Replica evolution of classical field in 4+1 dimensional spacetime toward real time dynamics of quantum field

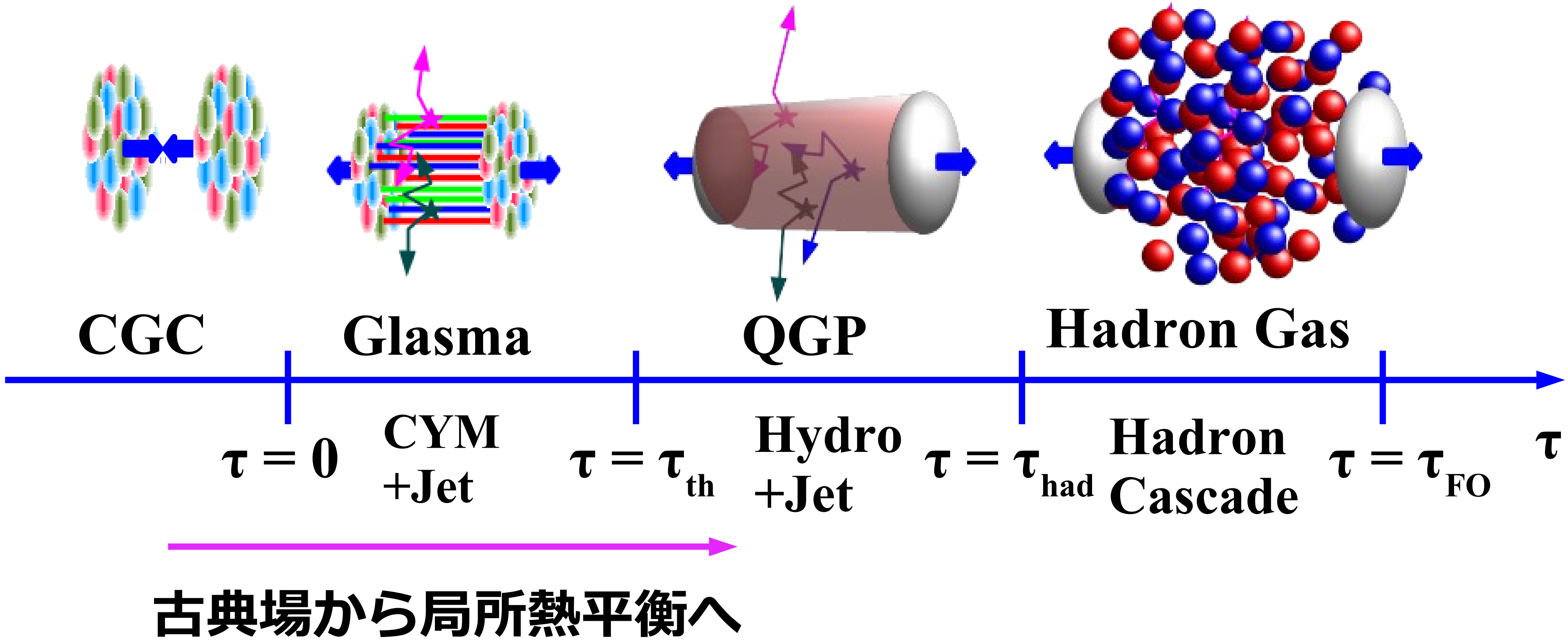
Akira Ohnishi [1],
H. Matsuda [2], T. Kunihiro [1], and T. T. Takahashi [3]
1. YITP, Kyoto U., 2. Dept. of Phys., Kyoto U., 3. Gunma NCT

arXiv:2008.09556 [hep-lat]

Seminar in Osaka University, Oct. 12, 2020

格子場の理論で重イオン衝突？

- 高エネルギー重イオン衝突の初期段階
  古典場が支配的な段階 (CGC, Glasma) から熱平衡 (Hydro., QGP) へ
  → 古典的な背景場の中での量子場の緩和過程の記述が必要



- しかし量子場の時間発展には強烈な符号問題が ....
  叶わぬ夢か？

$$S_{fi} = \mathcal{N} \int \mathcal{D}\phi \, \langle \Psi(t_f) | \exp\left(iS[\phi]\right) | \Psi(t_i) \rangle$$

量子場の時間発展への取り組み

- 古典場によるダイナミクスの記述
  - 最小作用の原理 δS=0 → 符号問題なし
  - 多くの非平衡現象で有用
    condensate (Time dep. Gross-Pitaevski), nuclei (TD Hartree-Fock), Inflation, high-energy heavy-ion collisions (classical Yang-Mills), …
  - ただし古典平衡に近づき、連続極限でエネルギー密度は発散

$$n_{\boldsymbol{k}} = T/\omega_{\boldsymbol{k}} \text{(Classical)}, \quad n_{\boldsymbol{k}} = [\exp(\omega_{\boldsymbol{k}}/T) \mp 1]^{-1} \text{(Quantum)}$$

- (有限の古典場を含む) 量子場の実時間発展への取り組み
  - Closed Time Path+ 2PI 作用による記述 (Kadanoff-Baym 方程式)
    Aarts, Berges ('02), Hatta, Nishiyama ('12)
    → 背景場が一様でない場合には計算量が膨大
  - 高運動量自由度を先に積分 and/or 別扱い
    Bodeker, McLerran, Smilga ('95), Greiner, B.Muller ('97), Dumitru, Nara, (Strickland) ('05,'07) …
    → 古典場部分は古典統計、古典場と粒子の変換は non-trivial

平衡で"量子統計性"をもつ古典場の枠組みがあれば …

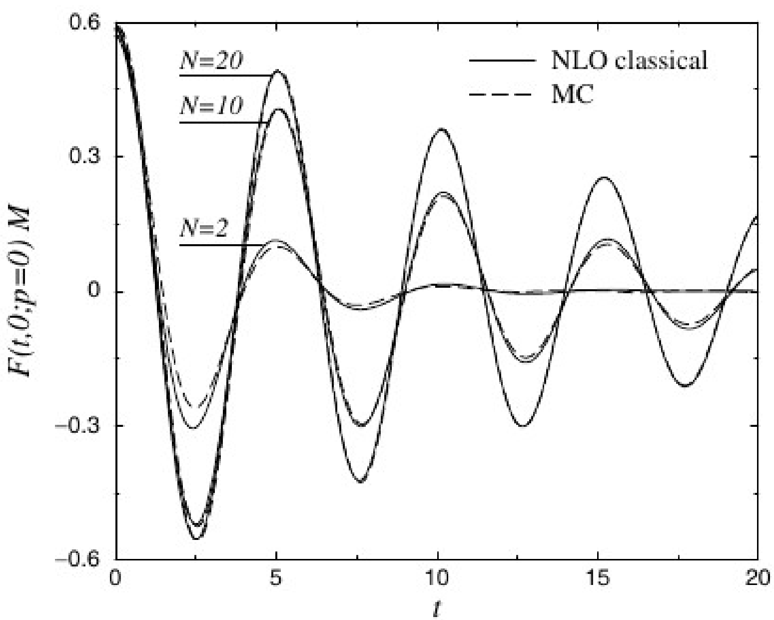
Approaches using 2PI effective action

VOLUME 88, NUMBER 4 PHYSICAL REVIEW LETTERS 28 JANUARY 2002

Classical Aspects of Quantum Fields Far from Equilibrium

Gert Aarts and Jürgen Berges
Institut für Theoretische Physik, Philosophenweg 16, 69120 Heidelberg, Germany
(Received 16 July 2001; published 15 January 2002)

We consider the time evolution of nonequilibrium quantum scalar fields in the $O(N)$ model, using the next-to-leading order $1/N$ expansion of the two-particle irreducible effective action. A comparison with exact numerical simulations in 1 + 1 dimensions in the classical limit shows that the $1/N$ expansion gives quantitatively precise results already for moderate values of N. For sufficiently high initial occupation numbers the time evolution of quantum fields is shown to be accurately described by classical physics. Eventually the correspondence breaks down due to the difference between classical and quantum thermal equilibrium.



