?**





Commutator in Classical Dynamics

- **Classical-Quantum Correspondence**

$$[A, B] \to i\hbar\{A, B\}_{\mathrm{PB}} + \mathcal{O}(\hbar^3)$$

- **Unequal-time Poisson bracket** *Aarts ('01)*

$$\langle \frac{1}{2}[\hat{x}_H(t), \hat{x}_H(0)]\rangle \simeq \left\langle \frac{i}{2}\{x(t), x(0)\}_{\mathrm{PB}} \right\rangle$$

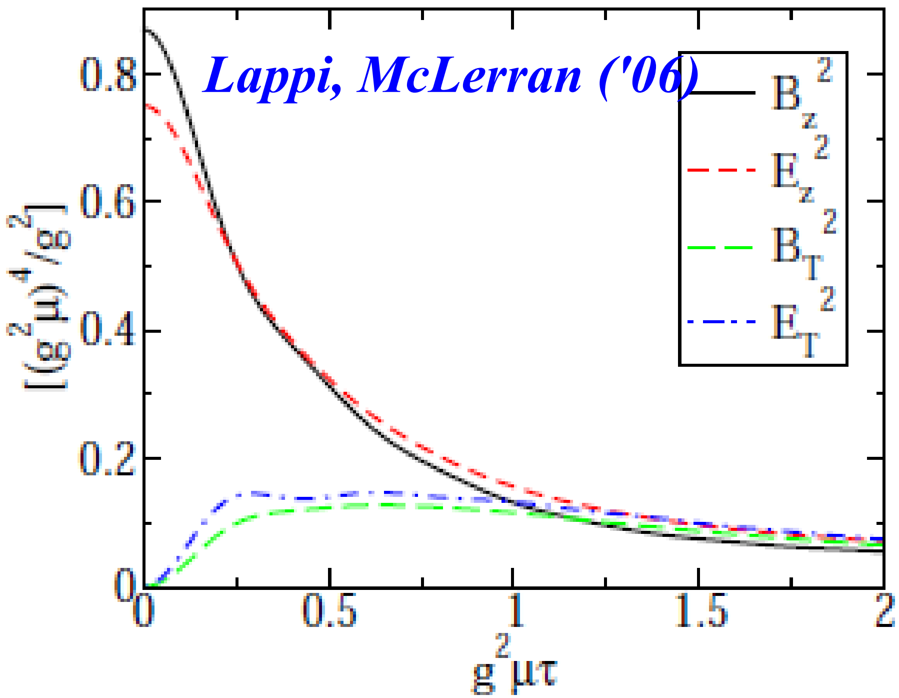
$$= \frac{i}{2}\langle \sum_{n,n'} \left[\frac{\partial \bar{x}_n(t)}{\partial \bar{x}_{n'}(t_0)} \frac{\partial \bar{x}_n(0)}{\partial \bar{p}_{n'}(t_0)} - \frac{\partial \bar{x}_n(t)}{\partial \bar{p}_{n'}(t_0)} \frac{\partial \bar{x}_n(0)}{\partial \bar{x}_{n'}(t_0)} \right] \rangle$$

