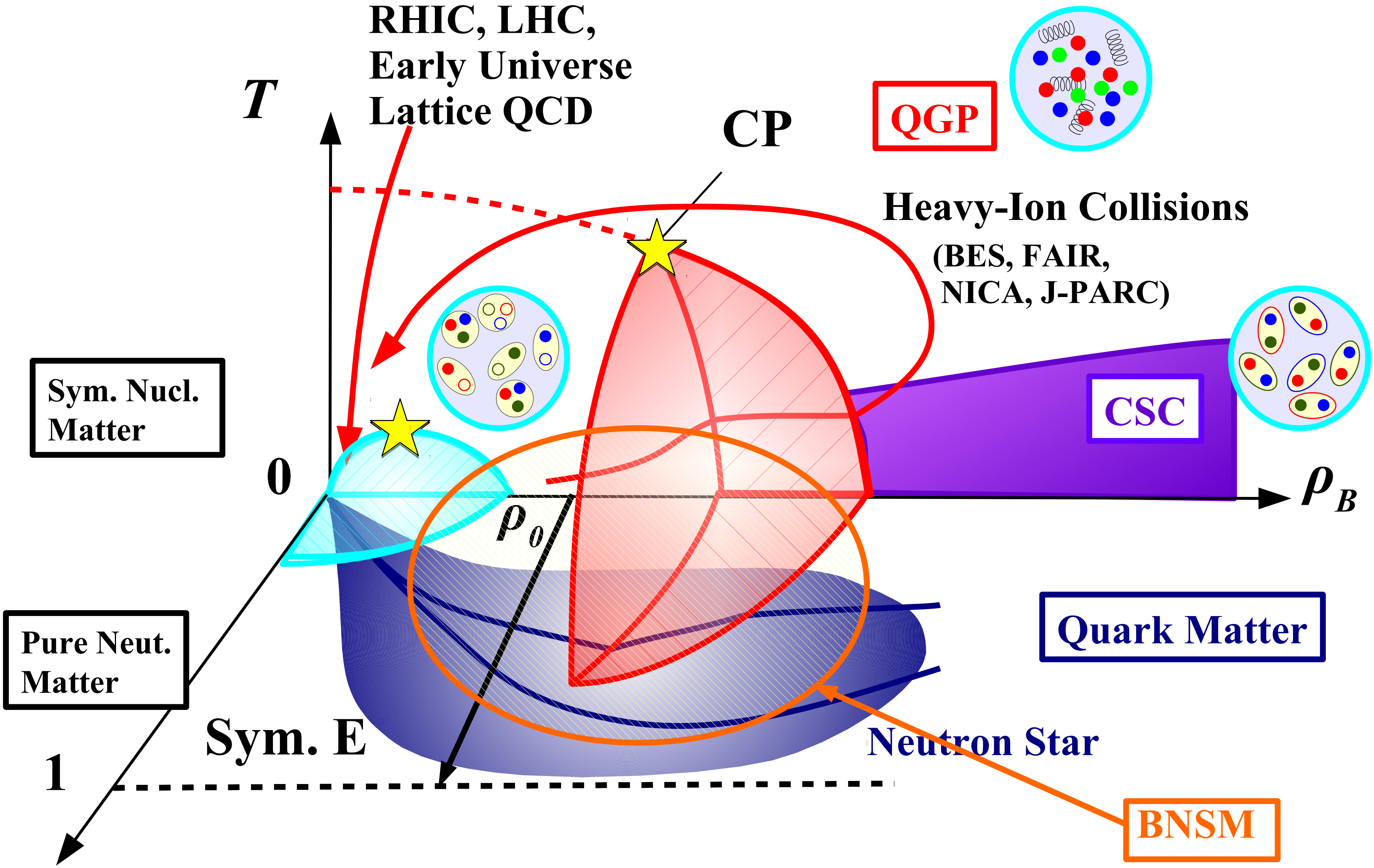
Replica evolution of classical field in 3+1+1 dimensional spacetime toward real time dynamics of quantum field

Akira Ohnishi (YITP, Kyoto U.)
based on the work in progress in collaboration with
H. Matsuda (Kyoto U.), T. Kunihiro (RCNP/YITP),
and T. T. Takahashi (Gunma NCT)