Equation of motion for particle distribution function.
Coupling with classical field is still hard.

摂動論と格子 *QCD* で十分では？

- **摂動論 and/or 格子 QCD から求めた EOS+ 輸送係数で Hydro**
  R. Baier, A.H. Mueller, D. Schiff, D.T. Son ('01, pQCD, τ_{th}), P.Arnold, D.G.Moore, L.G.Yaffe ('03, pQCD, η); A. Nakamura, S. Sakai ('05, LQCD, η); A. Bazavov et al. [HotQCD] ('14, LQCD, EOS); S. Borsanyi et al. ('14, LQCD, EOS)
  - **熱平衡化する前では、おそらく不十分 : early thermalization puzzle, large η (pQCD), large uncertainty in η (LQCD)**
- **なぜか？背景場 (古典場) の効果**
  - **Anomalous viscosity under strong disordered field**
    M. Asakawa, S. A. Bass, B. Müller ('06)
    乱雑な背景場中の運動により運動量移行が促進される (小さな η)
  - **古典場発展でも見られる**
    H. Matsuda, T. Kunihiro, AO, T.T.Takahashi ('20)
    $$\eta \propto (g^4 \log(1/g))^{-1}\ \text{(pQCD)} \to \eta \propto g^{-3/2} \text{(ABM, CYM)}$$
  - **低エネルギーの原子核反応では自然 (Wall-Window formula)**
    One-body dissipation (古典場) > Two-body dissipation (衝突)

非一様・非平衡な背景場の下での量子場の発展の記述が必要

異常なずり粘性

- **Anomalous viscosity under strong disordered field**
  M. Asakawa, S. A. Bass, B. Müller, PRL96 ('06)252301; PTP116 ('07) 725.
  乱雑な背景場中の運動により運動量移行が促進される (小さな η)

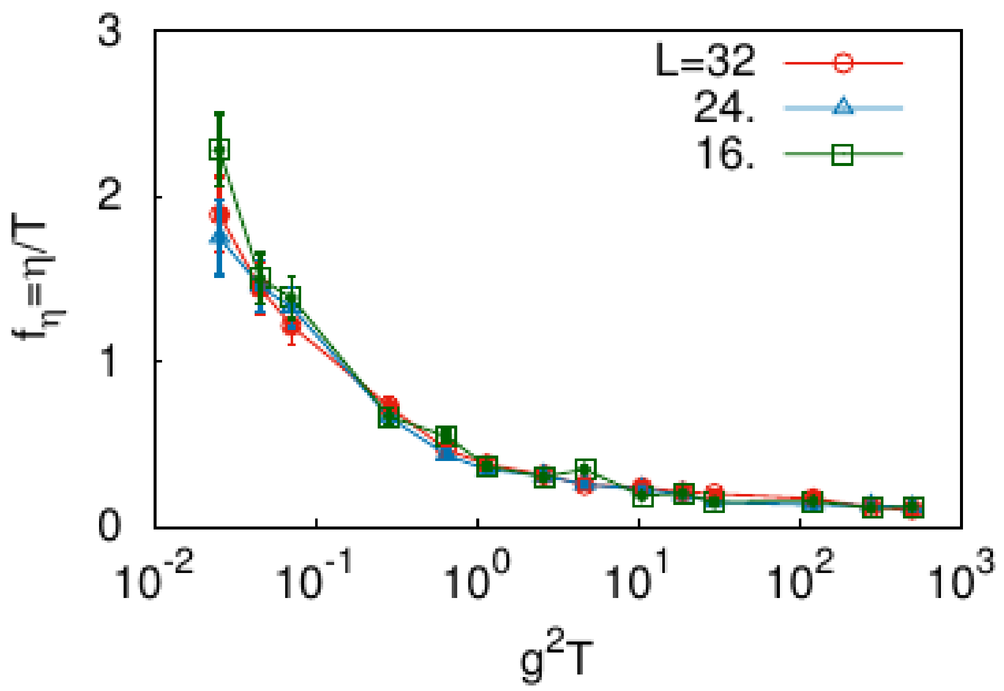
$$\eta_{\rm A} = \left(\frac{2(N_c^2-1)\nu_4\zeta(4)T\tau}{25b_0N_c\nu_2'\zeta(2)} \right)^{1/2} \frac{s}{g^{3/2}}.$$

- **古典場発展でも見られる**
  H. Matsuda, T. Kunihiro, AO, T.T.Takahashi (arXiv:2007.06886)

$$\alpha x^{-\beta/2} + \gamma x^{-\delta/2}$$

$$\alpha = 0.09 \pm 0.07, \quad \boxed{\beta = 1.49 \pm 0.39,}$$
$$\gamma = 0.33 \pm 0.06, \quad \delta = 0.35 \pm 0.07.$$

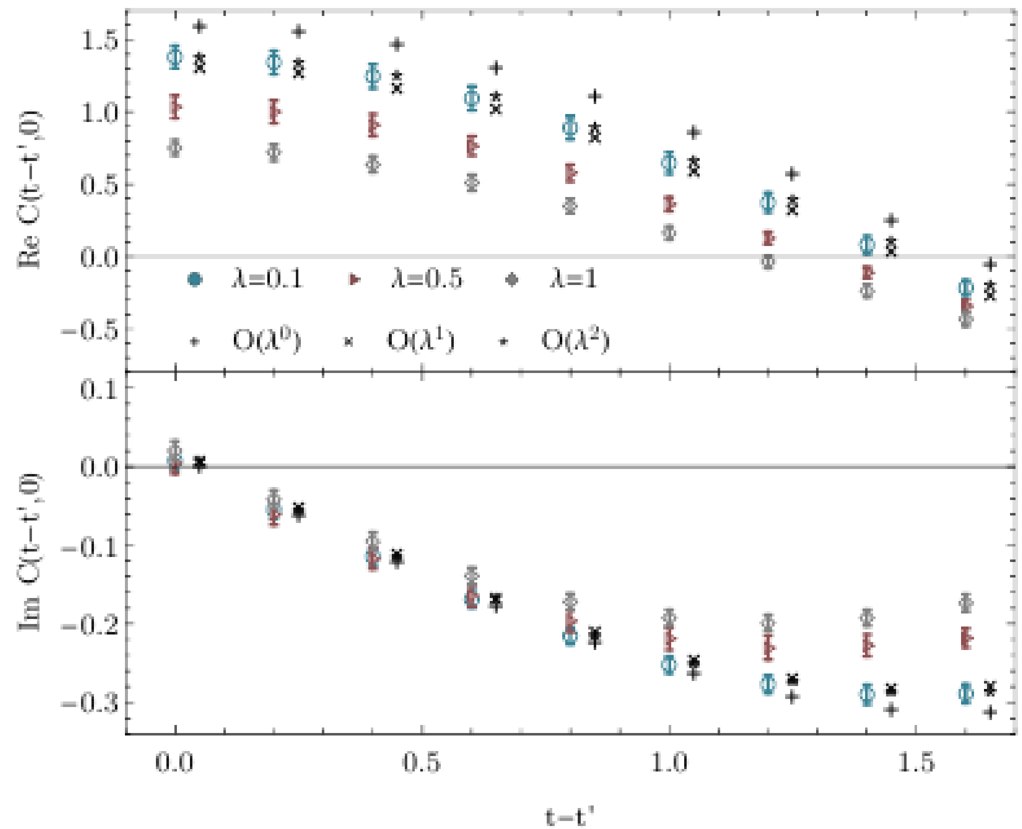
$x = g^2T$



変数を複素化した経路積分で計算できないか？

- **Lefshetz thimble approach to real time evolution**
  - **２点関数　E.g. Alexandru+(‘17)**
    - Closed time path での経路積分を Lefschetz thimble 上で実行
    - 1+1 次元、 1/4 周期程度まで、 λ=0.1 → 現象論的には不十分
    - 虚部が求まるのは魅力的