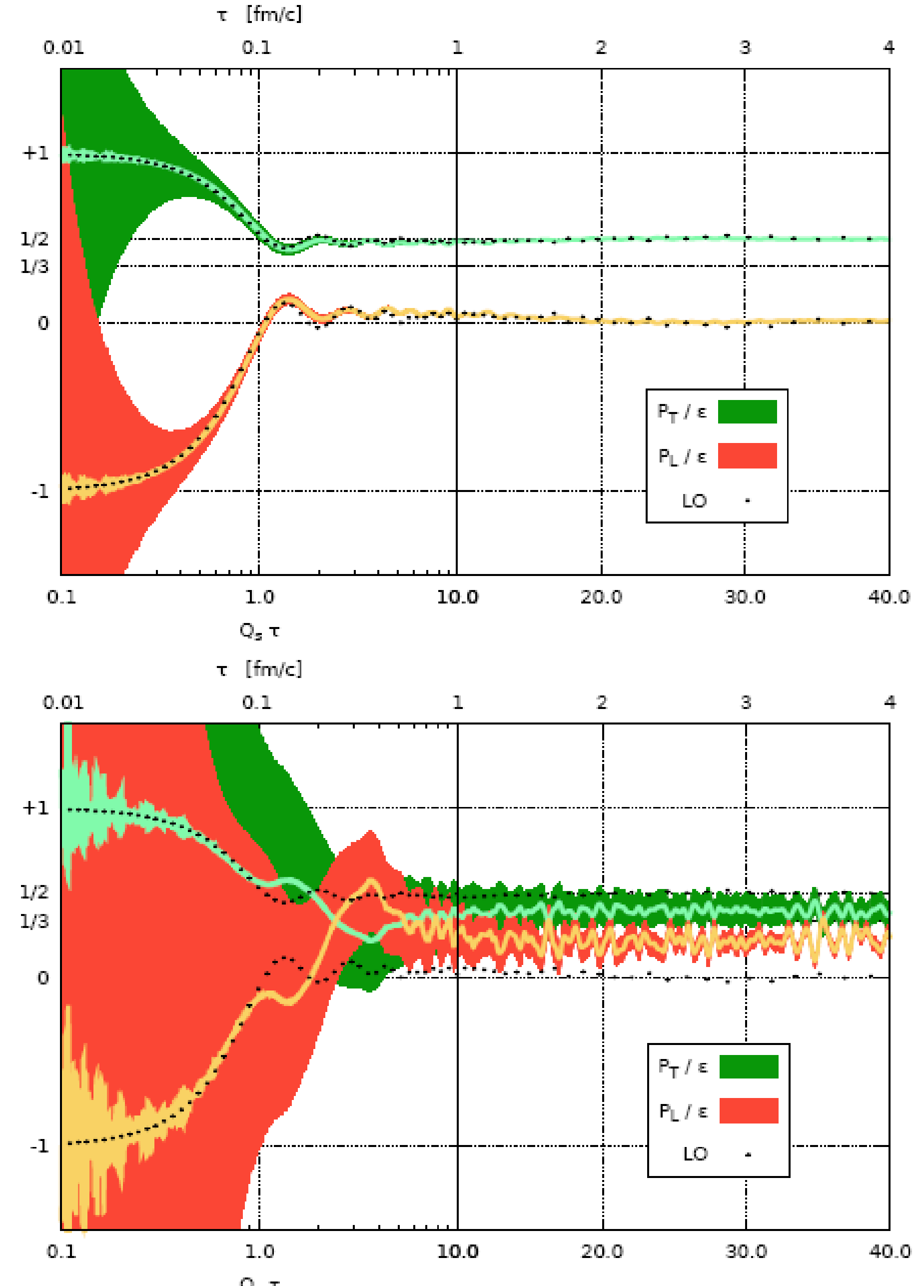
$$\xrightarrow[\text{Free}]{} -\frac{i}{2} \sum_n \frac{1}{M_n} \sin M_n t$$

  - **n=0 term reproduces quantum mechanical result in a HO.**
  - **Unequal-time derivative can be obtained by using the Trotter formula together with Hessian matrix.**
    Kunihiro, Muller, AO, Schafer, Takahashi, Yamamoto ('10)

Real-time evolution of Classical Yang-Mills field

- **Classical Statistical Simulation**
  McLerran, Venugopalan ('94), Romatschke, Venugopalan ('06), Lappi, McLerran ('06), Berges, Scheffler, Sexty ('08), Fukushima ('11), Fukushima, Gelis ('12), Epelbaum, Gelis ('13)





Epelbaum, Gelis ('13)

