4+1次元時空でのレプリカ発展による量子統計性をもつ古典場理論の提案 (17pSH−14)

大西明[A]、松田英史[B]、国広悌二[A]、高橋徹[C]
京大基研[A]、京大理[B]、群馬高専[C]

arXiv:2008.09556 [hep-lat], 5 August 2020

・イントロダクション

・レプリカ発展法

・スカラー場理論への適用

- レイリー・ジーンズ発散、熱質量(thermal mass)

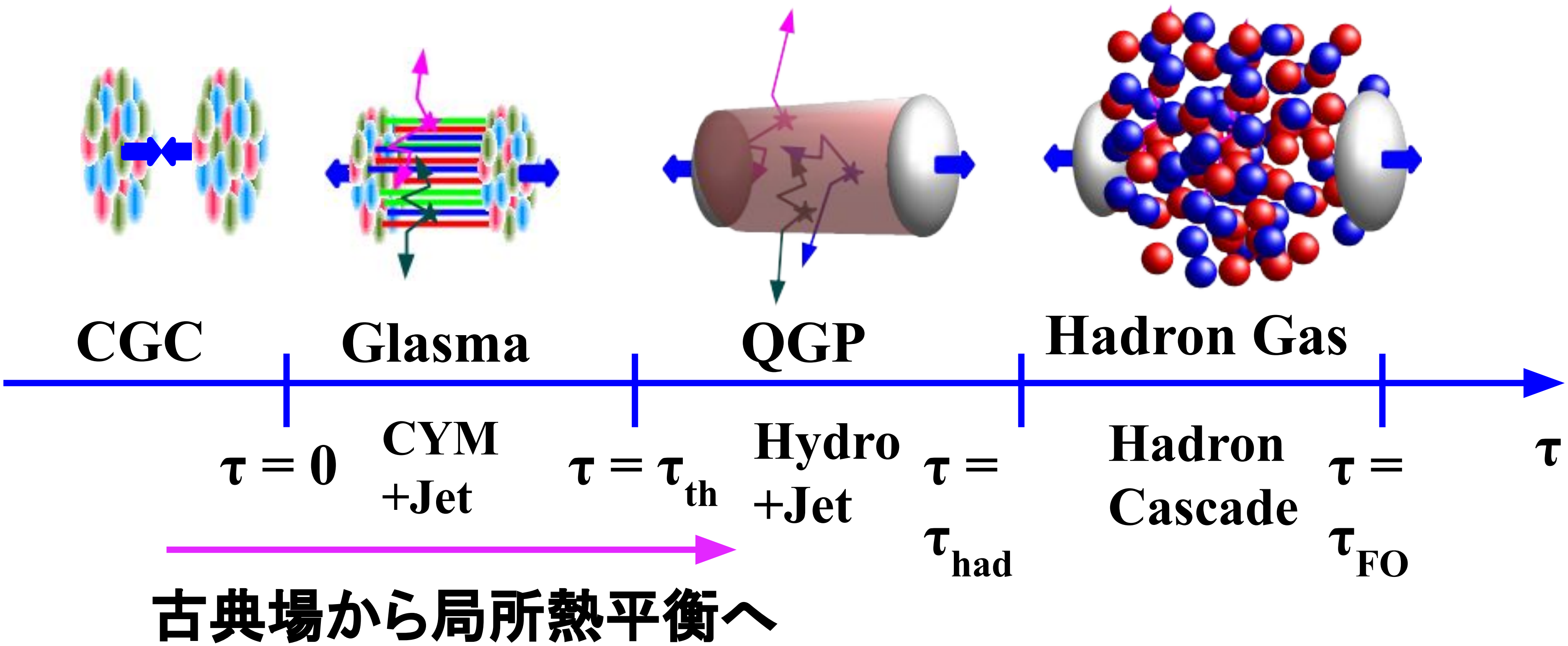
・まとめ

日本物理学会2020年秋季大会

Sep.14-17, 2020 (Tsukuba → Zoom).

格子上の場の理論で重イオン衝突？

- **高エネルギー重イオン衝突の初期段階**
  古典場が支配的な段階(CGC, Glasma)から熱平衡(Hydro., QGP)へ
  → 古典的な背景場の中での量子場の緩和・発展過程



- **しかし量子場の時間発展には強烈な符号問題が….**
  叶わぬ夢か？

$$S_{fi} = \mathcal{N} \int \mathcal{D}\phi \, \langle \Psi(t_f) | \exp(iS[\phi]) | \Psi(t_i) \rangle$$

量子場の時間発展への取り組み

- 古典場発展: 最小作用の原理δS=0 → 符号問題なし
  - 多くの非平衡現象で有用
    condensate (Time dep. Gross-Pitaevski), nuclei (TD Hartree-Fock), Inflation, high-energy heavy-ion collisions (classical Yang-Mills), …
  - ただし古典平衡に近づき、連続極限でエネルギー密度は発散
    $$n_{\boldsymbol{k}} = T/\omega_{\boldsymbol{k}} (\text{Classical}), \quad n_{\boldsymbol{k}} = [\exp(\omega_{\boldsymbol{k}}) \mp 1]^{-1} (\text{Quantum})$$

- (有限の古典場を含む)量子場の実時間発展への取り組み
  - Closed Time Path+2PI作用による記述(Kadanoff-Baym方程式)
    Aarts, Berges ('02), Hatta, Nishiyama ('12)
    → 背景場が一様でない場合には計算量が膨大
  - 高運動量自由度を先に積分and/or別扱い
    Bodeker, McLerran, Smilga ('95), Greiner, B.Muller ('97),
    Dumitru, Nara, (Strickland) ('05,'07) …
    → 古典場部分は古典統計、古典場と粒子の変換はnon-trivial

平衡で"量子統計性"をもつ古典場の枠組みがあれば… → レプリカ発展

レプリカ発展
(Replica evolution)

レプリカ発展での分配関数(量子力学)

- 古典力学での分配関数

$$H=\frac{p^2}{2}+U(x) \qquad \mathcal{Z}_C(T)=\int\frac{dxdp}{2\pi}\exp\left[-\frac{H(x,p)}{T}\right]$$

- 量子力学での分配関数(虚時間法)

$$S=\frac{1}{\xi}\left[\mathcal{V}+\sum_{\tau=1}^{N}U(x_\tau)\right] \qquad \mathcal{Z}_Q(T)=\int\mathcal{D}x\exp\left(-S[x]\right)$$

$$\mathcal{V}=\sum_{\tau=1}^{N}\frac{\xi^2}{2}(x_{\tau+1}-x_\tau)^2\simeq\xi\int_0^{1/T}d\bar{\tau}\frac{1}{2}\left[\frac{\partial x}{\partial\bar{\tau}}\right]^2, \quad \xi=NT$$

- レプリカ発展(虚時間微分項を相互作用とみなす)での分配関数

$$\mathcal{H}=\sum_{\tau=1}^{N}\left[\frac{p_\tau^2}{2}+U(x_\tau)\right]+\mathcal{V}$$

$H(x_\tau,p_\tau)$ $\qquad \xi S[\phi]$

$$\mathcal{Z}_R(\xi)=\int\frac{\mathcal{D}x\mathcal{D}p}{2\pi}\exp\left(-\frac{\mathcal{H}[x,p]}{\xi}\right)$$
$$=(2\pi\xi)^{NL^3/2}Z_Q(T)$$

虚時間微分項で相互作用するN個の古典場の分配関数(温度 ξ)
∝量子力学での分配関数(温度T=ξ/N)

レプリカ指標平均～古典運動

- レプリカ座標 (x_τ, p_τ) の運動方程式=正準運動方程式

$$\frac{dx_\tau}{dt} = \frac{\partial \mathcal{H}}{\partial p_\tau} = p_\tau$$

$$\frac{dp_\tau}{dt} = -\frac{\partial \mathcal{H}}{\partial x_\tau} = -\frac{\partial U(x_\tau)}{\partial x_\tau} + \xi^2(x_{\tau+1} + x_{\tau-1} - 2x_\tau)$$

τ-derivative terms

- レプリカ指標平均

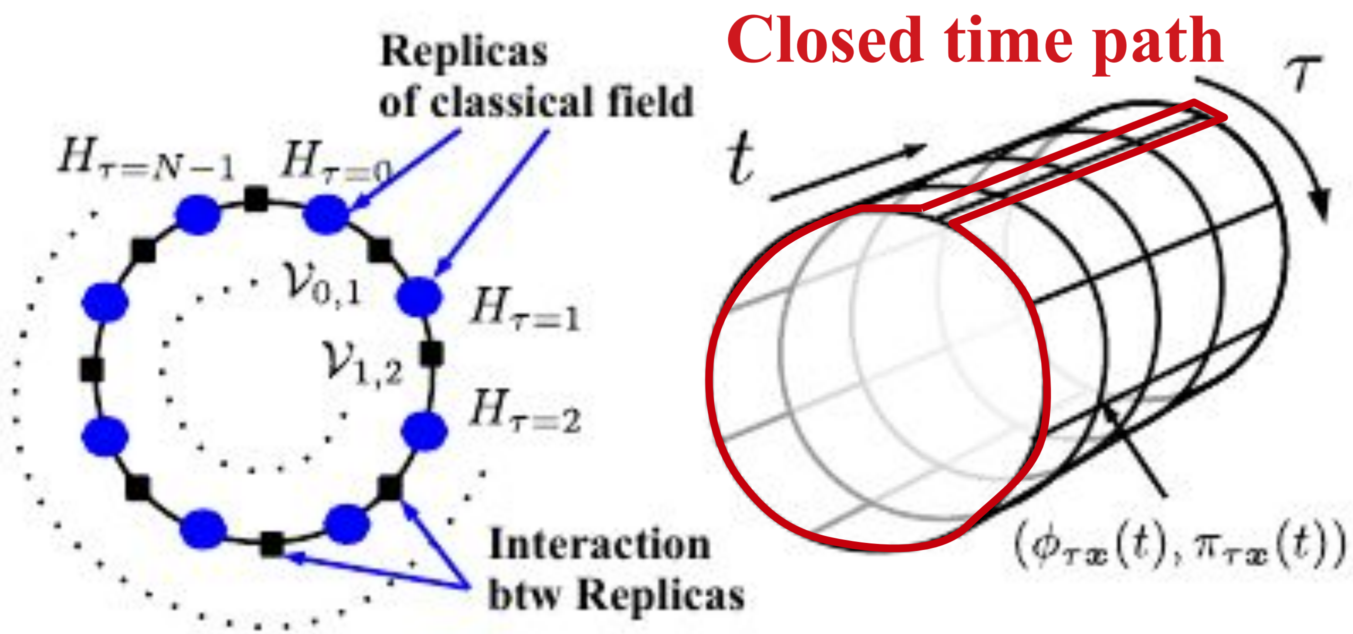
$$\frac{d\widetilde{x}}{dt} = \frac{1}{N}\sum_\tau \frac{dx_\tau}{dt} = \widetilde{p}$$

$$\frac{d\widetilde{p}}{dt} = \frac{1}{N}\sum_\tau \frac{dp_\tau}{dt} = -\frac{1}{N}\sum_\tau \frac{\partial U(x_\tau)}{\partial x_\tau} + 0 \quad \text{(Ehrenfest's theorem)}$$

$$= -\frac{\partial U(\widetilde{x})}{\partial \widetilde{x}} + \mathcal{O}((\delta x)^2)$$