A. Alexandru, G. Basar, P. F. Bedaque, G. W. Ridgway, PRD95(‘17), 114501



非一様・非平衡な背景場の下での量子場の発展の記述
→ 量子統計性をもつ
古典場が使えれば ….
(Replica Evolution)

Contents

- Introduction
- レプリカ発展法
- 量子力学への適用
- スカラー場理論への適用
- まとめ

AO, H. Matsuda, T. Kunihiro, T. T. Takahashi, arXiv:2008.09556 [hep-lat]

レプリカ発展法
(Replica evolution)

レプリカ発展法での分配関数（量子力学）

古典力学での分配関数

$$H = \frac{p^2}{2} + U(x) \qquad \mathcal{Z}_C(T) = \int \frac{dxdp}{2\pi} \exp\left[-\frac{H(x,p)}{T}\right]$$

量子力学での分配関数（虚時間法）

$$S = \frac{1}{\xi}\left[\mathcal{V} + \sum_{\tau=1}^{N} U(x_\tau)\right] \qquad \mathcal{Z}_Q(T) = \int \mathcal{D}x \exp\left(-S[x]\right)$$

$$\mathcal{V} = \sum_{\tau=1}^{N} \frac{\xi^2}{2}(x_{\tau+1}-x_\tau)^2 \simeq \xi \int_0^{1/T} d\bar{\tau} \frac{1}{2}\left[\frac{\partial x}{\partial \bar{\tau}}\right]^2, \quad \xi = NT(= a/a_\tau)$$

レプリカ発展法（虚時間微分項を相互作用とみなす）での分配関数

$$\mathcal{H} = \sum_{\tau=1}^{N}\left[\frac{p_\tau^2}{2} + U(x_\tau)\right] + \mathcal{V}$$

$H(x_\tau, p_\tau)$ 　 $\xi S[\phi]$

$$\mathcal{Z}_R(\xi) = \int \frac{\mathcal{D}x\mathcal{D}p}{2\pi} \exp\left(-\frac{\mathcal{H}[x,p]}{\xi}\right) = (2\pi\xi)^{NL^3/2} Z_Q(T)$$

虚時間微分項で相互作用する N 個の古典場の分配関数（温度 ξ）
$\propto$ 量子力学での分配関数（温度 $T=\xi/N$）

レプリカ指標平均〜古典運動

- **レプリカ座標 (x_τ, p_τ) の運動方程式**

$$\frac{dx_\tau}{dt} = \frac{\partial \mathcal{H}}{\partial p_\tau} = p_\tau$$

$$\frac{dp_\tau}{dt} = -\frac{\partial \mathcal{H}}{\partial x_\tau} = -\frac{\partial U(x_\tau)}{\partial x_\tau} + \xi^2(x_{\tau+1} + x_{\tau-1} - 2x_\tau)$$

- **レプリカ指標平均**

τ-derivative terms

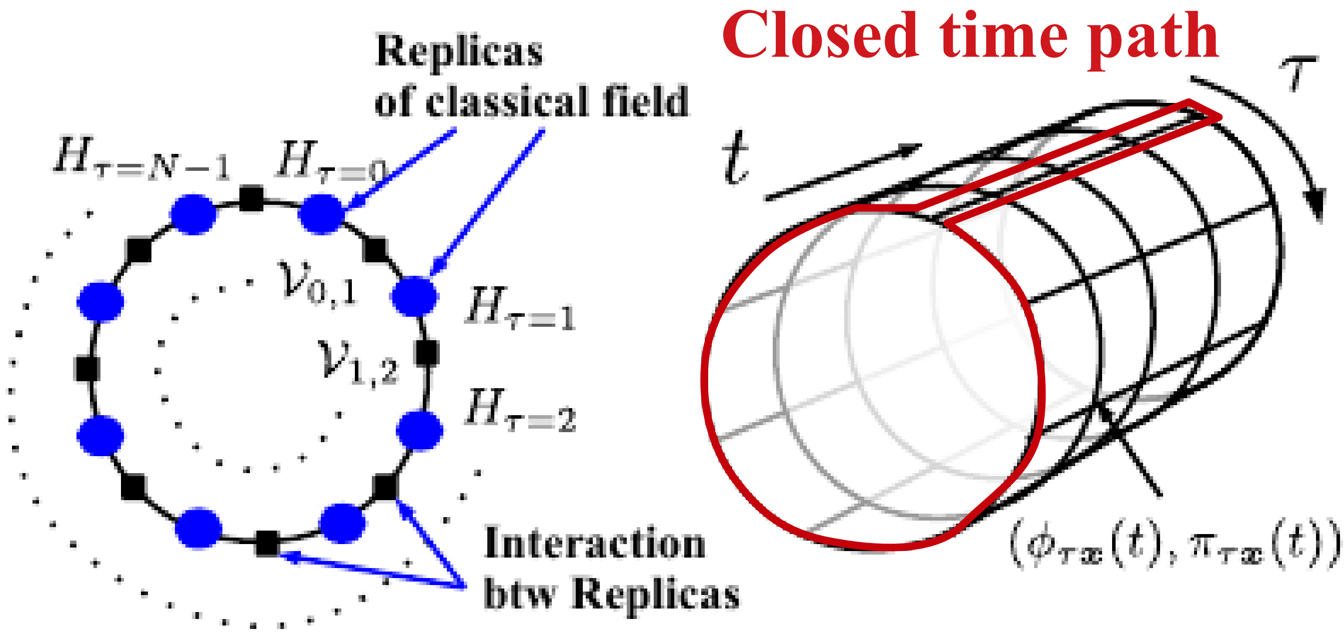
$$\frac{d\widetilde{x}}{dt} = \frac{1}{N}\sum_\tau \frac{dx_\tau}{dt} = \widetilde{p}$$

$$\frac{d\widetilde{p}}{dt} = \frac{1}{N}\sum_\tau \frac{dp_\tau}{dt} = -\frac{1}{N}\sum_\tau \frac{\partial U(x_\tau)}{\partial x_\tau} + 0 \text{ (Ehrenfest's theorem)}$$

$$= -\frac{\partial U(\widetilde{x})}{\partial \widetilde{x}} + \mathcal{O}((\delta x)^2)$$

レプリカ間のゆらぎが小さい場合には、
レプリカ指標平均（τ 平均）は古典運動方程式に従う

レプリカ発展法 *(Replica Evolution method)*



$$\mathcal{H} = \sum_{\tau} H_{\tau} + \sum_{\tau} \mathcal{V}_{\tau,\tau+1} = \frac{1}{2}\sum_{\tau,\boldsymbol{x}} \pi^2_{\tau,\boldsymbol{x}} + \xi S[\phi]$$

$$\mathcal{Z}_R = \int \mathcal{D}\pi\, \mathcal{D}\phi\, \exp(-\mathcal{H}/\xi) \propto \int \mathcal{D}\phi\, \exp(-S[\phi])$$

レプリカ発展法＝量子統計性をもつ古典力学

Replica evolution of harmonic oscillator (quantum mechanics)