QH Seminar, May 29, 2020

QCD Phase Diagram

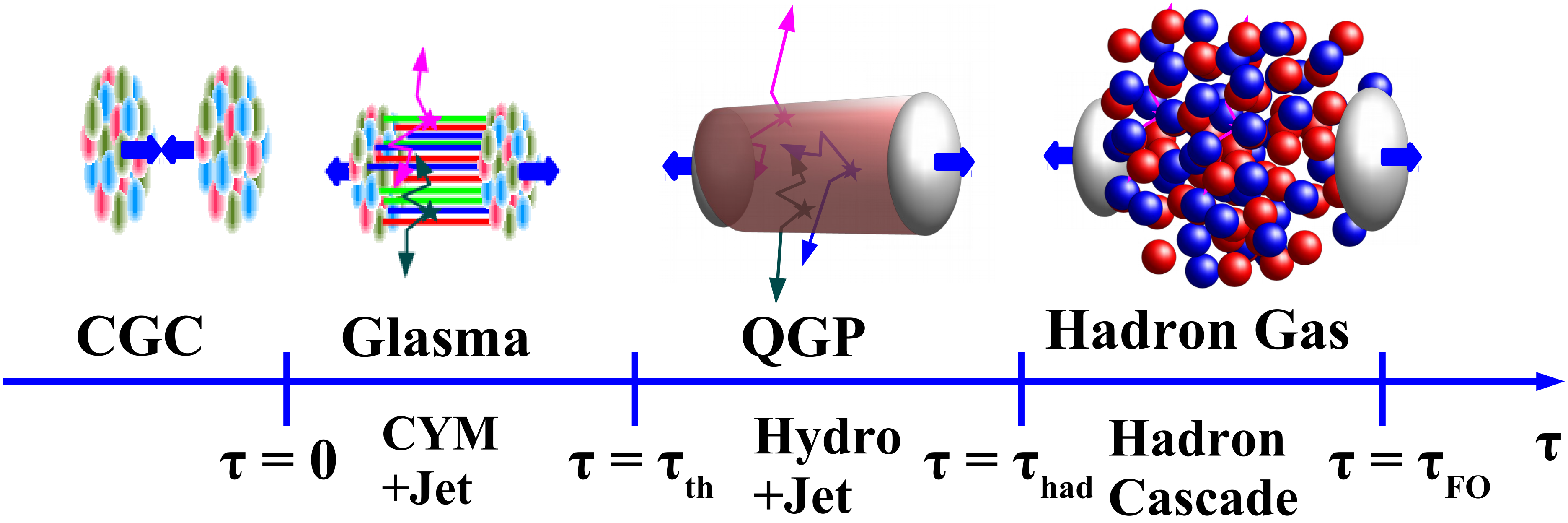


δ=(N-Z)/A (or Y_Q(hadron)=Q_h/B~(1-δ)/2)

AO, JPS Conf. Proc. 20 (2018), 011035

Various Stages in High-Energy Heavy-Ion Collisions

- Initial Condition: Color Glass Condensate (CGC)
- Early Stage: Glasma
- Main Stage: Quark Gluon Plasma (QGP)
- Final Stage: Hadron Gas



Real time evolution of quantum field

- **Static (equilibrium) problem →MC simulation of Lattice QFT**

$$\mathcal{Z} = \int \mathcal{D}\phi\, e^{-\int d^4x \mathcal{L}_E}$$

- **Path integral → Strong sign problem**

$$|\Psi(t)\rangle = \mathcal{N} \int \mathcal{D}\phi\, T \exp\left[i \int d^4x \mathcal{L} \right] |\Psi(t_0)\rangle$$

  - **Real time simulation of quantum field is difficult.**

- **Classical field simulation**

$$H = \frac{1}{2} \sum_{\boldsymbol{x}} \pi_{\boldsymbol{x}}^2 + V \rightarrow \frac{d\phi_{\boldsymbol{x}}}{dt} = \pi_{\boldsymbol{x}} \ , \ \frac{d\pi_{\boldsymbol{x}}}{dt} = -\frac{\partial V}{\partial \phi_{\boldsymbol{x}}} \ ,$$

  - **Phase is stationary w.r.t. the variation of (φ,π) → No cancellation**
  - **CF describes the growth of most unstable mode precisely.**
    (Classical Statistical Simulation;
    ***S. Y. Khlebnikov and I. I. Tkachev, PRL77('96)219*)**
  - **But CF evolution reaches classical equilibrium.**