ゆらぎが小さい場合には、
レプリカ指標平均(τ平均)は古典運動方程式に従う

レプリカ発展 (Replica Evolution)



$$\mathcal{H} = \sum_{\tau} H_{\tau} + \sum_{\tau} \mathcal{V}_{\tau,\tau+1} = \frac{1}{2}\sum_{\tau,\boldsymbol{x}} \pi^2_{\tau,\boldsymbol{x}} + \xi S[\phi]$$

$$\mathcal{Z}_R = \int \mathcal{D}\pi\,\mathcal{D}\phi\,\exp(-\mathcal{H}/\xi) \propto \int \mathcal{D}\phi\,\exp(-S[\phi])$$

レプリカ発展＝量子統計性をもつ古典力学

スカラー場理論への適用

スカラー場のレプリカ発展

- 場の理論でのレプリカ発展
  - レプリカ変数 $(x_\tau, p_\tau) \to (\phi_{\tau\boldsymbol{x}}, \pi_{\tau\boldsymbol{x}})$
  - 場の理論 → 質量くりこみ、ゼロ点エネルギーの除去

- 例: Φ^4 theory

$$\mathcal{H} = \sum_{\tau,\boldsymbol{x}} \left[\frac{\pi_{\tau\boldsymbol{x}}^2}{2} + \frac{1}{2}(\nabla\phi_{\tau\boldsymbol{x}})^2 + \frac{m^2}{2}\phi_{\tau\boldsymbol{x}}^2 + \frac{\lambda}{24}\phi_{\tau\boldsymbol{x}}^4 + \frac{\xi^2}{2}(\phi_{\tau+1,\boldsymbol{x}} - \phi_{\tau\boldsymbol{x}})^2 \right]$$

$\mathcal{V}$

$H(\phi_{\tau\boldsymbol{x}}, \pi_{\tau\boldsymbol{x}})$

$\xi S[\phi]$

$-\frac{\delta m^2}{2}\phi_{\tau\boldsymbol{x}}^2$

- 質量くりこみ (1 loop)

Aarts, Smit ('97), Kapusta, Gale (textbook)

$$\delta m^2 = \frac{\lambda}{2}\langle\phi^2\rangle_{\rm div}$$

$$\langle\phi^2\rangle_{\rm div} = \frac{1}{L^3}\sum_{\boldsymbol{k}} \frac{1}{2\omega_{\boldsymbol{k}}\sqrt{1+(\omega_{\boldsymbol{k}}/2\xi)^2}}$$

運動量分布

- レプリカでの運動量分布（自由場）

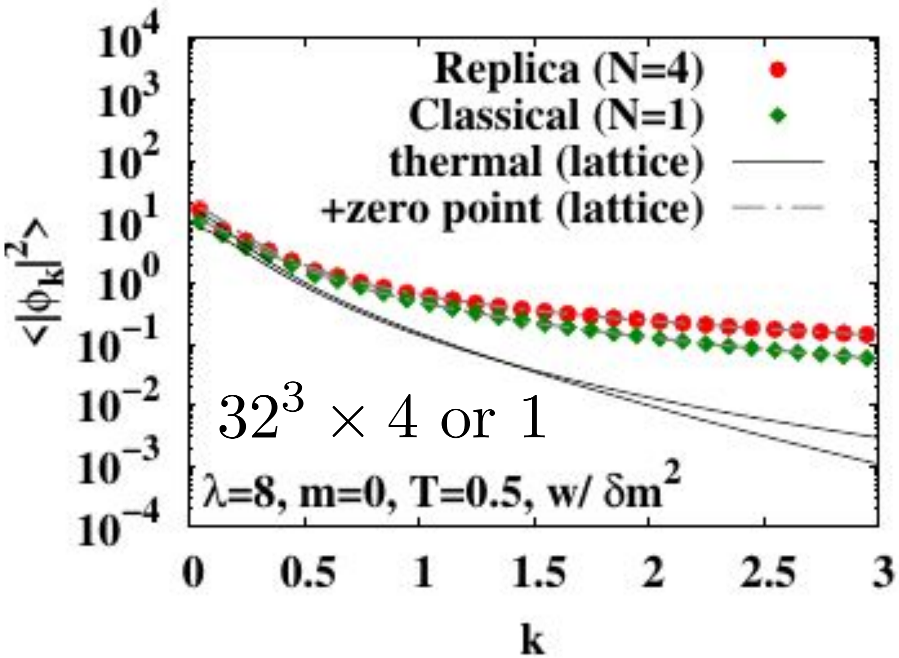
$$\langle|\phi_{\boldsymbol{k}}|^2\rangle = \frac{1}{N}\sum_n \langle\phi_{n\boldsymbol{k}}\phi^*_{n\boldsymbol{k}}\rangle = \frac{1}{\omega_{\boldsymbol{k}}\sqrt{1+(\omega_{\boldsymbol{k}}/2\xi)^2}}\left[\frac{1}{2}+\frac{1}{e^{\Omega_{\boldsymbol{k}}/T}-1}\right]$$

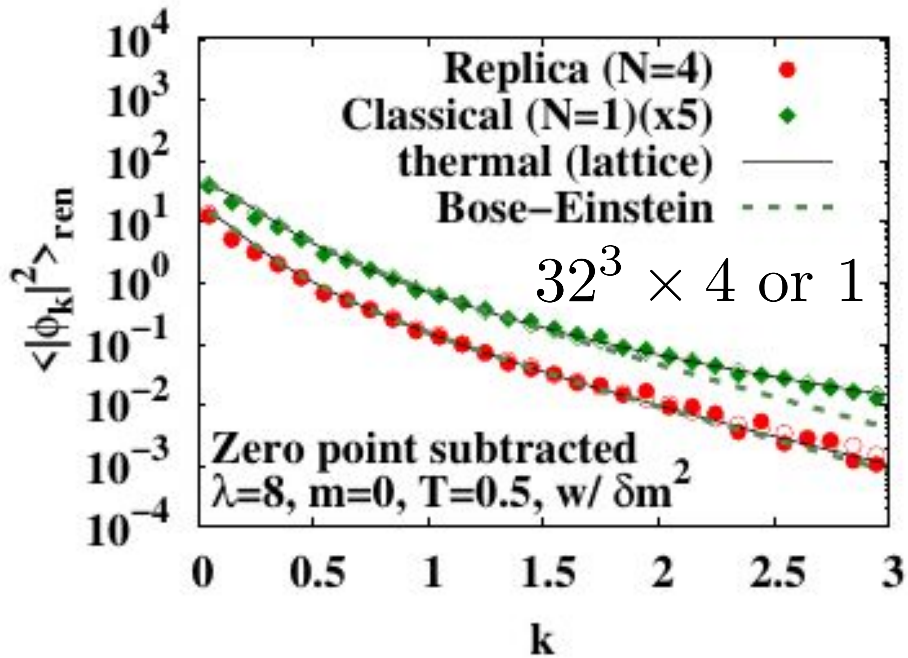
Free field, Matsubara sum

Zero point

Thermal → Bose-Einstein

- ゼロ点部分を取り除くと、レイリー・ジーンズ発散が取り除ける。（N=1（通常の古典場）では完全には取り除けない）



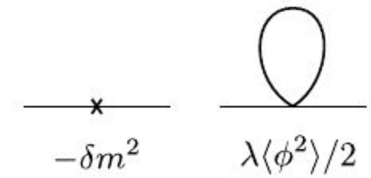


レプリカ発展法での熱質量

Kapusta, Gale (textbook)
Parwani ('92, '93)

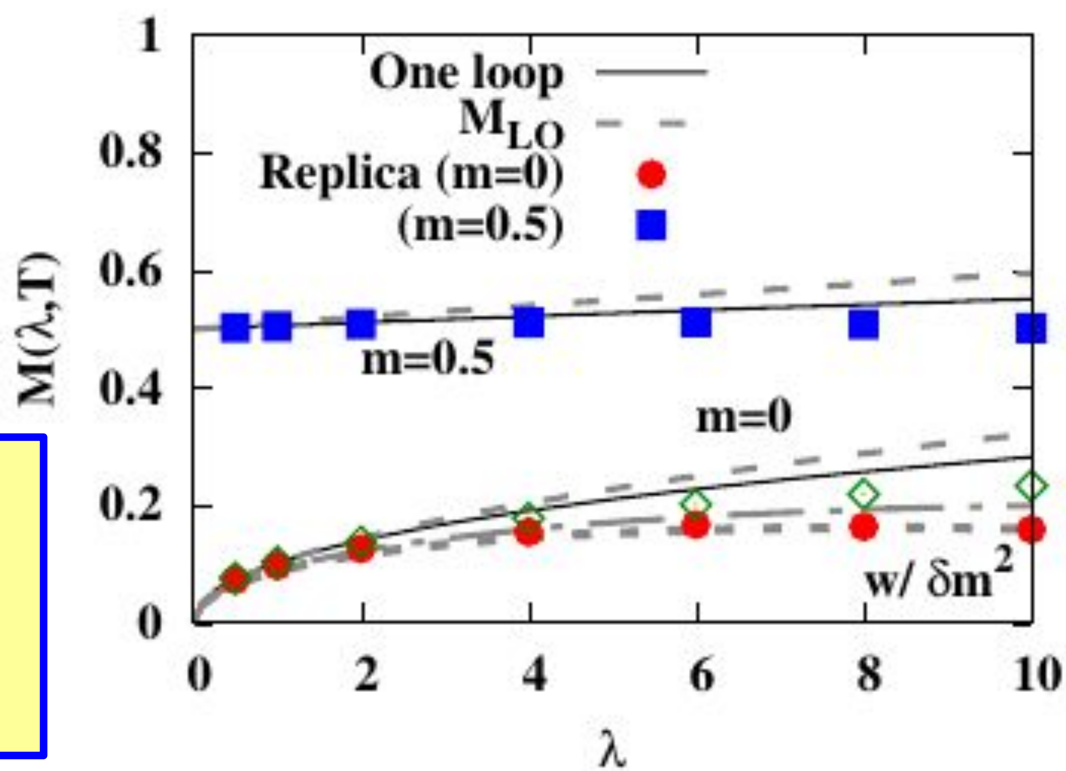
- くりこみ後の熱質量
  - **Leading Order (one-loop)** $M_{\rm LO}^2 = m^2 + \lambda T^2/24.$
  - **Resummed One-Loop** $M_{\rm resum}^2 = \dfrac{\lambda T^2}{24}\left[1 - \dfrac{3}{\pi}\sqrt{\dfrac{\lambda}{24}}\right]$
  - **Two-Loop**

$$M_{2-\rm loop}^2 = \frac{\lambda T^2}{24}\left\{1 - \frac{3}{\pi}\sqrt{\frac{\lambda}{24}} + \frac{\lambda}{(4\pi)^2}\left[\frac{3}{2}\log\left(\frac{T^2}{4\pi\mu^2}\right) + 2\log\left(\frac{\lambda}{24}\right) + \alpha\right]\right\},$$