Harmonic Oscillator

- **Hamiltonian**

Fourier transf. of x_τ, p_τ

$$H(x,p) = \frac{p^2}{2} + \frac{\omega^2 x^2}{2}$$

$$\mathcal{H} = \sum_\tau H(x_\tau, p_\tau) + \mathcal{V} = \sum_n \left[\frac{\bar{p}_n^2}{2} + \frac{M_n^2 \bar{x}_n^2}{2}\right]$$

$$\mathcal{V} = \sum_\tau \frac{\xi^2}{2}(x_{\tau+1} - x_\tau)^2 \ , \quad M_n^2 = \omega^2 + 4\xi^2 \sin^2(n\pi/N)$$

- **Replica partition function**

Matsubara freq. summation tech.

$$\mathcal{Z}_R(\xi) = \prod_n (\xi/M_n) = [2\sinh(\Omega/2T)]^{-1} \underset{N\to\infty}{\longrightarrow} [2\sinh(\omega/2T)]^{-1} = \mathcal{Z}_Q(T)$$

$$\Omega = 2\xi \,\mathrm{arcsinh}\,(\omega/2\xi) = 2NT \,\mathrm{arcsinh}\,(\omega/2NT) \underset{N\to\infty}{\longrightarrow} \omega$$

Correct quantum partition function is obtained in the large N limit

Time-Correlation Function

- **Unequal time two-point function (Time-correlation function)**
  - **Quantum mechanics**

Good exercise for UG students

$$C_Q(t) \equiv \langle x_H(t) x_H(0) \rangle_T = \frac{1}{2\omega} \left[\coth\left(\frac{\omega}{2T}\right) \cos\omega t - i \sin\omega t \right]$$

  - **Replica evolution**

Solution of EOM

$$\bar{x}_n(t) = \bar{x}_n(0) \cos M_n t + \frac{\bar{p}_n(0)}{M_n} \sin M_n t \;, \qquad \frac{\bar{p}_n(t)}{M_n} = -\bar{x}_n(0) \sin M_n t + \frac{\bar{p}_n(0)}{M_n} \cos M_n t \;.$$

Thermal expectation values

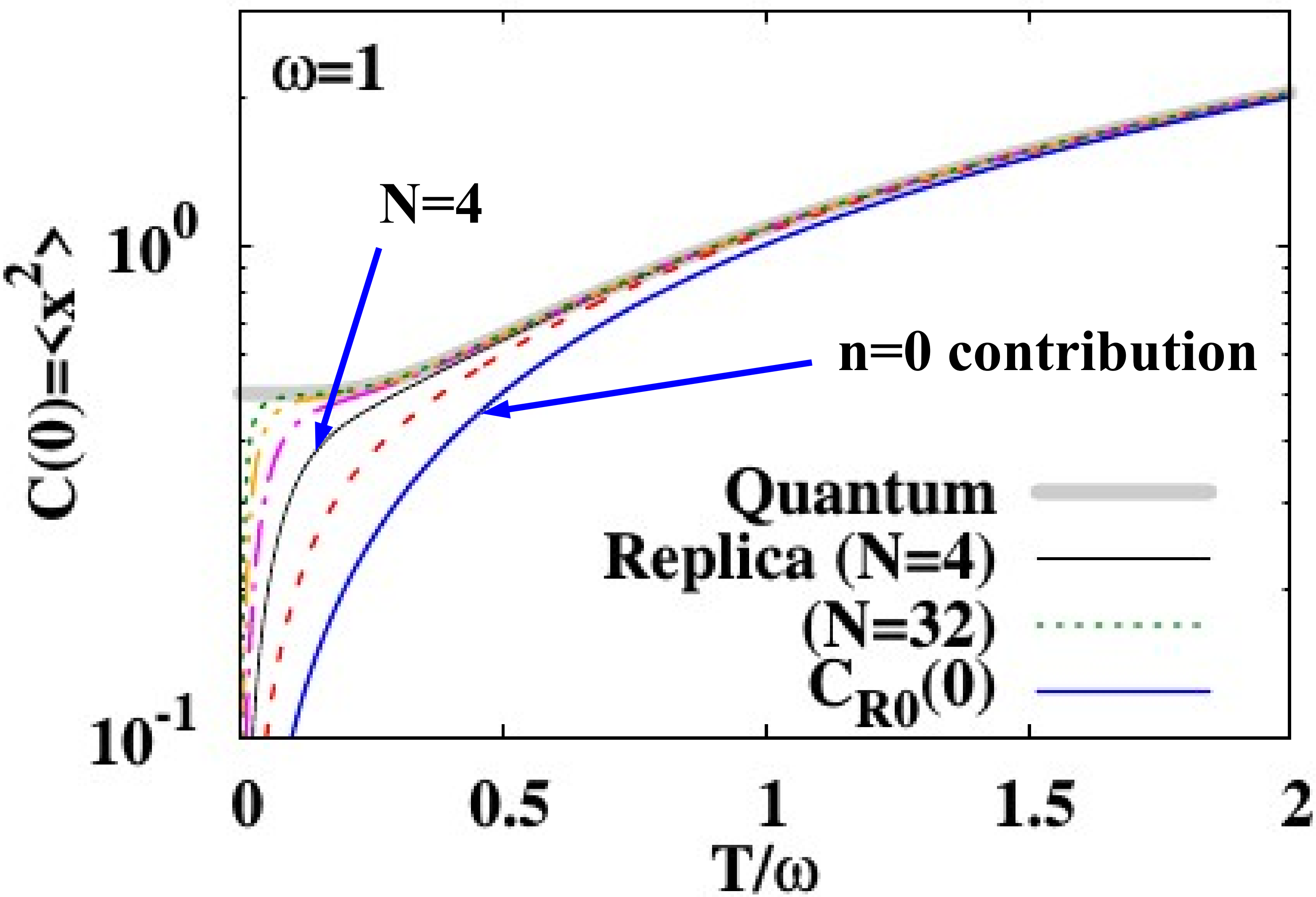
$$\langle \bar{x}_n(0) \bar{x}_{n'}(0) \rangle_T = \xi / M_n^2 \delta_{nn'} , \qquad \langle \bar{p}_n(0) \bar{p}_{n'}(0) \rangle_T = \xi \delta_{nn'}$$

Time-corr. fn.

$$C_R(t) = \langle x(t) x(0) \rangle_T \equiv \frac{1}{N} \sum_\tau \langle x_\tau(t) x_\tau(0) \rangle_T = \frac{1}{N} \sum_n \langle \bar{x}_n(t) \bar{x}_n(0) \rangle_T$$