Previous Attempts

- Can we construct a "classical field" framework which satisfies quantum statistical properties ?
- Separate soft and hard modes
  Soft modes still have classical statistics, cutoff needs to be small.
  - Effective action of soft modes by integrating hard modes
    → dissipation and fluctuation from integrated hard modes
    D. Bodeker, L. D. McLerran and A. V. Smilga, PRD ('95)52;
    C. Greiner and B. Muller, PRD 55 ('97)1026.
  - Introducing mass counterterm → Similar results with 2PI
    e.g. G. Aarts and J. Smit, PLB 393 ('97) 395.
- Coupled equation of field and particles
  - Solve coupled equation of field and particles → faster equilibration
    A. Dumitru and Y. Nara, PLB 621 ('05) 89.
  - Two particle irreducible (2PI) effective action approach
    → Large numerical cost to simulate 3+1D fields
    J. Berges, AIP Conf. Proc. 739('04)1; G. Aarts, J. Berges, PRL 88('02)041603; Y. Hatta, A. Nishiyama, NPA 873('12)47.

“Classical” evolution to “Quantum” equilibrium

- **Example: φ^4 theory on a lattice at T=ξ/N**

$$S_E = \frac{1}{\xi}\sum_{\tau=1}^{N}\sum_{\boldsymbol{x}}\left[\underline{\frac{1}{2}\left(\partial_\tau\phi_{\tau\boldsymbol{x}}\right)^2} + \frac{1}{2}(\boldsymbol{\nabla}\phi_{\tau\boldsymbol{x}})^2 + \frac{m^2}{2}\phi_{\tau\boldsymbol{x}}^2 + \frac{\lambda}{24}\phi_{\tau\boldsymbol{x}}^4\right] \to \mathcal{Z}_{\mathrm{Q}} = \underline{\int \mathcal{D}\phi e^{-S_E}}$$

- **Classical Hamiltonian**

$$H(\phi,\pi) = \sum_{\boldsymbol{x}}\left[\frac{1}{2}\pi_{\boldsymbol{x}}^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi_{\boldsymbol{x}})^2 + \frac{m^2}{2}\phi_{\boldsymbol{x}}^2 + \frac{\lambda}{24}\phi_{\boldsymbol{x}}^4\right] \to \mathcal{Z}_{\mathrm{cl}} = \int \mathcal{D}\phi\mathcal{D}\pi e^{-H(\phi,\pi)/T}$$

- **Replica Evolution: Simultaneous evolution of N configs of CF**

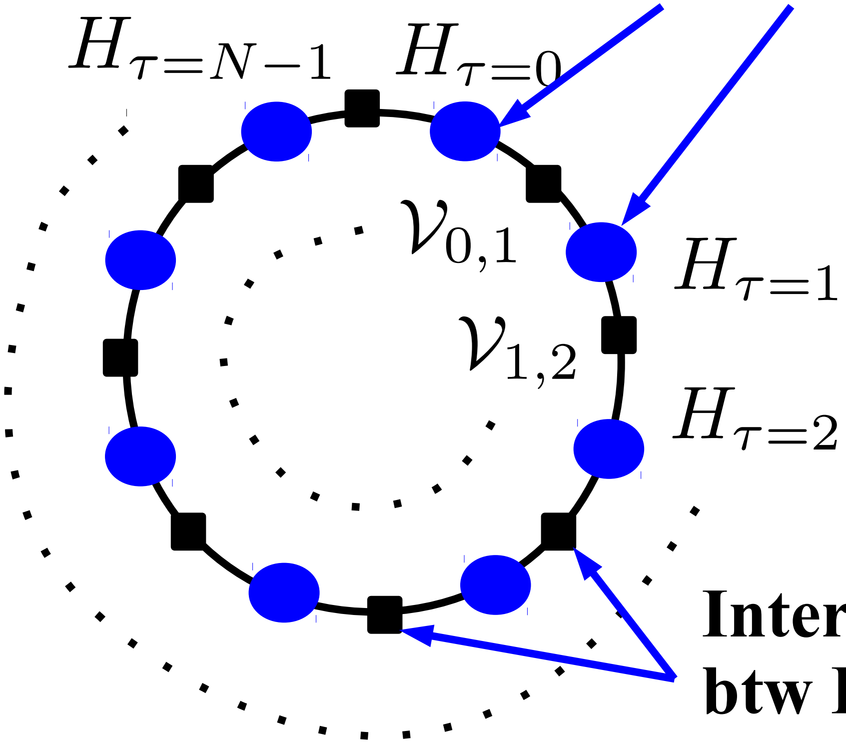
$$\mathcal{H} = \sum_{\tau} H(\phi_\tau,\pi_\tau) + \underline{\sum_{\tau,\boldsymbol{x}}\frac{\xi^2}{2}\left(\phi_{\tau+1,\boldsymbol{x}} - \phi_{\tau\boldsymbol{x}}\right)^2} = \sum_{\tau,\boldsymbol{x}}\frac{1}{2}\pi_{\tau\boldsymbol{x}}^2 + \xi S_E$$