- レプリカ発展
  → 時間相関関数から得られた
  熱質量

$$C(t) = \langle \phi_{p=0}(t)\phi_{p=0}(0)\rangle$$



レプリカ発展法での熱質量
→1ループを越える
相互作用効果を含む

まとめ

- **量子統計性をもつ古典力学(古典場理論)である レプリカ発展法を提案した。**
  - 虚時間座標をレプリカ指標(replica index)として、虚時間形式で現れる τ 微分項をポテンシャルとしてみなす、4(＝3＋1)次元空間での場の変数の古典的時間発展
  - ・ 技術的にはHMCの分子動力学部分
  - ・ 古典場の性質と量子統計性を併せ持つ。
  - ・ レイリー・ジーンズ発散が無く、1ループを越えた相互作用効果を示す

・ **今後の課題**

- これまでに提案されている手法との比較
  Hard mode effects [Bodeker, McLerran, Smilga ('95), Greiner, B.Muller ('97)], Field-particle sim. [Dumitru, Nara ('05), Dumitru, Nara, Strickland ('07)], 2PI [Aarts, Berges ('02), Hatta, Nishiyama ('12)], …
- 輸送方程式との関連　*Classical field → Boltzmann eq., A.Muller, Son ('04).*
- ・ ずり粘性、非平衡発展(エントロピー生成、量子クエンチなど)、…

ご清聴ありがとうございました。

Hidefumi Matsuda **Toru T. Takahashi**

AO **Teiji Kunihiro**

AO, H. Matsuda, T. Kunihiro, T. T. Takahashi,
Replica evolution of classical field in 4+1 dimensional spacetime toward real time dynamics of quantum field,
arXiv:2008.09556 [hep-lat]

Replica Evolution

- Replicas = N classical systems interacting with τ-derivative terms (V)

$$\mathcal{V} = \sum_{\tau=1}^{N} \frac{\xi^2}{2}(x_{\tau+1} - x_\tau)^2 \simeq \xi \int_0^{1/T} d\bar{\tau} \frac{1}{2} \left[\frac{\partial x}{\partial \bar{\tau}}\right]^2$$

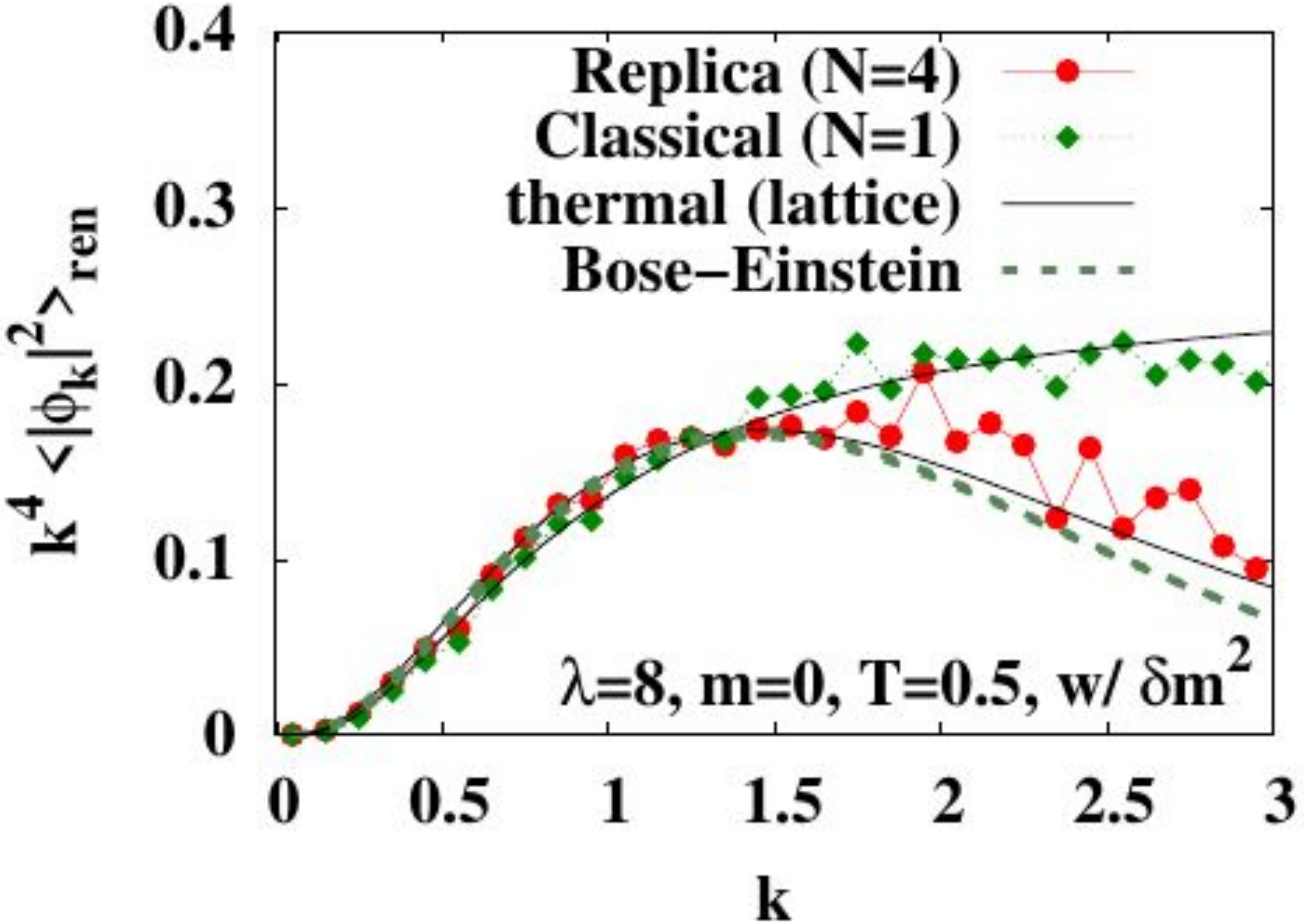
  - Replica variables (x_τ, p_τ) are assumed to evolve with canonical EOM
  - Long (real) time evolution in 4+1D spacetime (space, τ, t) samples correct quantum statistical configurations of x_τ.
    ~ MD part of HMC
  - Replica index (τ) average of (x_τ, p_τ) obeys the classical EOM.
- Replica evolution of field
  - Replace variables $(x_\tau, p_\tau) \to (\phi_{\tau \boldsymbol{x}}, \pi_{\tau \boldsymbol{x}})$
  - Mass renormalization & Subtracting zero point contribution
- How about dynamical properties of replica evolution ?

I will skip this page...

Lattice Setup

- **Lattice size: 32^3 x 4 (L=32, N=4)**
- **Temperature: T=0.5, Coupling: λ=0.5-10, bare mass:m=0, 0.5**
- **Average over replica index (τ) and replica ensemble (N_{conf}=1000)**
- **Thermal ensemble is prepared by solving the Langevin equation at temperature ξ=NT=2.**
- **EOM is solved in the leap-frog method (reversible !) with the time step of Δt=0.025 until t=500 after equilibration.**
- **A few hours for each (λ, m) on iCore7 PC (w/o MP).**

Momentum Distribution



$$\frac{1}{N}\sum_{\tau\boldsymbol{x}}\frac{1}{2}(\boldsymbol{\nabla}\phi_{\tau\boldsymbol{x}})^2=\frac{1}{N}\sum_{n\boldsymbol{k}}\frac{1}{2}\boldsymbol{k}^2|\phi_{n\boldsymbol{k}}|^2\to L^3\int\frac{d\boldsymbol{k}}{(2\pi)^3}\boldsymbol{k}^2|\phi_{n\boldsymbol{k}}|^2$$

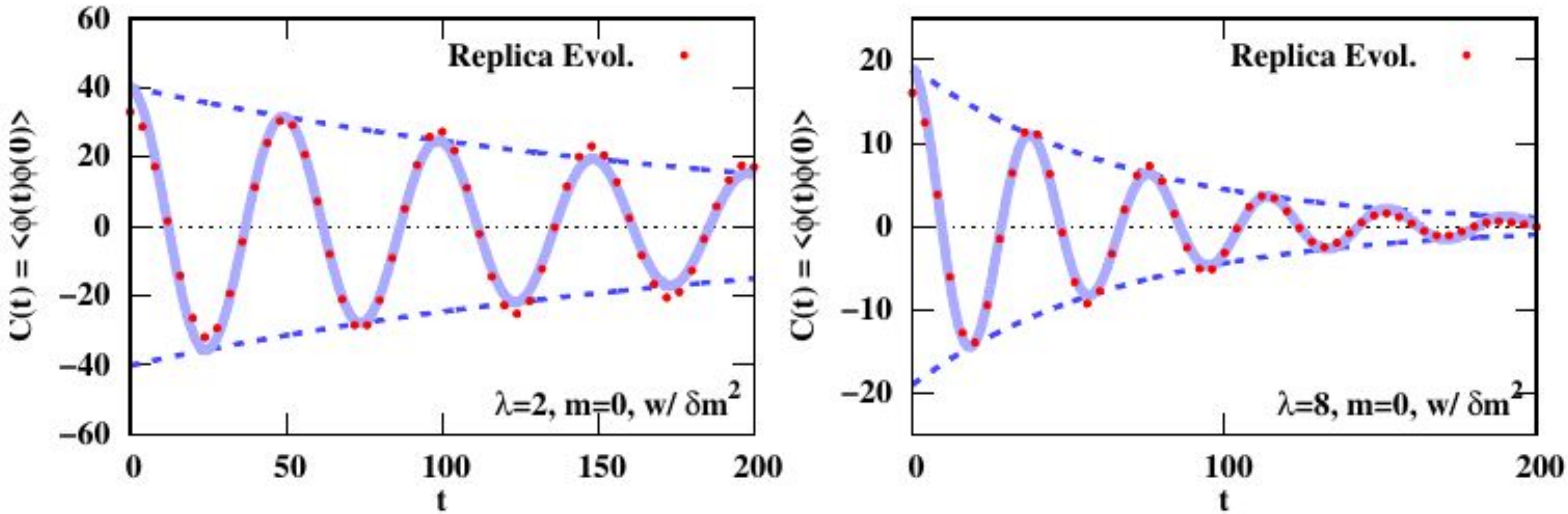
By subtracting the zero point part,
we can avoid the Rayleigh-Jeans divergence of energy.