Approaches using 2PI effective action

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Classical Aspects of Quantum Fields Far from Equilibrium

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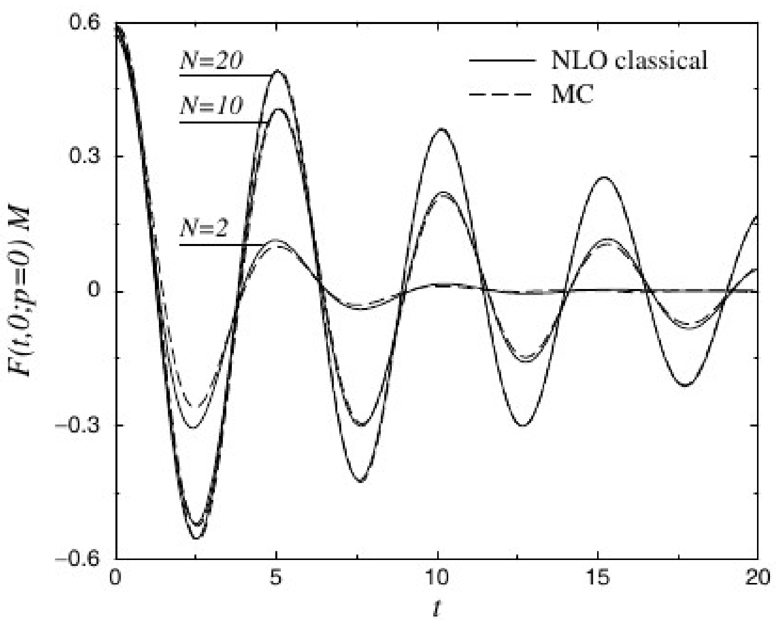
(Received 16 July 2001; published 15 January 2002)

We consider the time evolution of nonequilibrium quantum scalar fields in the $O(N)$ model, using the next-to-leading order $1/N$ expansion of the two-particle irreducible effective action. A comparison with exact numerical simulations in $1 + 1$ dimensions in the classical limit shows that the $1/N$ expansion gives quantitatively precise results already for moderate values of N. For sufficiently high initial occupation numbers the time evolution of quantum fields is shown to be accurately described by classical physics. Eventually the correspondence breaks down due to the difference between classical and quantum thermal equilibrium.



Time-correlation function is reasonably described by classical field, but statistics in equilibrium is problematic.

Isn't it enough to use perturbative and lattice field theory

- **Hydrodynamics with EOS and transport coeff. from pQCD and/or LQCD**
  R. Baier, A.H. Mueller, D. Schiff, D.T. Son ('01, pQCD, τ_{th}), P.Arnold, D.G.Moore, L.G.Yaffe ('03, pQCD, η); A. Nakamura, S. Sakai ('05, LQCD, η); A. Bazavov et al. [HotQCD] ('14, LQCD, EOS); S. Borsanyi et al. ('14, LQCD, EOS)
  - **Not enough: early thermalization puzzle, large η (pQCD), large uncertainty in η (LQCD)**
- **Why ? Background field effect ?**
  - **Anomalous viscosity under strong disordered field**
    M. Asakawa, S. A. Bass, B. Müller ('06)
    Under disordered background field, momentum transfer is promoted more than perturbation predicts → Small η
  - **Classical field evolution also predict small η**
    H. Matsuda, T. Kunihiro, AO, T.T.Takahashi ('20)

$$\eta \propto (g^4 \log(1/g))^{-1} \ (\mathrm{pQCD}) \to \eta \propto g^{-3/2} (\mathrm{ABM,\ CYM})$$