$$\to \mathcal{Z}_R(T_{\mathrm{repl}}) = \int \mathcal{D}\phi\mathcal{D}\pi e^{-\mathcal{H}/\xi} \propto \underline{\int \mathcal{D}\phi e^{-S_E}}$$

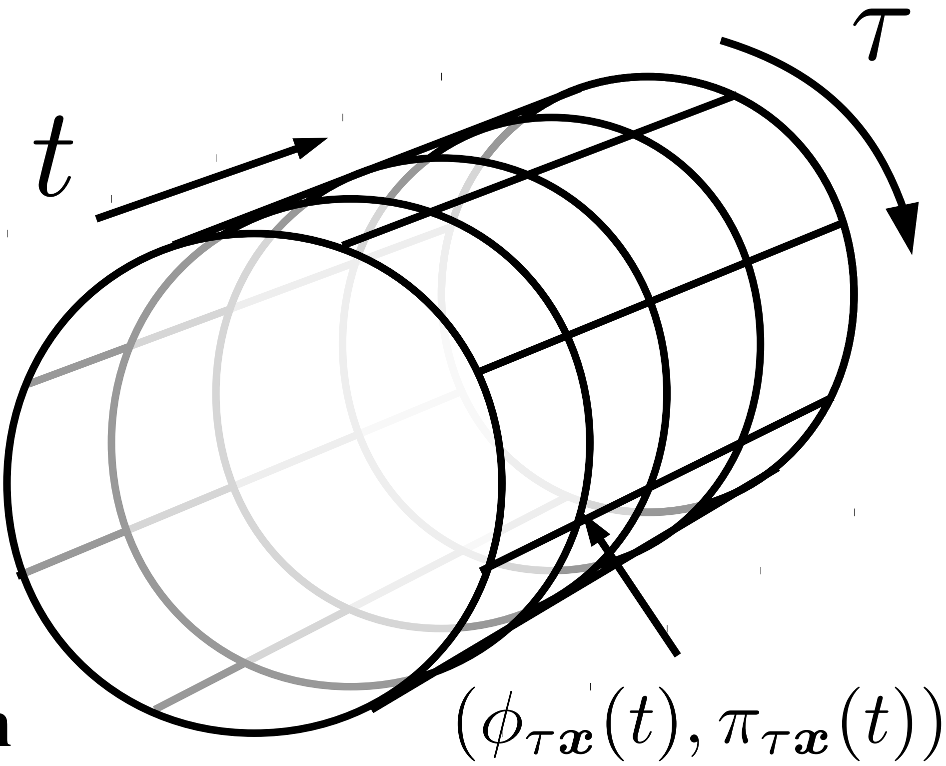
Discussed in QH seminar, Dec. 2019
AO, H. Matsuda, T. Kunihiro, T.T.Takahasi, in prep.

Replica Evolution

Replicas of classical field



Interaction btw Replicas



$$\mathcal{H} = \sum_{\tau} H_{\tau} + \sum_{\tau} \mathcal{V}_{\tau,\tau+1} = \frac{1}{2} \sum_{\tau,\boldsymbol{x}} \pi^2_{\tau,\boldsymbol{x}} + \xi S[\phi]$$

$$\mathcal{Z}_R = \int \mathcal{D}\pi\, \mathcal{D}\phi\, \exp(-\mathcal{H}/\xi) \propto \int \mathcal{D}\phi\, \exp(-S[\phi])$$

Replica Evolution

- We consider N classical field configurations, dubbed as replicas,
  which interact with each other
  via the τ-derivative potential terms.
  (~ Molecular dynamics part of the hybrid Monte-Carlo)
- Replica evolves according to the classical EOM.
- In the replica ensemble at temperature ξ=NT,
  classical field distribution is described
  by the quantum partition function in the imag. time formalism
  after the long real-time evolution.