Time-Correlation Function

- **Time-correlation function (unequal-time two-point function at zero momentum)**

$$C(t) = \frac{1}{L^3}\sum_{\boldsymbol{x},\boldsymbol{y}}\langle\phi_{\boldsymbol{x}}(t)\phi_{\boldsymbol{y}}(0)\rangle \underset{\text{free}}{\longrightarrow} \sum_n \frac{T}{M_n^2}\cos M_n t$$

- **With interaction (non-zero λ), C(t) shows damped oscillatory behavior. → Thermal mass & damping rate**



Replica Evolution of a Single Harmonic Oscillator

Replica Evolution of a Harmonic Oscillator

- **Replica Hamiltonian = N free HO Hamiltonian**

$$\mathcal{H} = \sum_\tau \left[\frac{p_\tau^2}{2} + \frac{\omega^2 x_\tau^2}{2} + \underbrace{\frac{\xi^2}{2}(x_{\tau+1} - x_\tau)^2}_{\mathcal{V}}\right] = \sum_n \left[\frac{\bar{p}_n^2}{2} + \frac{M_n^2 \bar{x}_n^2}{2}\right]$$

Fourier transf.

$$M_n^2 = \omega^2 + 4\xi^2 \sin^2(\pi n/N)$$

- **Expectation value of x^2 in Replica**

Matsubara freq. sum

$\exp(-\mathcal{H}/\xi)$

$$\langle x^2 \rangle = \frac{1}{N}\sum_\tau \langle x_\tau^2 \rangle = \frac{1}{N}\sum_n \langle \bar{x}_n^2 \rangle = \frac{1}{N}\sum_n \frac{\xi}{M_n^2} = \frac{\coth(\Omega/2T)}{2\omega\sqrt{1+\omega^2/4\xi^2}}$$

$$\Omega = 2\xi \mathrm{arcsinh}(\omega/2\xi)$$

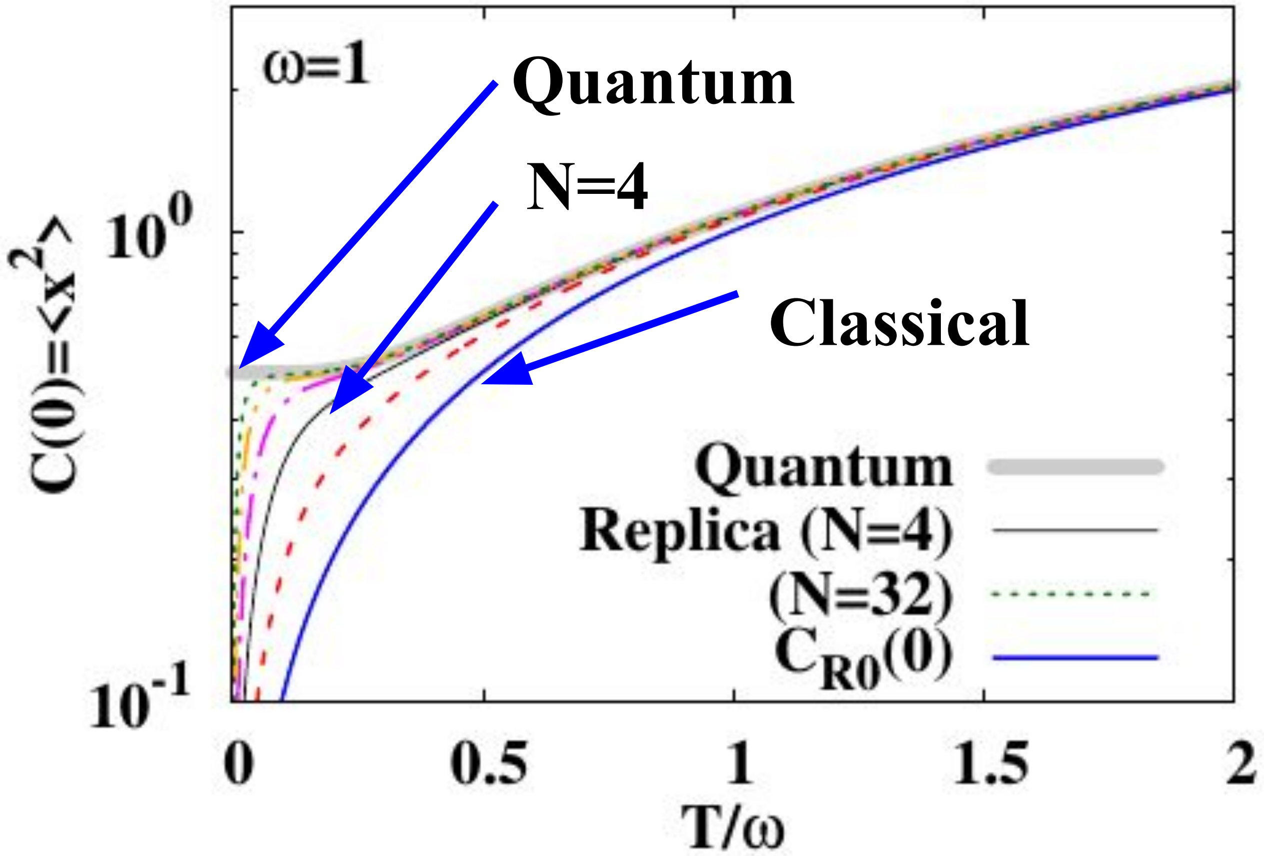
$$\frac{T}{\omega^2} (N = 1, \mathrm{Classical})$$

zero point **thermal**

$$\to \frac{\coth(\omega/2T)}{2\omega} = \frac{1}{\omega}\left[\frac{1}{2} + \frac{1}{e^{\omega/T} - 1}\right] (N \to \infty, \mathrm{Quantum})$$

Equal time observables of x are reproduced at $N \to \infty$

Expectation value of x^2



$$\frac{T}{\omega^2}(N=1,\mathrm{Classical}) \to \frac{\coth(\omega/2T)}{2\omega}(N\to\infty,\mathrm{Quantum})$$

Equal time observables of x are reproduced at $N \to \infty$

Time-Correlation Function

- **Time-correlation function (Unequal-time two-point function)**

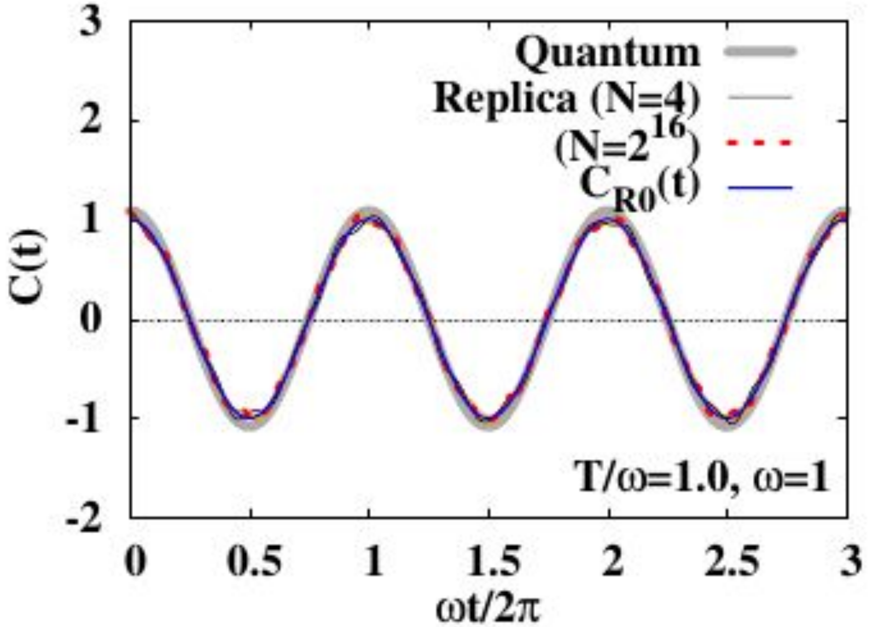
$$C(t) = \left\langle \frac{1}{2}\left[x(t)x(0) + x(0)x(t)\right] \right\rangle$$