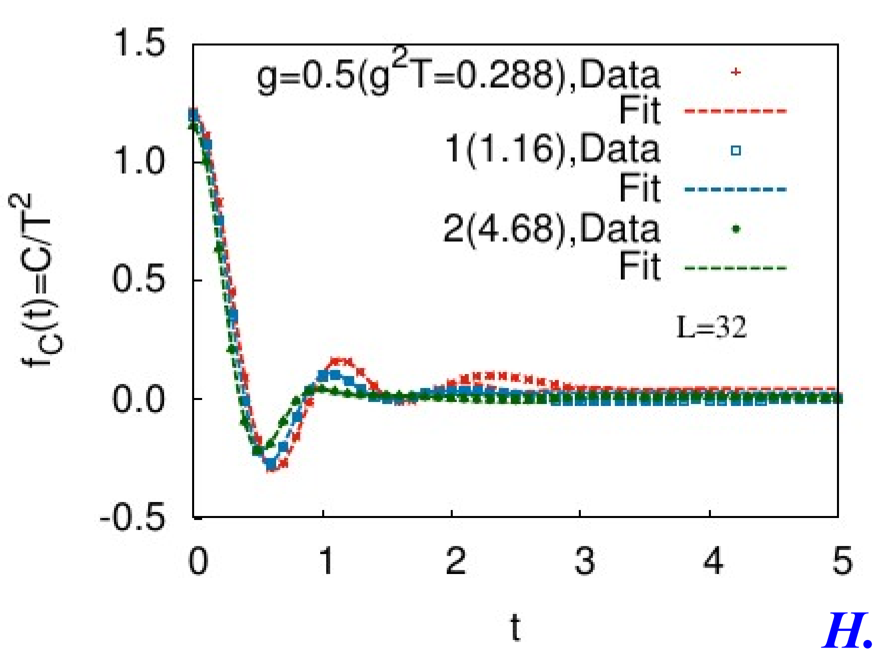
We need evolution of quantum field under inhomogeneous And non-equilibrium background field.

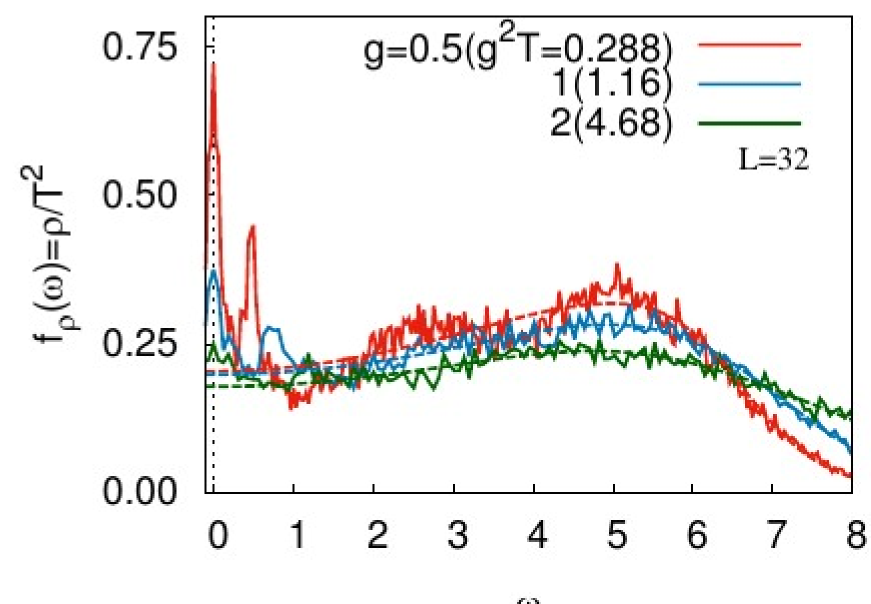
Shear Viscosity of Classical Yang-Mills field

- **Green-Kubo formula**

$$\eta = \frac{1}{T}\int_0^\infty dt\, C(t), \quad C(t) = \frac{V}{3}\sum_{i<j}\tau_{ij}(t)\tau_{ij}(0)$$

  - **Numerical integration of the time-correlation function of the energy-momentum tensor of classical field.**
  - **This should be simulating η of IP-glasma model.**