- Question
  - Does the replica evolution give correct real-time evolution ?
  - Does it describe the thermal mass correctly ?

Replica evolution of harmonic oscillator (quantum mechanics)

Harmonic Oscillator

- **Hamiltonian**

Fourier transf. of x_τ, p_τ

$$H(x,p) = \frac{p^2}{2} + \frac{\omega^2 x^2}{2}$$

$$\mathcal{H} = \sum_\tau H(x_\tau, p_\tau) + \mathcal{V} = \sum_n \left[\frac{\bar{p}_n^2}{2} + \frac{M_n^2 \bar{x}_n^2}{2} \right]$$

$$\mathcal{V} = \sum_\tau \frac{\xi^2}{2}(x_{\tau+1} - x_\tau)^2 \ , \quad M_n^2 = \omega^2 + 4\xi^2 \sin^2(n\pi/N)$$

- **Replica partition function**

Matsubara freq. summation tech.

$$\mathcal{Z}_R(\xi) = \prod_n (\xi/M_n) = [2\sinh(\Omega/2T)]^{-1} \underset{N\to\infty}{\longrightarrow} [2\sinh(\omega/2T)]^{-1} = \mathcal{Z}_Q(T)$$

$$\Omega = 2\xi \,\mathrm{arcsinh}\,(\omega/2\xi) = 2NT \,\mathrm{arcsinh}\,(\omega/2NT) \underset{N\to\infty}{\longrightarrow} \omega$$

Correct quantum partition function is obtained in the large N limit

Time-Correlation Function

- **Unequal time two-point function (Time-correlation function)**
  - **Quantum mechanics**

Good exercise for UG students

$$C_Q(t) \equiv \langle x_H(t) x_H(0) \rangle_T = \frac{1}{2\omega}\left[\coth\left(\frac{\omega}{2T}\right)\cos\omega t - i\sin\omega t\right]$$

  - **Replica evolution**
    Solution of EOM

$$\bar{x}_n(t) = \bar{x}_n(0)\cos M_n t + \frac{\bar{p}_n(0)}{M_n}\sin M_n t\ , \qquad \frac{\bar{p}_n(t)}{M_n} = -\bar{x}_n(0)\sin M_n t + \frac{\bar{p}_n(0)}{M_n}\cos M_n t\ .$$