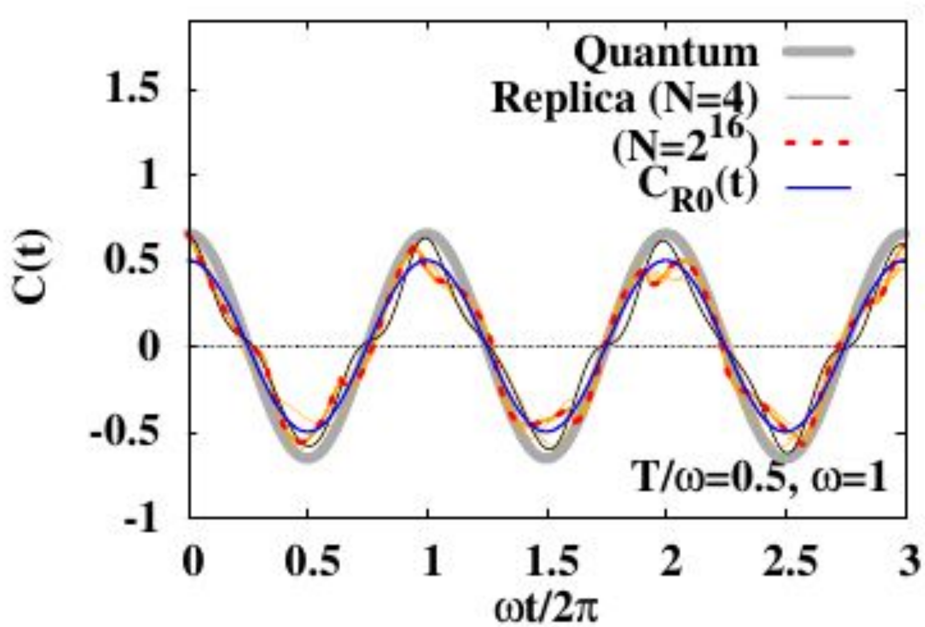
- **Quantum**

$$C_Q(t) = \frac{\coth(\omega/2T)}{2\omega} \cos\omega t$$

Replica

$$C_R(t) = \sum_n \frac{T}{M_n^2} \cos M_n t$$





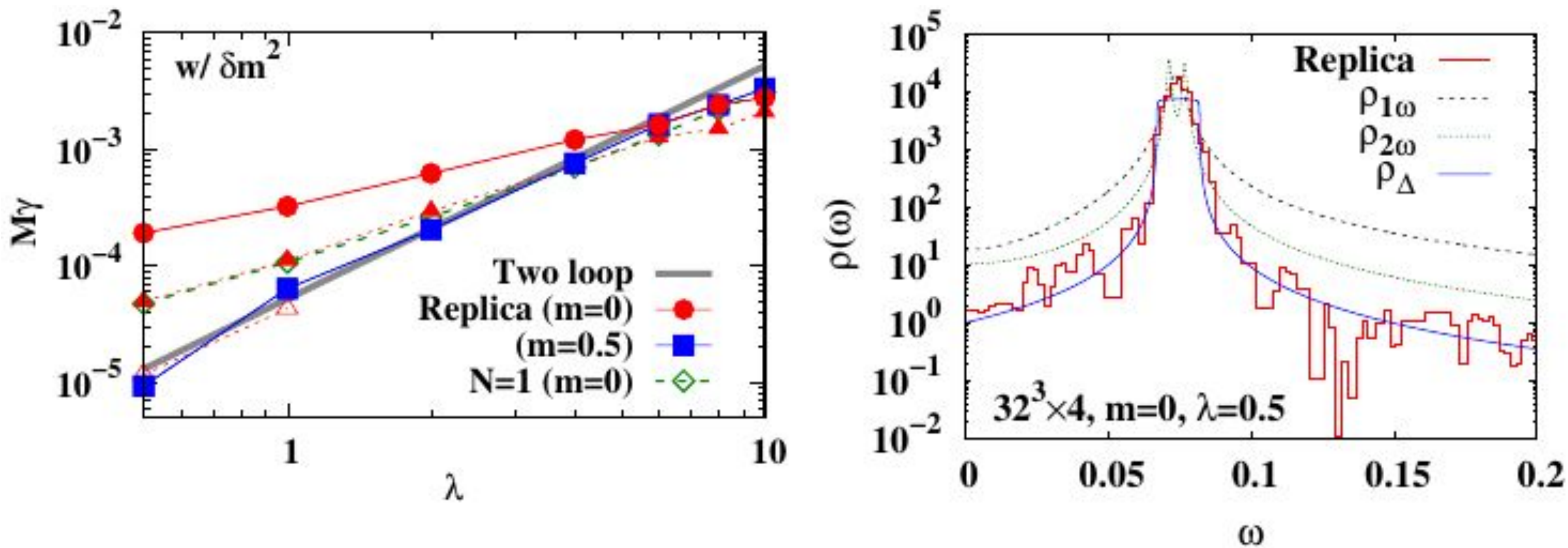
Not perfect, but $C_R(t)$ roughly explains $C_Q(t)$ at $T/\omega > 0.5$

Application to Gauge theories and Fermion Systems

- **Gauge theory**
  - **Temporal component of the gauge field is Wick-rotated in the imaginary time formalism, and replica evolution cannot be applied as it is, except for the case in the temporal gauge (A_0=0).**
- **Fermions**
  - **We do not know (yet) how to handle Grassman number in replica.**
  - **Time-dependent Hartree-Fock theory may help.**

Damping Rate

- **Apparent damping rate in replica evolution is larger than 2-loop results at small coupling. Why ?**
  - **Classical results (N=1) better agrees with 2-loop results.**
    Aarts ('01)
  - **Power spectrum shows wide spread of the mass, but falls off quickly. Fragmentation of zero-momentum single particle mode ?**



Commutator in Classical Dynamics

- **Classical-Quantum Correspondence**

$$[A,B] \to i\hbar\{A,B\}_{\mathrm{PB}} + \mathcal{O}(\hbar^3)$$

- **Unequal-time Poisson bracket** *Aarts ('01)*

$$\langle \frac{1}{2}[\hat{x}_H(t),\hat{x}_H(0)]\rangle \simeq \left\langle \frac{i}{2}\{x(t),x(0)\}_{\mathrm{PB}} \right\rangle$$

$$= \frac{i}{2}\langle \sum_{n,n'} \left[\frac{\partial \bar{x}_n(t)}{\partial \bar{x}_{n'}(t_0)} \frac{\partial \bar{x}_n(0)}{\partial \bar{p}_{n'}(t_0)} - \frac{\partial \bar{x}_n(t)}{\partial \bar{p}_{n'}(t_0)} \frac{\partial \bar{x}_n(0)}{\partial \bar{x}_{n'}(t_0)} \right] \rangle$$

$$\xrightarrow[\text{Free}]{} -\frac{i}{2} \sum_n \frac{1}{M_n} \sin M_n t$$

al result in a HO.

  - **Unequal-time derivative can be obtained by using the Trotter formula together with Hessian matrix.**
    Kunihiro, Muller, AO, Schafer, Takahashi, Yamamoto ('10)

Previous Attempts

- **Separate soft and hard modes**
  Soft modes still have classical statistics, cutoff needs to be small.
  - **Effective action of soft modes by integrating hard modes**
    → dissipation and fluctuation from integrated hard modes
    D. Bodeker, L. D. McLerran and A. V. Smilga, PRD ('95)52;
    C. Greiner and B. Muller, PRD 55 ('97)1026.
  - **Introducing mass counterterm → Similar results with 2PI**
    e.g. G. Aarts and J. Smit, PLB 393 ('97) 395.
- **Coupled equation of field and particles**
  - **Solve coupled equation of field and particles → faster equilibration**
    A. Dumitru and Y. Nara, PLB 621 ('05) 89.
  - **Two particle irreducible (2PI) effective action approach**
    → Large numerical cost to simulate 3+1D fields
    J. Berges, AIP Conf. Proc. 739('04)1; G. Aarts, J. Berges, PRL 88('02)041603; Y. Hatta, A. Nishiyama, NPA 873('12)47.

"Classical" evolution to "Quantum" equilibrium

- **Example: φ^4 theory on a lattice at T=ξ/N**

$$S_E = \frac{1}{\xi}\sum_{\tau=1}^{N}\sum_{\boldsymbol{x}}\left[\frac{1}{2}\left(\partial_\tau \phi_{\tau\boldsymbol{x}}\right)^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi_{\tau\boldsymbol{x}})^2 + \frac{m^2}{2}\phi_{\tau\boldsymbol{x}}^2 + \frac{\lambda}{24}\phi_{\tau\boldsymbol{x}}^4\right] \to \mathcal{Z}_{\mathrm{Q}} = \int \mathcal{D}\phi e^{-S_E}$$

- **Classical Hamiltonian**

$$H(\phi,\pi) = \sum_{\boldsymbol{x}}\left[\frac{1}{2}\pi_{\boldsymbol{x}}^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi_{\boldsymbol{x}})^2 + \frac{m^2}{2}\phi_{\boldsymbol{x}}^2 + \frac{\lambda}{24}\phi_{\boldsymbol{x}}^4\right] \to \mathcal{Z}_{\mathrm{cl}} = \int \mathcal{D}\phi\mathcal{D}\pi e^{-H(\phi,\pi)/T}$$

- **Replica Evolution: Simultaneous evolution of N configs of CF**

$$\mathcal{H} = \sum_{\tau} H(\phi_\tau,\pi_\tau) + \sum_{\tau,\boldsymbol{x}}\frac{\xi^2}{2}\left(\phi_{\tau+1,\boldsymbol{x}} - \phi_{\tau\boldsymbol{x}}\right)^2 = \sum_{\tau,\boldsymbol{x}}\frac{1}{2}\pi_{\tau\boldsymbol{x}}^2 + \xi S_E$$

$$\to \mathcal{Z}_R(T_{\mathrm{repl}}) = \int \mathcal{D}\phi\mathcal{D}\pi e^{-\mathcal{H}/\xi} \propto \int \mathcal{D}\phi e^{-S_E}$$

AO, H. Matsuda, T. Kunihiro, T.T.Takahasi, in prep.

Replica Evolution

- **We consider N classical field configurations, dubbed as replicas, which interact with each other via the τ-derivative potential terms. (~ Molecular dynamics part of the hybrid Monte-Carlo)**
- **Replica evolves according to the classical EOM.**
- **In the replica ensemble at temperature ξ=NT, classical field distribution is described by the quantum partition function in the imag. time formalism after the long real-time evolution.**
- **Question**
  - **Does the replica evolution give correct real-time evolution ?**
  - **Does it describe the thermal mass correctly ?**

Scalar theory (φ^4) on the lattice

- **Lagrangian & Hamiltonian**

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{24}\phi^4$$

$$H(\phi,\pi) = \sum_{\boldsymbol{x}}\left[\frac{1}{2}\pi_{\boldsymbol{x}}^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\phi_{\boldsymbol{x}}\right)^2 + \frac{m^2}{2}\phi_{\boldsymbol{x}}^2 + \frac{\lambda}{24}\phi_{\boldsymbol{x}}^4\right]$$

- **Replica Hamiltonian & Equation of Motion**

fluc. in replica index

$$\mathcal{H} = \sum_\tau H(\phi_\tau,\pi_\tau) + \mathcal{V}, \quad \mathcal{V} = \frac{\xi^2}{2}\sum_{\tau,\boldsymbol{x}}(\phi_{\tau+1,\boldsymbol{x}} - \phi)^2$$

- $$\frac{d\phi_{\tau\boldsymbol{x}}}{dt} = \pi_{\tau\boldsymbol{x}}\ , \quad \frac{d\pi_{\tau\boldsymbol{x}}}{dt} = \frac{\partial H(\phi_\tau,\pi_\tau)}{\partial\phi_{\tau\boldsymbol{x}}} + \xi^2(\phi_{\tau+1,\boldsymbol{x}} + \phi_{\tau-1,\boldsymbol{x}} - 2\phi_{\tau\boldsymbol{x}})$$

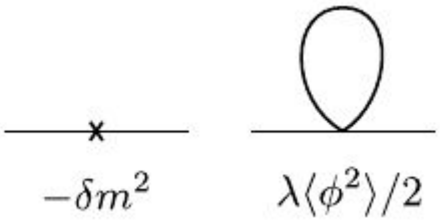
$$\widetilde{\phi}_{\boldsymbol{x}} \equiv \frac{1}{N}\sum_\tau\phi_{\tau\boldsymbol{x}} = \frac{1}{\sqrt{N}}\bar{\phi}_0 \;\to\; (\partial^\mu\partial_\mu + m^2)\widetilde{\phi}_{\boldsymbol{x}} + \frac{\lambda}{6}(\widetilde{\phi}_{\boldsymbol{x}})^3 = \mathcal{O}((\delta\phi_{\boldsymbol{x}})^2)$$

=zero in classical field eq.

Mass Counterterm

- **Leading order thermal mass**

$$M^2 = m^2 - \delta m^2 + \frac{\lambda}{2}\langle\phi^2\rangle_T = m^2 + \frac{\lambda}{2}\left\langle\phi^2\right\rangle_{\rm ren}$$



$$\frac{\lambda}{2}\langle\phi^2\rangle_T = \frac{\lambda}{2}\frac{1}{L^3}\sum_{\boldsymbol{k}}\frac{1}{\omega_{\boldsymbol{k}}\sqrt{1+(\omega_{\boldsymbol{k}}/2\xi)^2}}\left[\frac{1}{2}+\frac{1}{e^{\Omega_{\boldsymbol{k}}/T}-1}\right]$$

Matsubara sum.