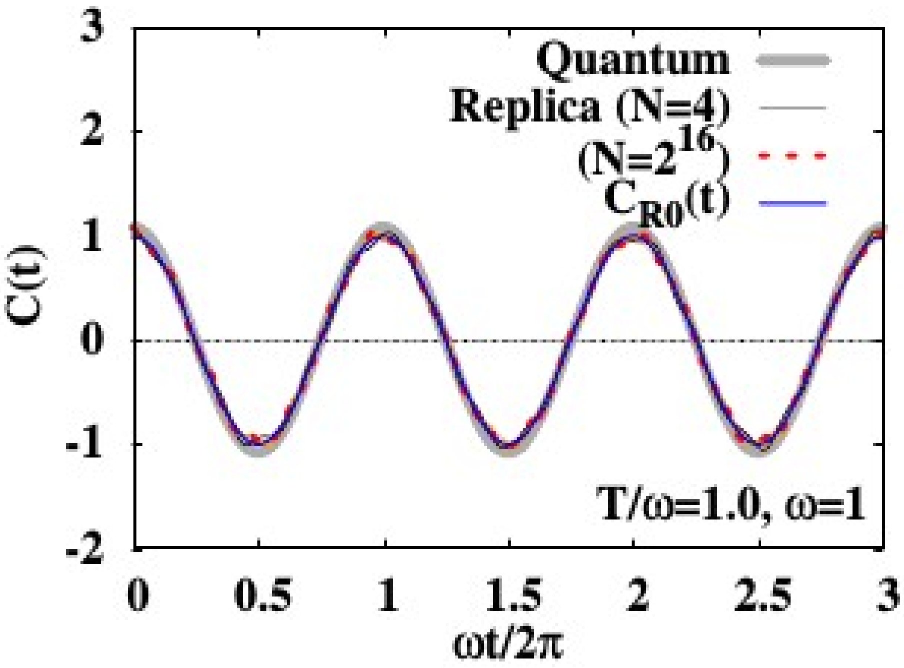
$$= \frac{1}{N} \sum_n \frac{\xi}{M_n^2} \cos M_n t = \sum_n \frac{T}{M_n^2} \cos M_n t \;.$$

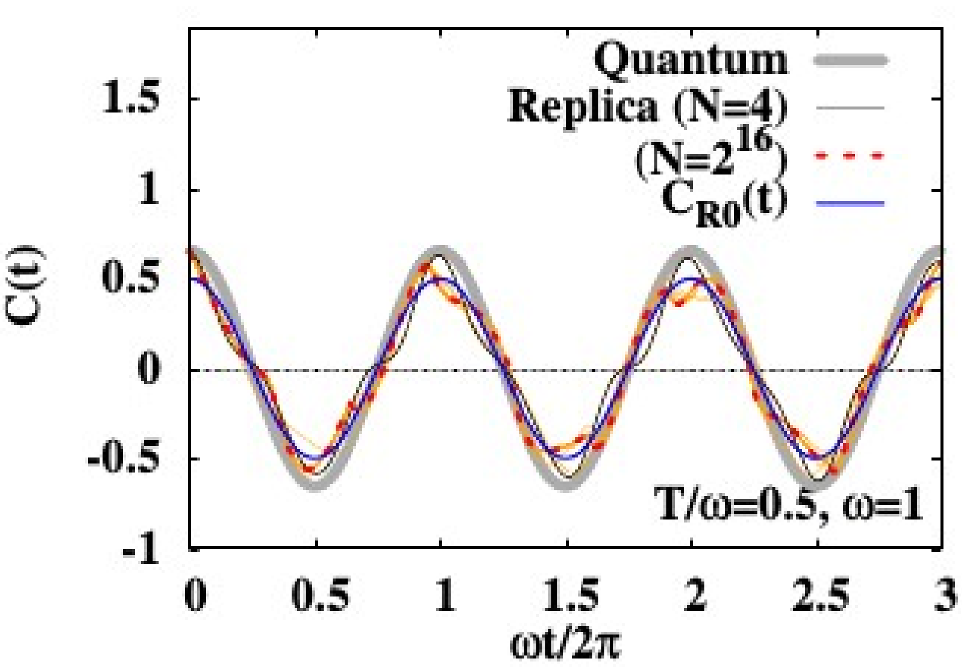
Equal time two-point function



Correct quantum amplitude is obtained in the large N limit.
N=4 is enough for $T/\omega > 0.3$

Unequal time two-point function





Good enough !

Well, we can extract ω from time-corr. fn.

For T/ω > 0.5, time-corr. fn. is dominated by n=0 component, and we can access the quantum TCF by the replica evolution.

スカラー場理論への適用

スカラー場のレプリカ発展

- **場の理論でのレプリカ発展**
  - **レプリカ変数** $(x_\tau, p_\tau) \to (\phi_{\tau\boldsymbol{x}}, \pi_{\tau\boldsymbol{x}})$
  - **場の理論 → 質量くりこみ、ゼロ点エネルギーの除去**

- **例 : Φ⁴ theory**

$$\mathcal{H} = \sum_{\tau,\boldsymbol{x}} \left[\frac{\pi_{\tau\boldsymbol{x}}^2}{2} + \frac{1}{2}(\nabla\phi_{\tau\boldsymbol{x}})^2 + \frac{m^2}{2}\phi_{\tau\boldsymbol{x}}^2 + \frac{\lambda}{24}\phi_{\tau\boldsymbol{x}}^4 + \frac{\xi^2}{2}(\phi_{\tau+1,\boldsymbol{x}} - \phi_{\tau\boldsymbol{x}})^2 \right]$$

$\mathcal{V}$

$H(\phi_{\tau\boldsymbol{x}}, \pi_{\tau\boldsymbol{x}})$

$\xi S[\phi]$

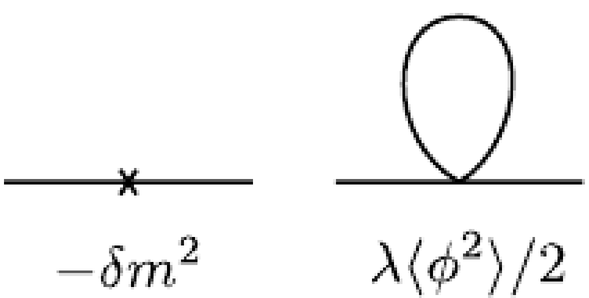
$$-\frac{\delta m^2}{2}\phi_{\tau\boldsymbol{x}}^2$$

- **質量くりこみ（1 loop)**

Aarts, Smit ('97), Kapusta, Gale (textbook)

$$\delta m^2 = \frac{\lambda}{2}\langle\phi^2\rangle_{\rm div}$$

$$\langle\phi^2\rangle_{\rm div} = \frac{1}{L^3}\sum_{\boldsymbol{k}} \frac{1}{2\omega_{\boldsymbol{k}}\sqrt{1+(\omega_{\boldsymbol{k}}/2\xi)^2}}$$



Numerical Calculation Setup

- **Lattice size = 32^3 x 4 (L=32, N=4)**
- **T=0.5 (ξ=NT=2); m=0, 0.5; λ=0.5, 1, 2, 4, 6, 8, 10.**
- **One loop renormalization of mass, no counterterm for λ.**
- **Initial conditions are obtained by solving the Langevin equation.**
- **Solve replica EOM until t=500 with the time step of Δt=0.025.**
- **Number of replica configurations = 1000**
  → 3-6 hours on one core of core i7 PC for a given (m, λ)