Thermal expectation values

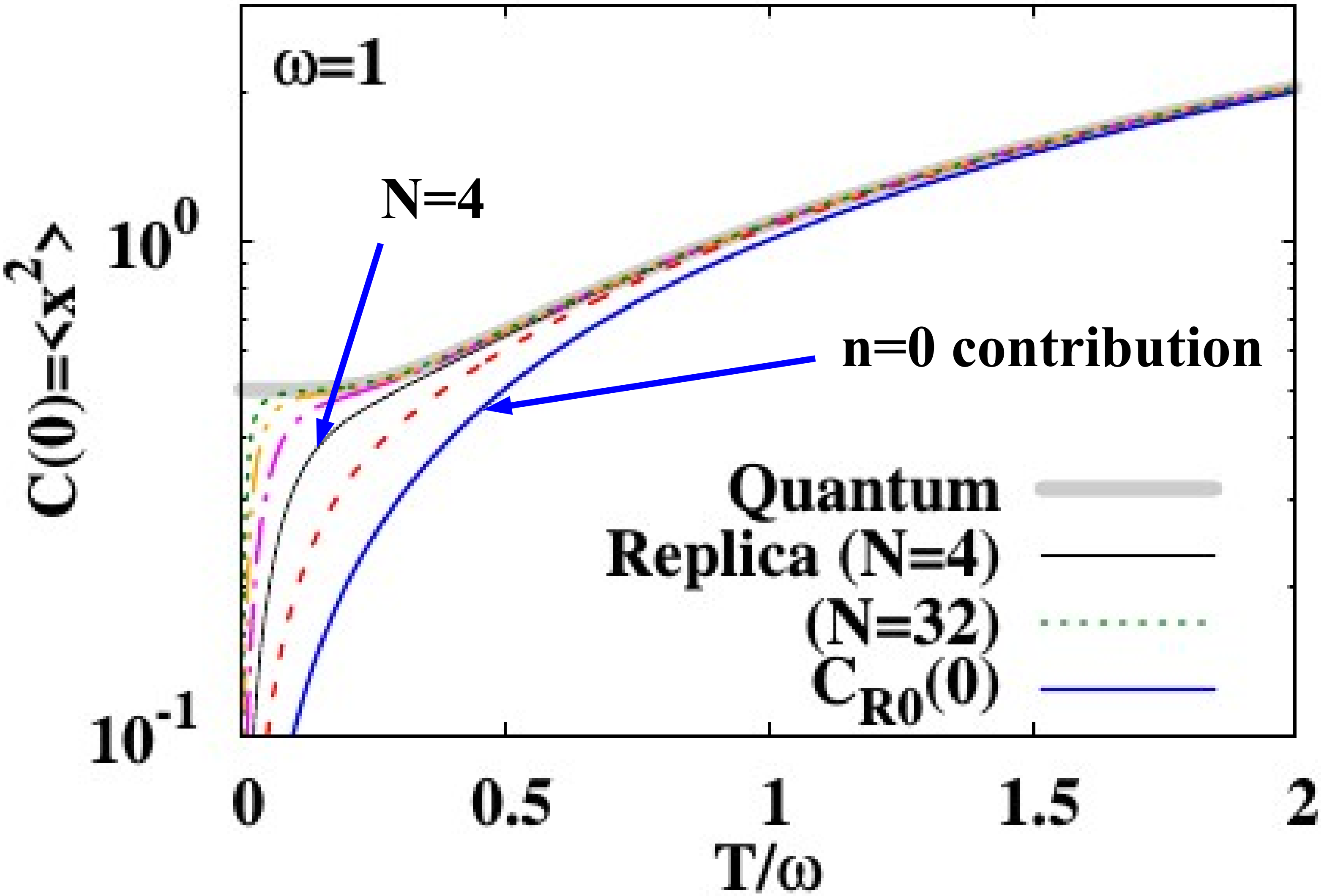
$$\langle \bar{x}_n(0)\bar{x}_{n'}(0)\rangle_T = \xi/M_n^2\delta_{nn'}, \qquad \langle \bar{p}_n(0)\bar{p}_{n'}(0)\rangle_T = \xi\delta_{nn'}$$

Time-corr. fn.

$$C_R(t) = \langle x(t)x(0)\rangle_T \equiv \frac{1}{N}\sum_\tau \langle x_\tau(t)x_\tau(0)\rangle_T = \frac{1}{N}\sum_n \langle \bar{x}_n(t)\bar{x}_n(0)\rangle_T$$

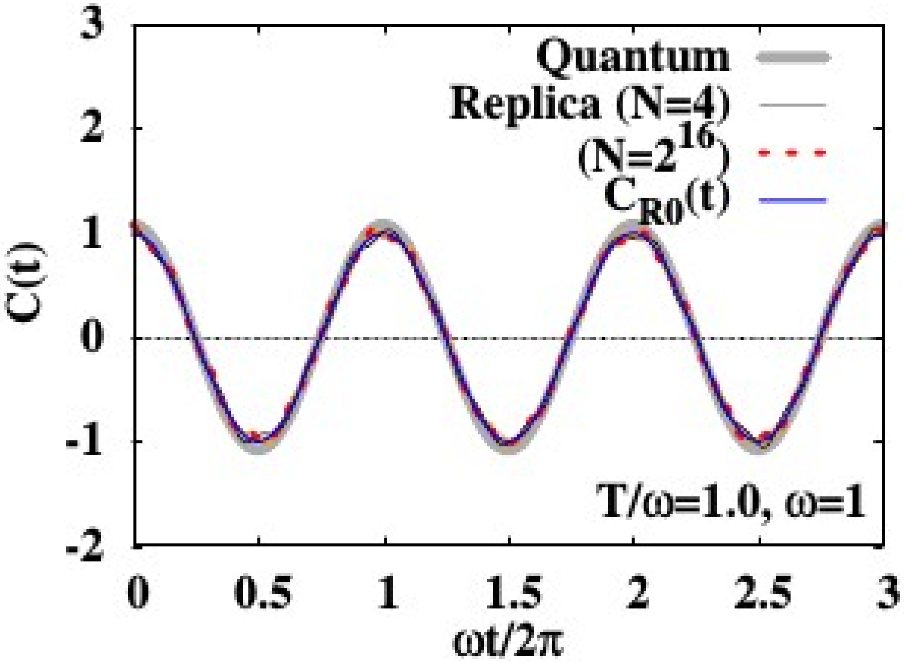
$$= \frac{1}{N}\sum_n \frac{\xi}{M_n^2}\cos M_n t = \sum_n \frac{T}{M_n^2}\cos M_n t\ .$$