δm^2

$\frac{\lambda}{2}\left\langle\phi^2\right\rangle_{\rm ren}$

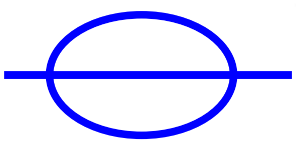
- **Perturbative Calc.**

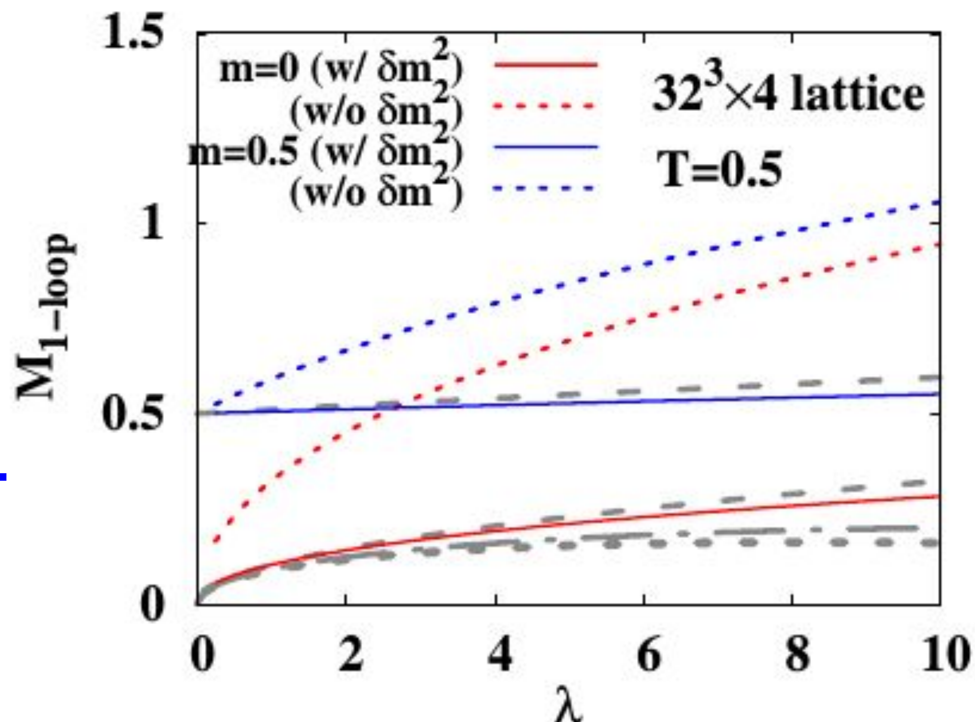
Kapusta, Gale

$$M^2_{\rm LO} = m^2 + \lambda T^2/24.$$

$$M^2_{\rm resum} = \frac{\lambda T^2}{24}\left[1-\frac{3}{\pi}\sqrt{\frac{\lambda}{24}}\right]$$

$$M^2_{2-\rm loop} = \frac{\lambda T^2}{24}\left\{1-\frac{3}{\pi}\sqrt{\frac{\lambda}{24}} + \frac{\lambda}{(4\pi)^2}\left[\frac{3}{2}\log\left(\frac{T^2}{4\pi\mu^2}\right)+2\log\left(\frac{\lambda}{24}\right)+\alpha\right]\right\}$$





R. R. Parwani, PRD 45 ('92)4695 [E:48('93)5965].

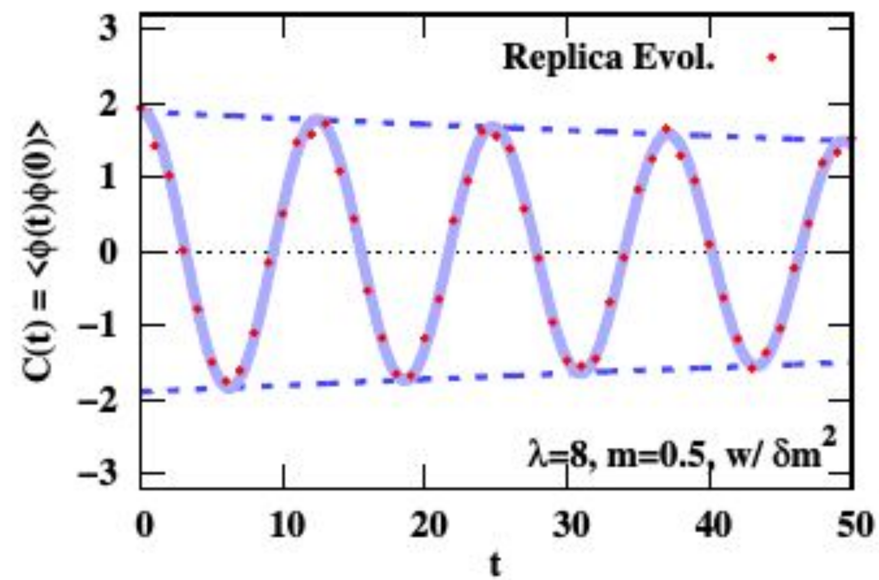
Numerical Calculation Setup

- **Lattice size = 32^3 x 4 (L=32, N=4)**
- **T=0.5 (ξ=NT=2); m=0, 0.5; λ=0.5, 1, 2, 4, 6, 8, 10.**
- **One loop renormalization of mass, no counterterm for λ.**
- **Initial conditions are obtained by solving the Langevin equation.**
- **Solve replica EOM until t=500 with the time step of Δt=0.025.**
- **Number of replica configurations = 1000**
  → 3-6 hours on one core of core i7 PC for a given (m, λ)

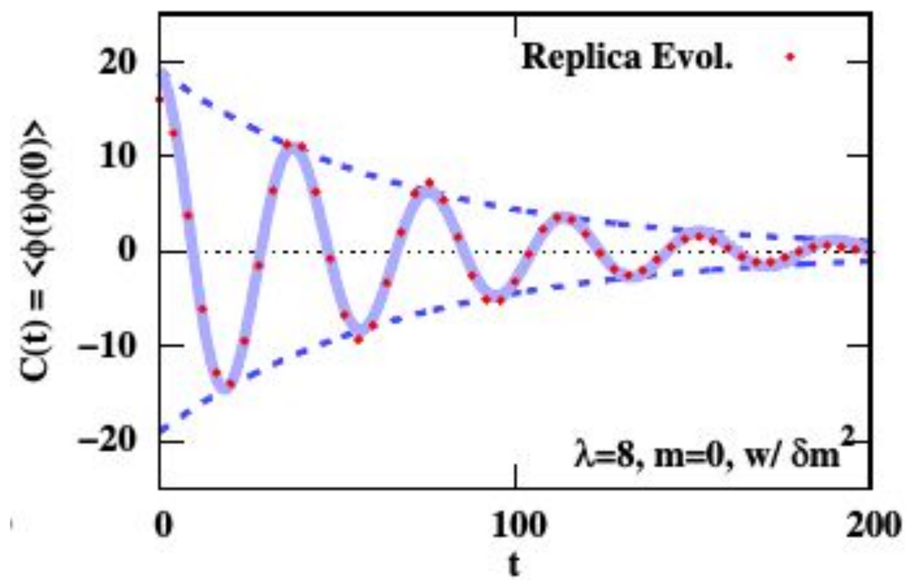
Time-correlation function

- **Time-correlation function of free field (zero momentum)**

$$C(t) = \frac{1}{L^3}\sum_{\boldsymbol{x},\boldsymbol{y}} \langle \phi_{\boldsymbol{x}}(t)\phi_{\boldsymbol{y}}(0)\rangle$$
$$= \frac{1}{NL^3}\sum_{\tau,\boldsymbol{x},\boldsymbol{y}} \langle \phi_{\tau\boldsymbol{x}}(t)\phi_{\tau\boldsymbol{y}}(0)\rangle$$
$$= \sum_n \frac{T}{M_n^2}\cos M_n t$$
$$(M_n^2 = \omega^2 + 4\xi^2 \sin^2(n\pi/N))$$



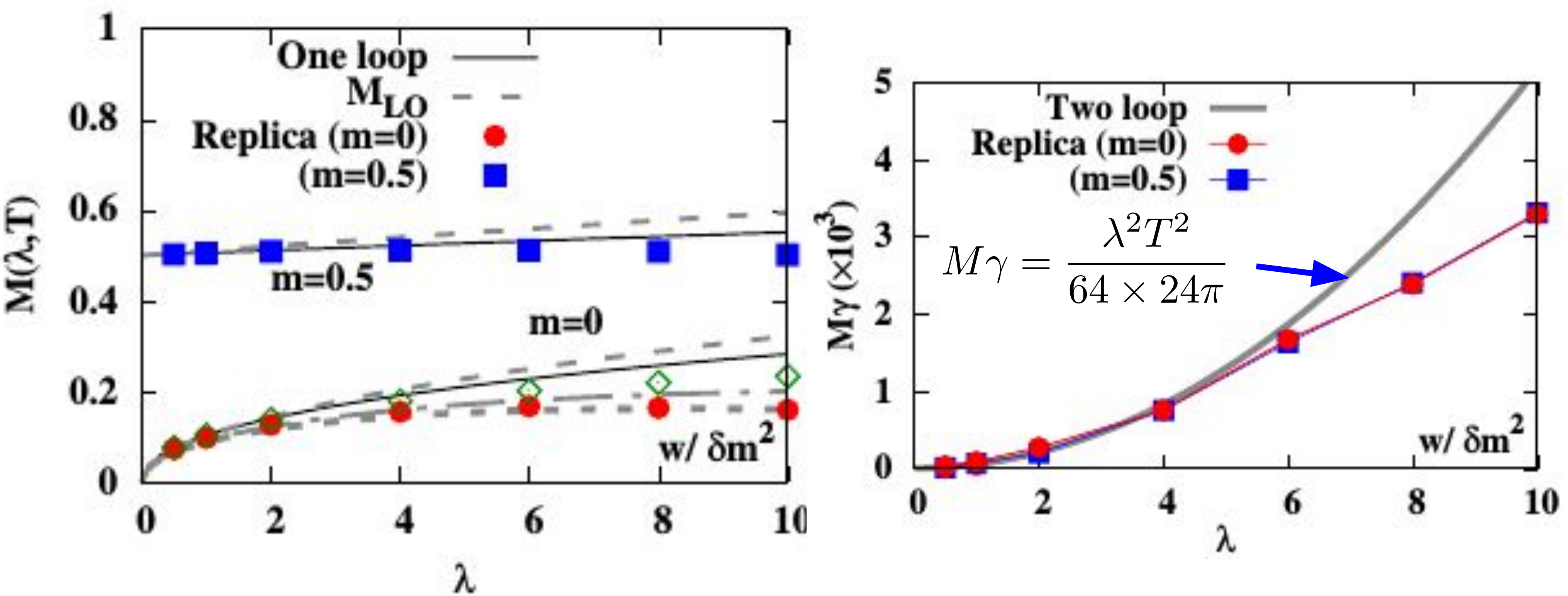
- **TCF of interacting field**
  - **Interaction induces thermal mass**
  - **Coupling of different momentum modes induces damping.**