H. Matsuda, T. Kunihiro, AO, T.T.Takahashi, arXiv:2007.06886 [hep-ph]

Anomalous Shear Viscosity

- **Anomalous viscosity under strong disordered field**
  M. Asakawa, S. A. Bass, B. Müller, PRL96 ('06)252301; PTP116 ('07) 725.
  Disordered background field promotes momentum transfer.

$$\eta_A = \left(\frac{2(N_c^2-1)\nu_4\zeta(4)T\tau}{25b_0N_c\nu_2^2\zeta(2)}\right)^{1/2}\frac{s}{g^{3/2}}.$$

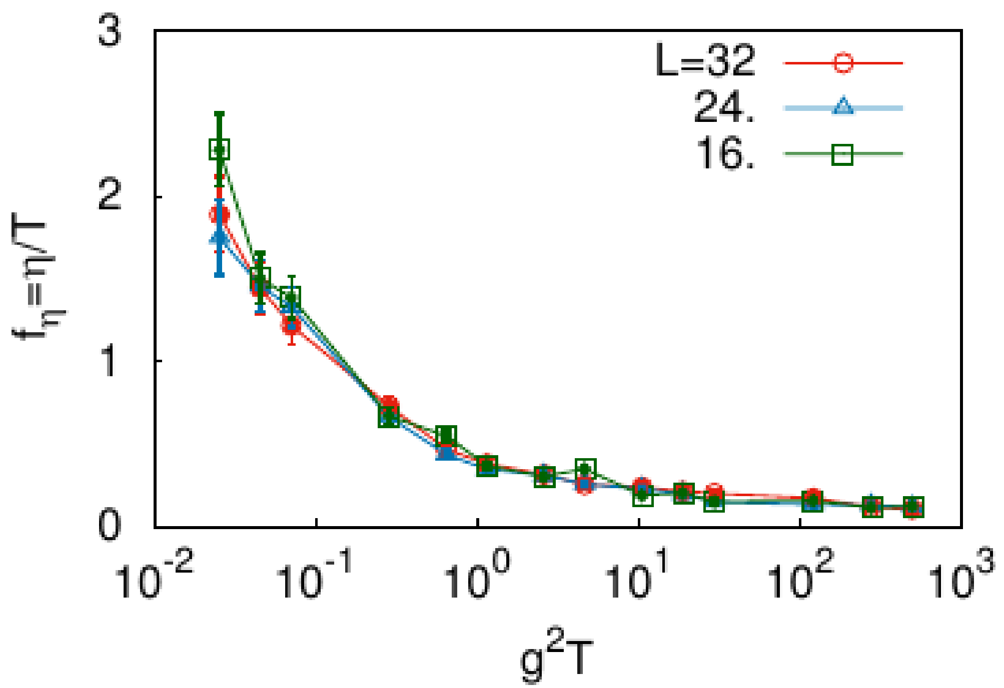
$\eta \propto (g^4\log(1/g))^{-1}$ (pQCD)

- **Classical Field simulation supports this idea.**
  H. Matsuda, T. Kunihiro, AO, T.T.Takahashi (arXiv:2007.06886)

$$\alpha x^{-\beta/2}+\gamma x^{-\delta/2}$$

$$\alpha = 0.09\pm0.07,\ \boxed{\beta = 1.49\pm0.39,}$$
$$\gamma = 0.33\pm0.06,\ \delta = 0.35\pm0.07.$$

x=g^{2}T



Application to Gauge theories and Fermion Systems

- **Gauge theory**
  - **Temporal component of the gauge field is Wick-rotated in the imaginary time formalism, and replica evolution cannot be applied as it is, except for the case in the temporal gauge (A_0=0).**
- **Fermions**
  - **We do not know (yet) how to handle Grassman number in replica.**
  - **Time-dependent Hartree-Fock theory may help.**