運動量分布

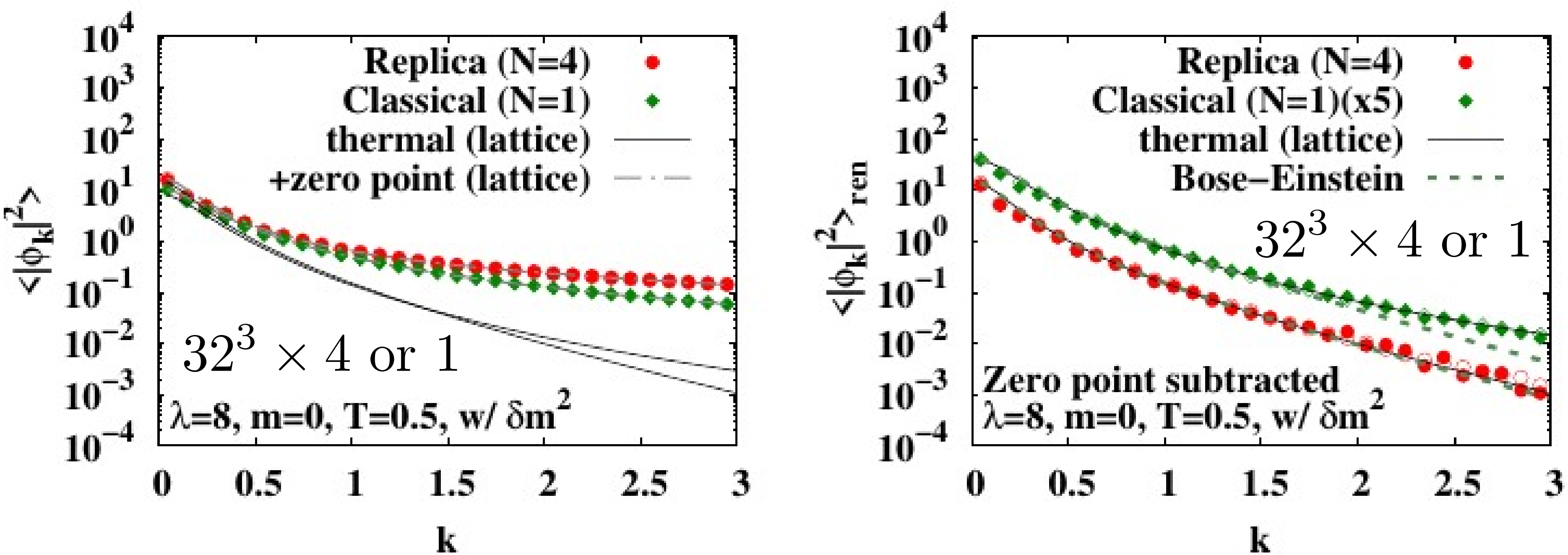
- レプリカでの運動量分布（自由場）

$$\langle|\phi_{\boldsymbol{k}}|^2\rangle = \frac{1}{N}\sum_n \langle\phi_{n\boldsymbol{k}}\phi^*_{n\boldsymbol{k}}\rangle = \frac{1}{\omega_{\boldsymbol{k}}\sqrt{1+(\omega_{\boldsymbol{k}}/2\xi)^2}}\left[\frac{1}{2}+\frac{1}{e^{\Omega_{\boldsymbol{k}}/T}-1}\right]$$

Free field, Matsubara sum

Zero point

Thermal → Bose-Einstein



Rayleigh-Jeans Divergence

- **With N >= 2, free field energy converges in the replica method.**

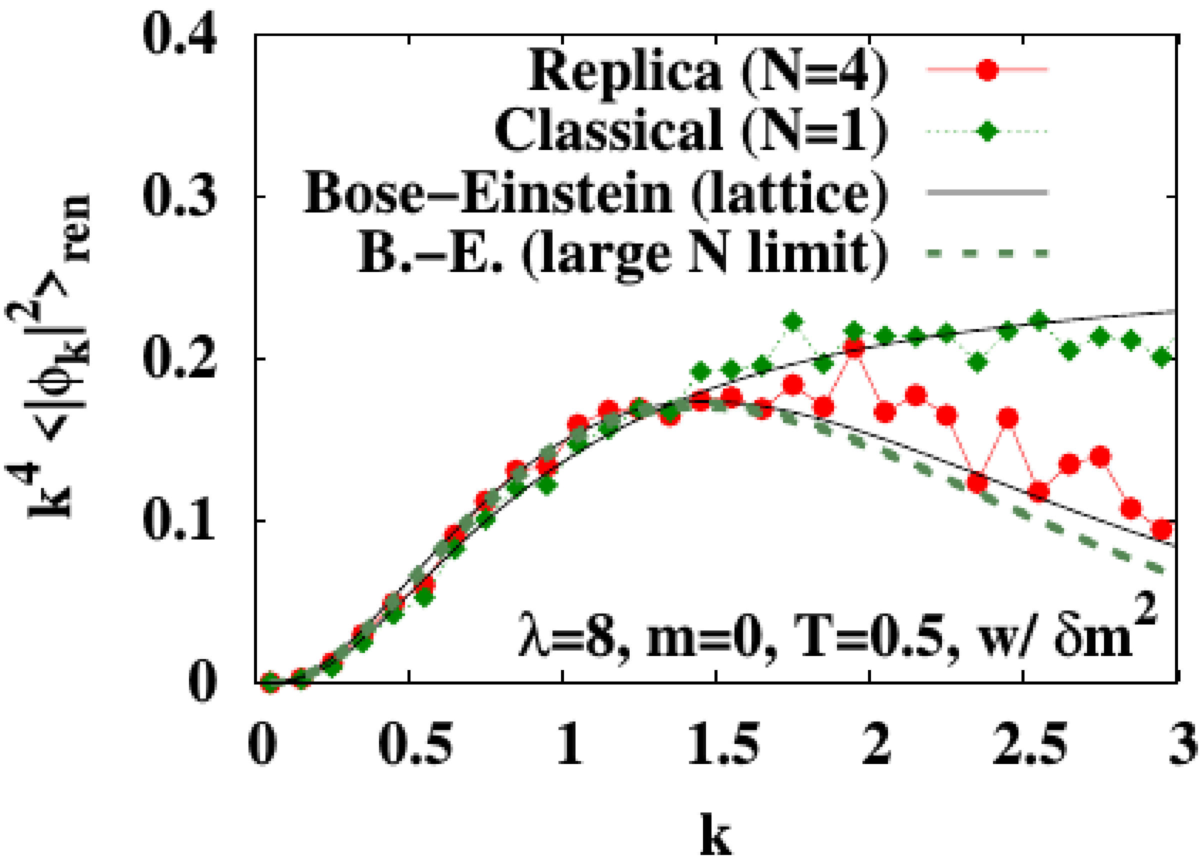
$$\Omega = 2NT\,\mathrm{arcsinh}\,(\omega/2NT) \underset{\omega \gg NT}{\longrightarrow} 2NT\log(\omega/NT)$$

$$\langle|\phi_{\boldsymbol{k}}|^2\rangle_{\mathrm{ren}} \simeq \frac{2NT}{k^2}\exp(-\Omega_{\boldsymbol{k}}/T) \to 2(NT)^{2N+1}k^{-2(N+1)}$$

$$k^4\,\langle|\phi_{\boldsymbol{k}}|^2\rangle_{\mathrm{ren}} \to 2(NT)^{2N+1}k^{-2(N-1)}$$

- **Convergence cond.**
  2(N-1) > 1 → N > 1.5

We can remove divergence of energy in the replica method (N>=2) with mass counterterm(s).

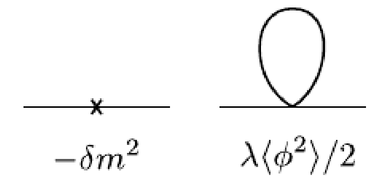


レプリカ発展法での熱質量

Kapusta, Gale (textbook)
Parwani ('92, '93)