Thermal mass and width

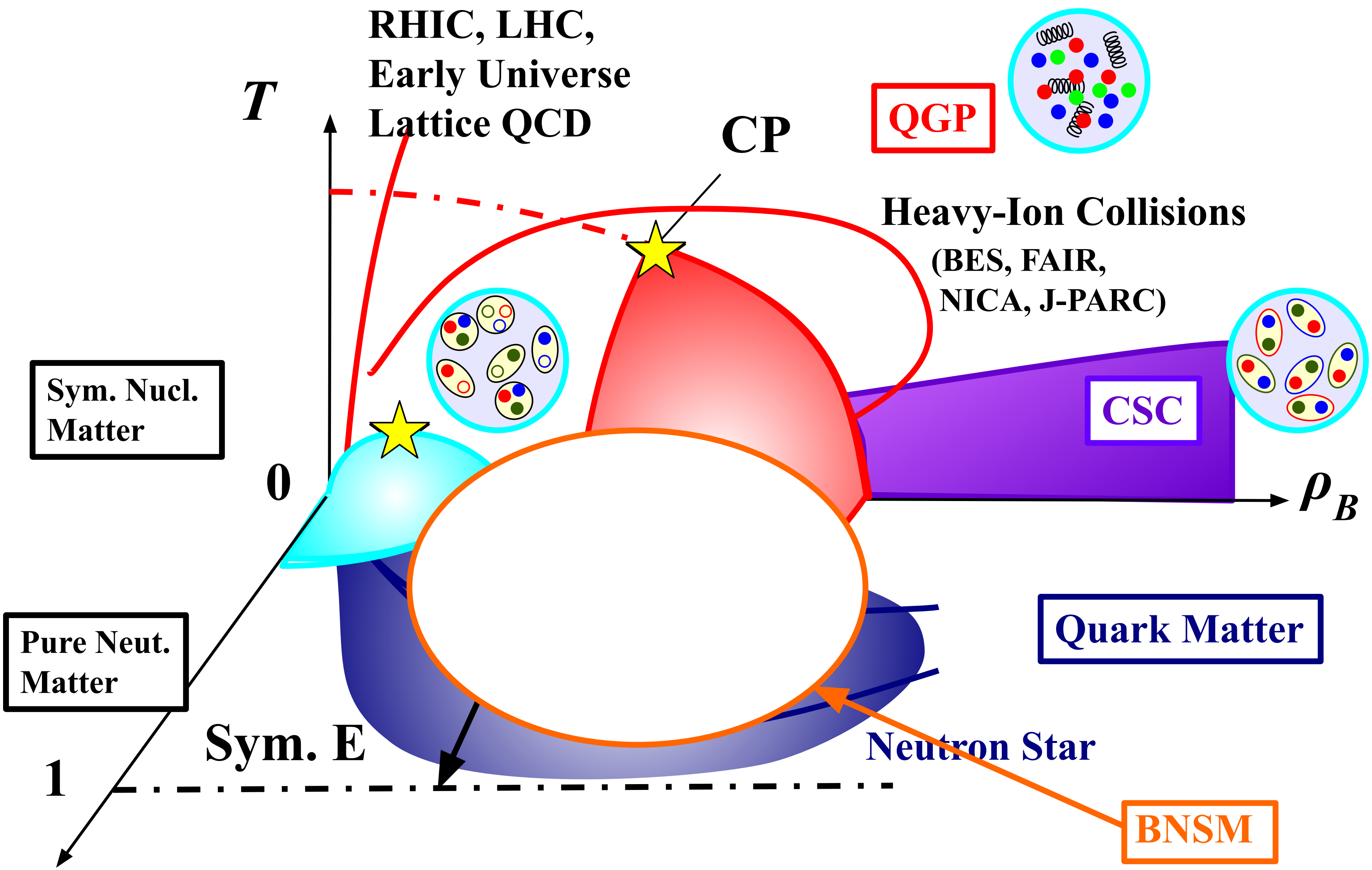
- **Thermal mass in replica method ~ 2-loop calc. results**
- **Thermal width in replica method ~ 2-loop calc. results at λ < 4**

Replica method takes account of higher-order interaction effects over one-loop.



$$M\gamma = \frac{\lambda^2 T^2}{64 \times 24\pi}$$

QCD Phase Diagram



$\delta=(N-Z)/A$ (or $Y_Q(\text{hadron})=Q_h/B\sim(1-\delta)/2$)

AO, JPS Conf. Proc. 20 (2018), 011035

Approaches using 2PI effective action

VOLUME 88, NUMBER 4 PHYSICAL REVIEW LETTERS 28 JANUARY 2002

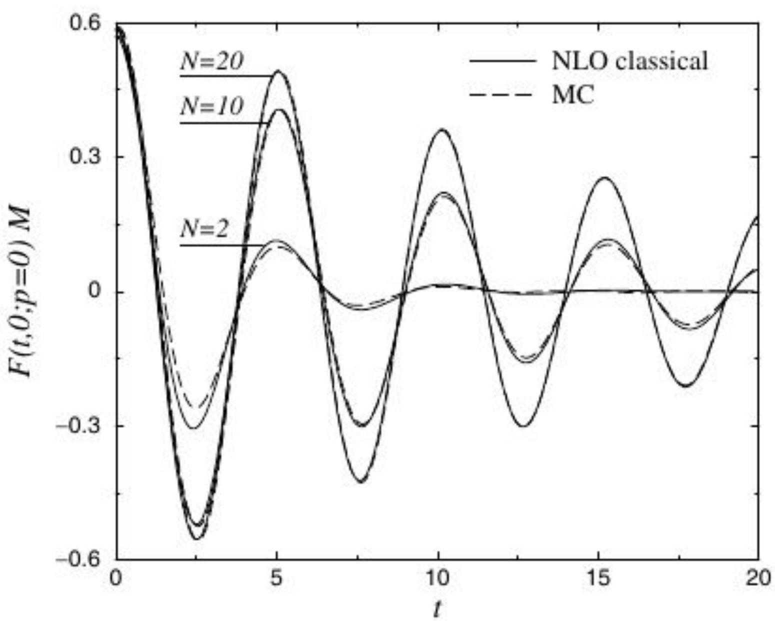
Classical Aspects of Quantum Fields Far from Equilibrium

Gert Aarts and Jürgen Berges

Institut für Theoretische Physik, Philosophenweg 16, 69120 Heidelberg, Germany

(Received 16 July 2001; published 15 January 2002)

We consider the time evolution of nonequilibrium quantum scalar fields in the $O(N)$ model, using the next-to-leading order $1/N$ expansion of the two-particle irreducible effective action. A comparison with exact numerical simulations in 1 + 1 dimensions in the classical limit shows that the $1/N$ expansion gives quantitatively precise results already for moderate values of N. For sufficiently high initial occupation numbers the time evolution of quantum fields is shown to be accurately described by classical physics. Eventually the correspondence breaks down due to the difference between classical and quantum thermal equilibrium.



Equation of motion for particle distribution function. Coupling with classical field is still hard.

Real time evolution of quantum field

- Static (equilibrium) problem → MC simulation of Lattice QFT

$$\mathcal{Z} = \int \mathcal{D}\phi \, e^{-\int d^4x \mathcal{L}_E}$$

- Path integral → Strong sign problem

$$|\Psi(t)\rangle = \mathcal{N} \int \mathcal{D}\phi \, T \exp\left[i \int d^4x \mathcal{L}\right] |\Psi(t_0)\rangle$$

  - Real time simulation of quantum field is difficult.

- Classical field simulation

$$H = \frac{1}{2} \sum_{\boldsymbol{x}} \pi_{\boldsymbol{x}}^2 + V \to \frac{d\phi_{\boldsymbol{x}}}{dt} = \pi_{\boldsymbol{x}} \ , \ \frac{d\pi_{\boldsymbol{x}}}{dt} = -\frac{\partial V}{\partial \phi_{\boldsymbol{x}}} \ ,$$

  - Phase is stationary w.r.t. the variation of (φ,π) → No cancellation
  - CF describes the growth of most unstable mode precisely. (Classical Statistical Simulation; *S. Y. Khlebnikov and I. I. Tkachev, PRL77('96)219*)
  - But CF evolution reaches classical equilibrium.

Harmonic Oscillator

- **Hamiltonian**

Fourier transf. of x_τ, p_τ

$$H(x,p) = \frac{p^2}{2} + \frac{\omega^2 x^2}{2}$$

$$\mathcal{H} = \sum_\tau H(x_\tau, p_\tau) + \mathcal{V} = \sum_n \left[\frac{\bar{p}_n^2}{2} + \frac{M_n^2 \bar{x}_n^2}{2}\right]$$

$$\mathcal{V} = \sum_\tau \frac{\xi^2}{2}(x_{\tau+1} - x_\tau)^2 \ , \qquad M_n^2 = \omega^2 + 4\xi^2 \sin^2(n\pi/N)$$

- **Re**

Matsubara freq. summation tech.

$$\mathcal{Z}_R(\xi) = \prod_n (\xi/M_n) = [2\sinh(\Omega/2T)]^{-1} \underset{N\to\infty}{\longrightarrow} [2\sinh(\omega/2T)]^{-1} = \mathcal{Z}_Q(T)$$

$$\Omega = 2\xi \operatorname{arcsinh}(\omega/2\xi) = 2NT \operatorname{arcsinh}(\omega/2NT) \underset{N\to\infty}{\longrightarrow} \omega$$

Correct quantum partition function is obtained in the large N limit

Time-Correlation Function

- **Unequal time two-point function (Time-correlation function)**
  - **Quantum mechanics**

Good exercise for UG students

$$C_Q(t) \equiv \langle x_H(t) x_H(0) \rangle_T = \frac{1}{2\omega}\left[\coth\left(\frac{\omega}{2T}\right)\cos\omega t - i\sin\omega t\right]$$

  - **Replica evolution**

    Solution of EOM

$$\bar{x}_n(t) = \bar{x}_n(0)\cos M_n t + \frac{\bar{p}_n(0)}{M_n}\sin M_n t\ , \qquad \frac{\bar{p}_n(t)}{M_n} = -\bar{x}_n(0)\sin M_n t + \frac{\bar{p}_n(0)}{M_n}\cos M_n t\ .$$

Thermal expectation values

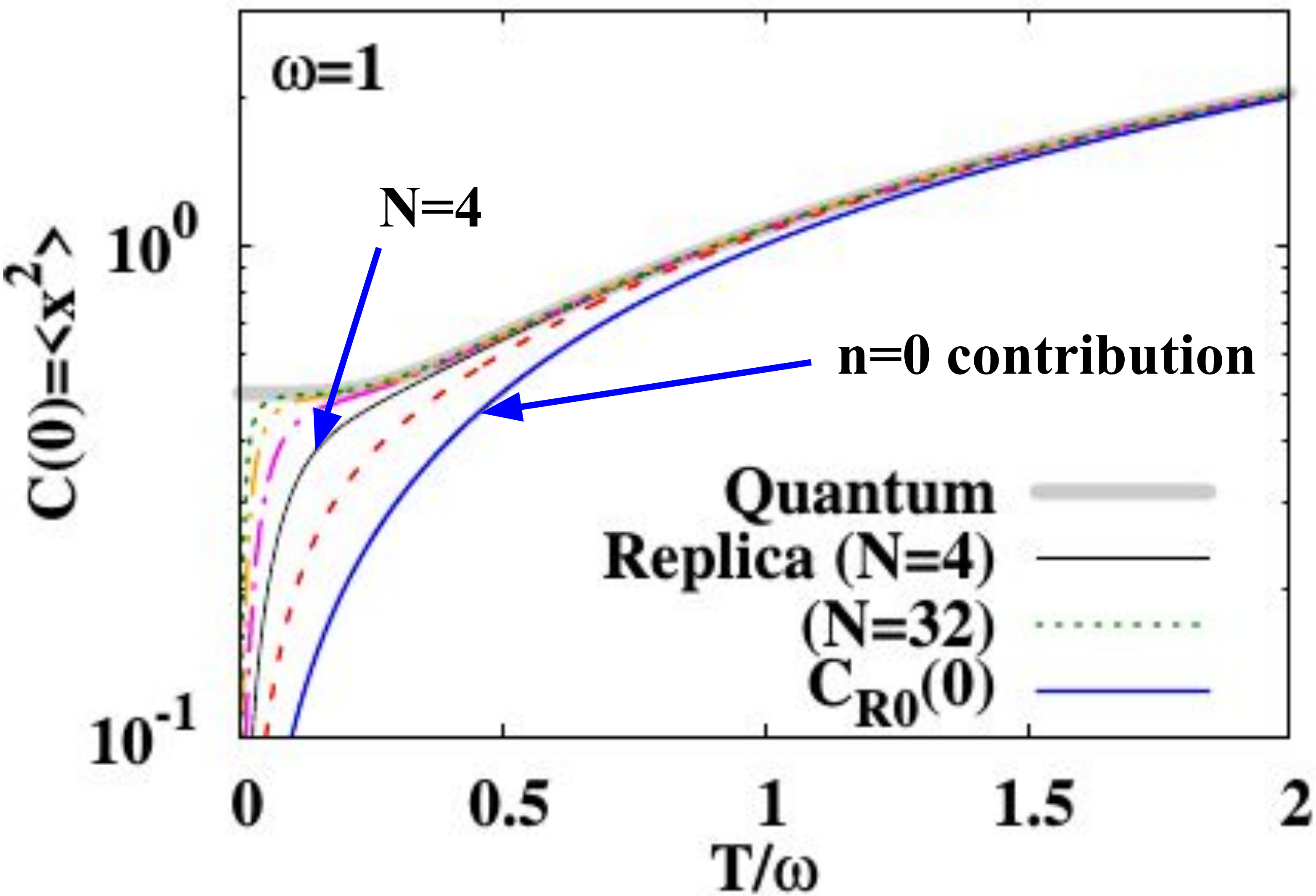
$$\langle \bar{x}_n(0)\bar{x}_{n'}(0)\rangle_T = \xi/M_n^2\delta_{nn'}, \qquad \langle \bar{p}_n(0)\bar{p}_{n'}(0)\rangle_T = \xi\delta_{nn'}$$

Time-corr. fn.

$$C_R(t) = \langle x(t)x(0)\rangle_T \equiv \frac{1}{N}\sum_\tau \langle x_\tau(t)x_\tau(0)\rangle_T = \frac{1}{N}\sum_n \langle \bar{x}_n(t)\bar{x}_n(0)\rangle_T$$

$$= \frac{1}{N}\sum_n \frac{\xi}{M_n^2}\cos M_n t = \sum_n \frac{T}{M_n^2}\cos M_n t\ .$$

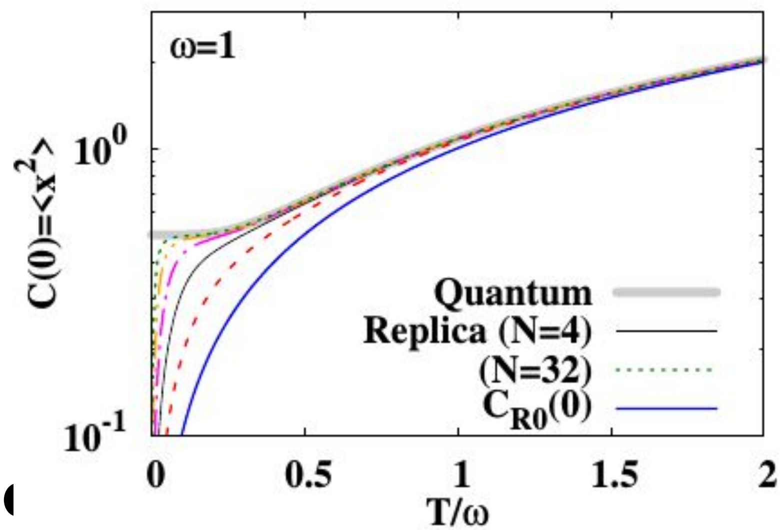
Equal time two-point function

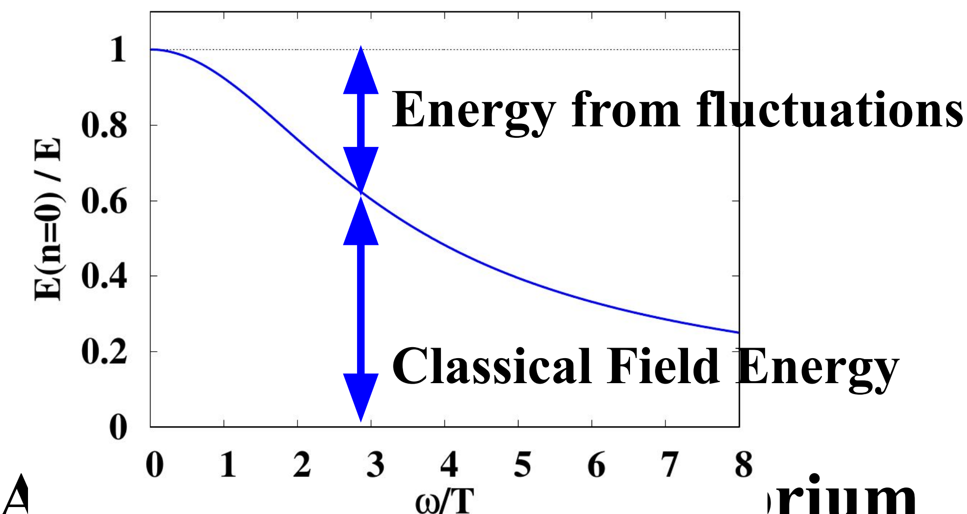


Correct quantum amplitude is obtained in the large N limit.
N=4 is enough for T/ω > 0.3

To do

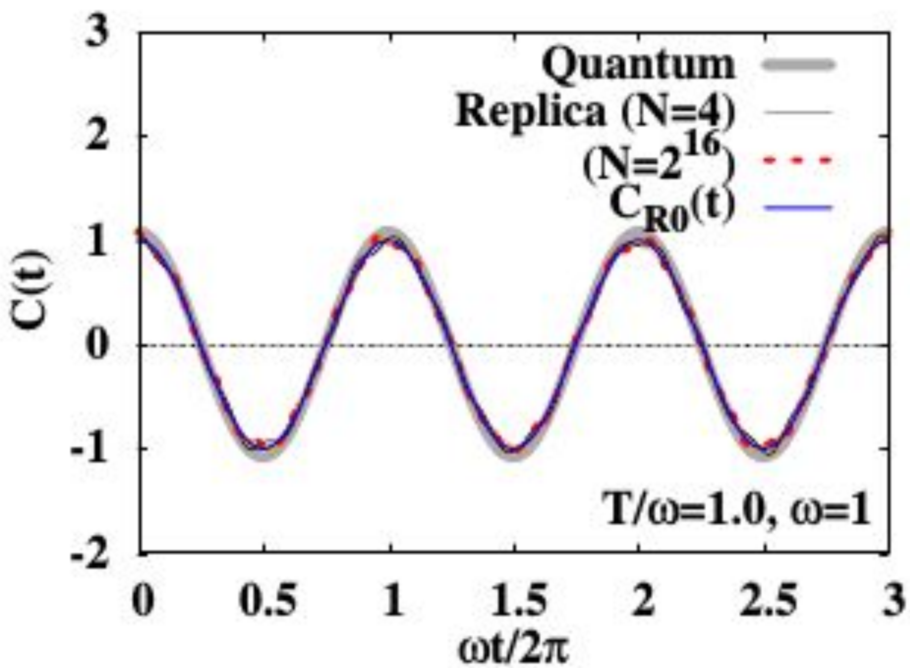
- Complete the manuscript.
- Understand the meaning of fluctuations among replicas. (Replica index average = Classical Field)

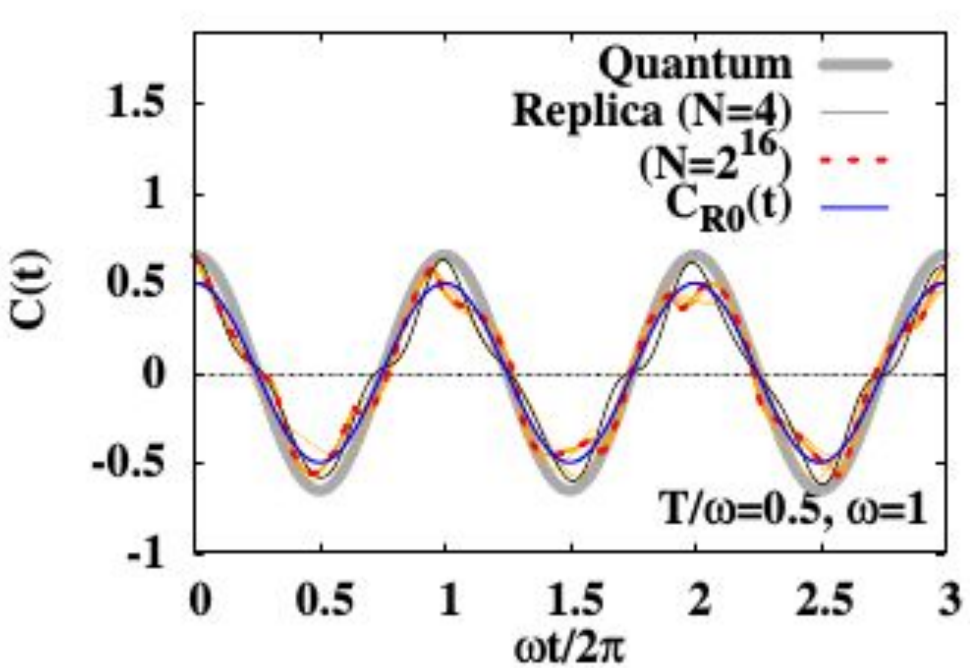




- C… , A… rium processes such as entropy production or pressure isotropization, Formal "derivation" of replica method (or its improved one), Coleman / Mermin-Wagner theorem, ...

Unequal time two-point function





Good enough !

Well, we can extract ω from time-corr. fn.

For T/ω > 0.5, time-corr. fn. is dominated by n=0 component, and we can access the quantum TCF by the replica evolution.