くりこみ後の熱質量

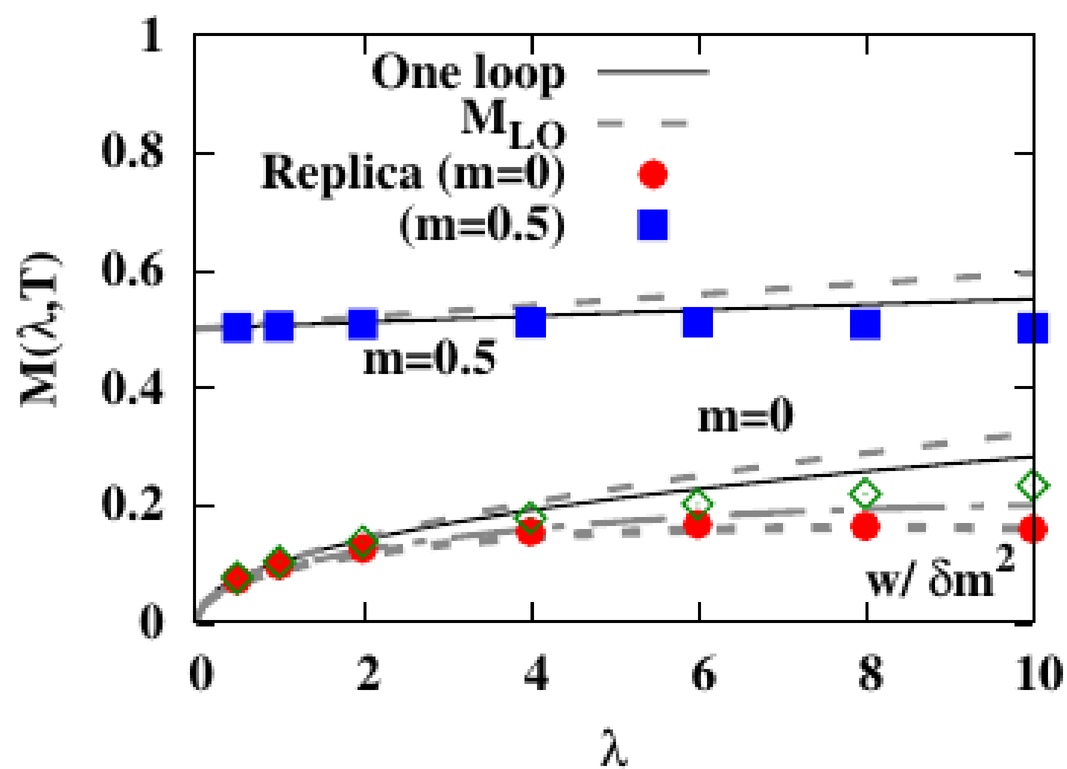
- **Leading Order (one-loop)** $M^2_{\rm LO} = m^2 + \lambda T^2/24.$
- **Resummed One-Loop** $M^2_{\rm resum} = \dfrac{\lambda T^2}{24}\left[1 - \dfrac{3}{\pi}\sqrt{\dfrac{\lambda}{24}}\right]$
- **Two-Loop**



$$M^2_{2-{\rm loop}} = \frac{\lambda T^2}{24}\left\{1 - \frac{3}{\pi}\sqrt{\frac{\lambda}{24}} + \frac{\lambda}{(4\pi)^2}\left[\frac{3}{2}\log\left(\frac{T^2}{4\pi\mu^2}\right) + 2\log\left(\frac{\lambda}{24}\right) + \alpha\right]\right\},$$

レプリカ発展法

→ 時間相関関数をフィット
して得られた熱質量

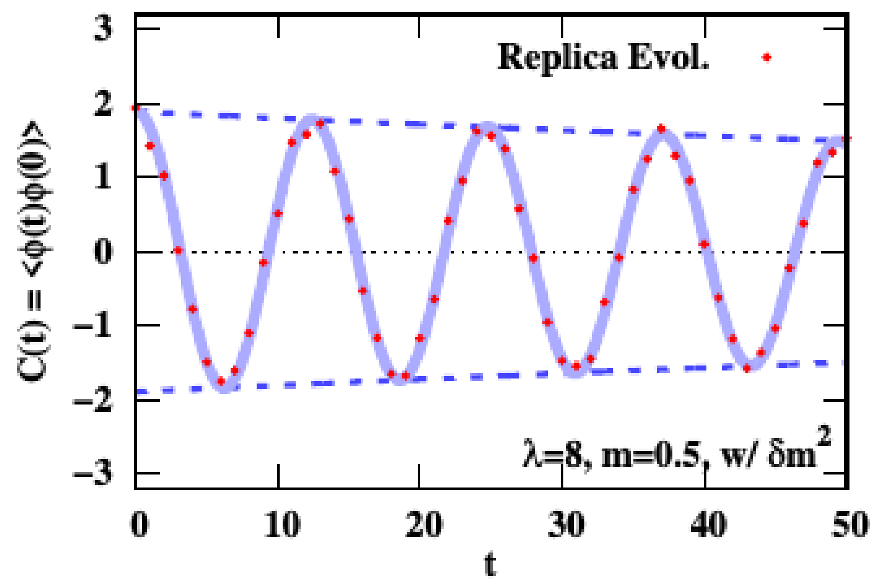


レプリカ発展法での熱質量
→ １ループを越える
相互作用効果を含む

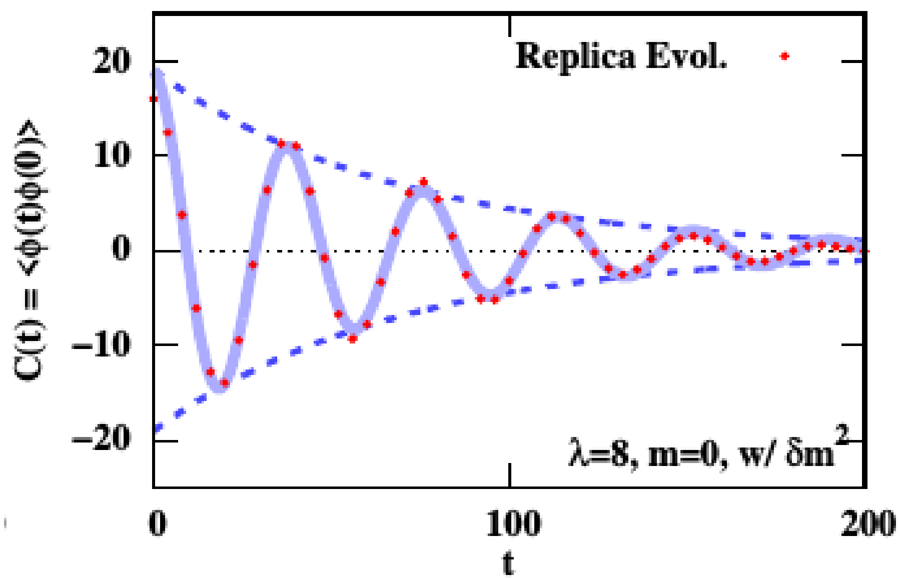
Time-correlation function

- **Time-correlation function of free field (zero momentum)**

$$C(t) = \frac{1}{L^3}\sum_{\boldsymbol{x},\boldsymbol{y}} \langle \phi_{\boldsymbol{x}}(t)\phi_{\boldsymbol{y}}(0)\rangle$$
$$= \frac{1}{NL^3}\sum_{\tau,\boldsymbol{x},\boldsymbol{y}} \langle \phi_{\tau\boldsymbol{x}}(t)\phi_{\tau\boldsymbol{y}}(0)\rangle$$
$$= \sum_n \frac{T}{M_n^2}\cos M_n t$$
$$(M_n^2 = \omega^2 + 4\xi^2 \sin^2(n\pi/N))$$



- **TCF of interacting field**
  - **Interaction induces thermal mass**
  - **Coupling of different momentum modes induces damping.**



まとめ

- 量子統計性をもつ古典力学（古典場理論）である
  レプリカ発展法を提案した。
  - 虚時間座標をレプリカ指標 (replica index)、
    虚時間形式で現れる τ 微分項をポテンシャルとみなした
    4 (=3+1) 次元空間での場の変数の古典的時間発展
  - 古典場の性質と量子統計性を併せ持つ。
  - 技術的には HMC の分子動力学部分
  - レイリー・ジーンズ発散を持たず、１ループを越えた相互作用効果を示す
- 今後の課題
  - これまでに提案されている手法との比較
    Hard mode effects [Bodeker, McLerran, Smilga ('95), Greiner, B.Muller ('97)], Field-particle sim. [Dumitru, Nara ('05), Dumitru, Nara, Strickland ('07)], 2PI [Aarts, Berges ('02), Hatta, Nishiyama ('12)],
  - 輸送方程式との関連
    Classical field → Boltzmann eq., A.Muller, Son ('04).
  - ずり粘性、非平衡発展 (エントロピー生成、量子クエンチなど) 、 ...

ご清聴ありがとうございました。

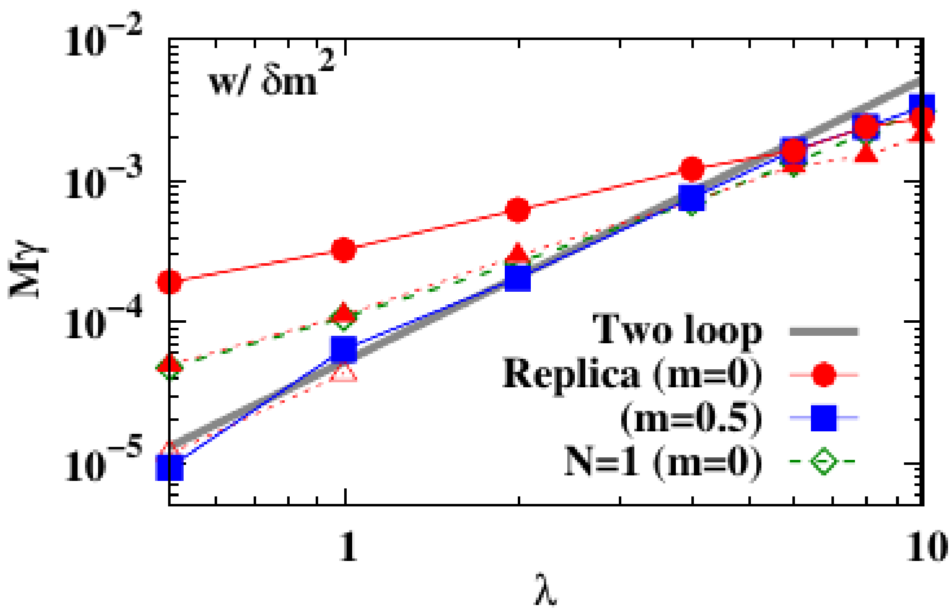
   

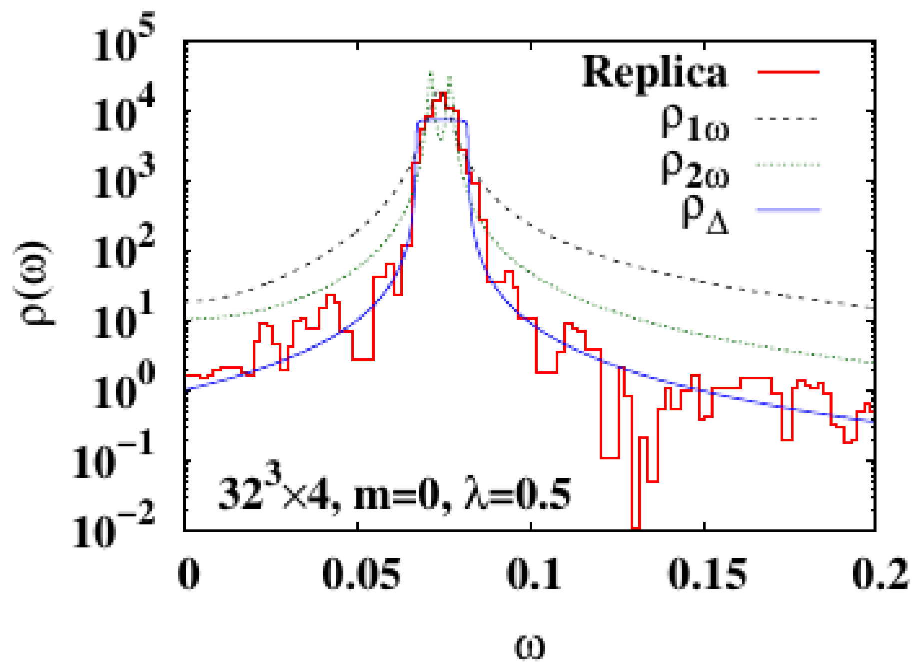
AO **Hidefumi Matsuda** **Teiji Kunihiro** **Toru T. Takahashi**

arXiv:2008.09556 [hep-lat]

Damping Rate

- **Apparent damping rate in replica evolution is larger than 2-loop results at small coupling. Why ?**
  - **Classical results (N=1) better agrees with 2-loop results.**
    Aarts ('01)
  - **Power spectrum shows wide spread of the mass, but falls off quickly. Fragmentation of zero-momentum single particle mode ?**





Commutator in Classical Dynamics

- **Classical-Quantum Correspondence**

$$[A, B] \to i\hbar\{A, B\}_{\mathrm{PB}} + \mathcal{O}(\hbar^3)$$

- **Unequal-time Poisson bracket** *Aarts ('01)*

$$\langle\frac{1}{2}[\hat{x}_H(t), \hat{x}_H(0)]\rangle \simeq \left\langle\frac{i}{2}\{x(t), x(0)\}_{\mathrm{PB}}\right\rangle$$

$$= \frac{i}{2}\langle\sum_{n,n'}\left[\frac{\partial\bar{x}_n(t)}{\partial\bar{x}_{n'}(t_0)}\frac{\partial\bar{x}_n(0)}{\partial\bar{p}_{n'}(t_0)} - \frac{\partial\bar{x}_n(t)}{\partial\bar{p}_{n'}(t_0)}\frac{\partial\bar{x}_n(0)}{\partial\bar{x}_{n'}(t_0)}\right]\rangle$$

$$\xrightarrow[\text{Free}]{} -\frac{i}{2}\sum_n \frac{1}{M_n}\sin M_n t$$

  - **n=0 term reproduces quantum mechanical result in a HO.**
  - **Unequal-time derivative can be obtained by using the Trotter formula together with Hessian matrix.**
    Kunihiro, Muller, AO, Schafer, Takahashi, Yamamoto ('10)

Application to Gauge theories and Fermion Systems

- **Gauge theory**
  - **Temporal component of the gauge field is Wick-rotated in the imaginary time formalism, and replica evolution cannot be applied as it is, except for the case in the temporal gauge (A_0=0).**
- **Fermions**
  - **We do not know (yet) how to handle Grassman number in replica.**
  - **Time-dependent Hartree-Fock theory may help.**

Previous Attempts

- **Separate soft and hard modes**
  Soft modes still have classical statistics, cutoff needs to be small.
  - **Effective action of soft modes by integrating hard modes**
    → dissipation and fluctuation from integrated hard modes
    D. Bodeker, L. D. McLerran and A. V. Smilga, PRD ('95)52;
    C. Greiner and B. Muller, PRD 55 ('97)1026.
  - **Introducing mass counterterm → Similar results with 2PI**
    e.g. G. Aarts and J. Smit, PLB 393 ('97) 395.
- **Coupled equation of field and particles**
  - **Solve coupled equation of field and particles → faster equilibration**
    A. Dumitru and Y. Nara, PLB 621 ('05) 89.
  - **Two particle irreducible (2PI) effective action approach**
    → Large numerical cost to simulate 3+1D fields
    J. Berges, AIP Conf. Proc. 739('04)1; G. Aarts, J. Berges, PRL 88('02)041603; Y. Hatta, A. Nishiyama, NPA 873('12)47.