Equal time two-point function

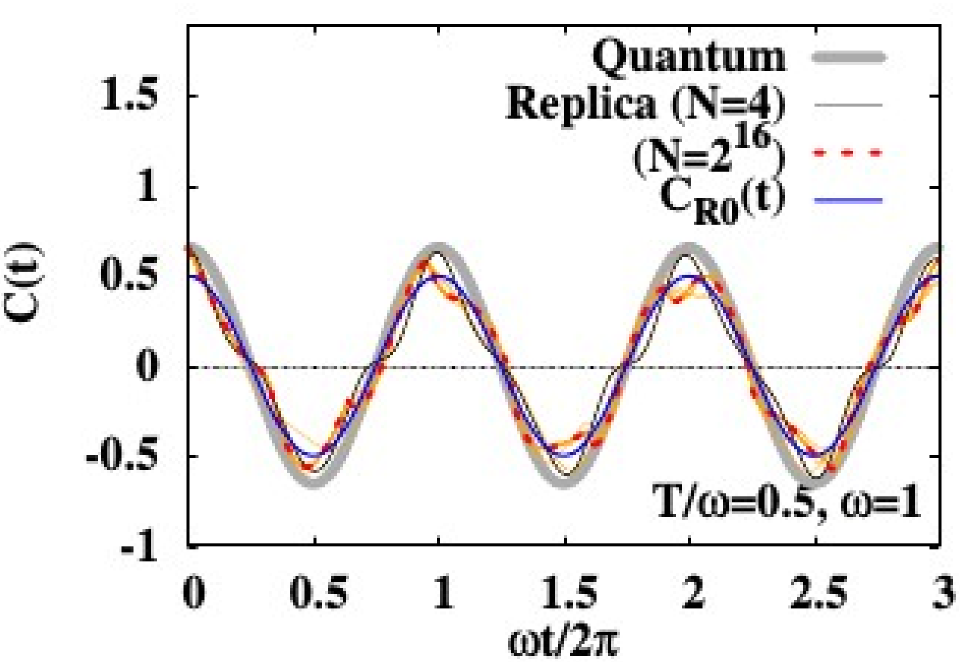


Correct quantum amplitude is obtained in the large N limit.
N=4 is enough for T/ω > 0.3

Unequal time two-point function



Good enough !



Well, we can extract ω from time-corr. fn.

For T/ω > 0.5, time-corr. fn. is dominated by n=0 component, and we can access the quantum TCF by the replica evolution.

Replica evolution of classical field (φ^4 theory)

Scalar theory (φ^4) on the lattice

- **Lagrangian & Hamiltonian**

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{24}\phi^4$$

$$H(\phi,\pi) = \sum_{\boldsymbol{x}}\left[\frac{1}{2}\pi_{\boldsymbol{x}}^2 + \frac{1}{2}\left(\boldsymbol{\nabla}\phi_{\boldsymbol{x}}\right)^2 + \frac{m^2}{2}\phi_{\boldsymbol{x}}^2 + \frac{\lambda}{24}\phi_{\boldsymbol{x}}^4\right]$$

- **Replica Hamiltonian & Equation of Motion**

fluc. in replica index

$$\mathcal{H} = \sum_\tau H(\phi_\tau,\pi_\tau) + \mathcal{V}, \quad \mathcal{V} = \frac{\xi^2}{2}\sum_{\tau,\boldsymbol{x}}(\phi_{\tau+1,\boldsymbol{x}} - \phi)^2$$

$$\frac{d\phi_{\tau\boldsymbol{x}}}{dt} = \pi_{\tau\boldsymbol{x}}\ , \quad \frac{d\pi_{\tau\boldsymbol{x}}}{dt} = \frac{\partial H(\phi_\tau,\pi_\tau)}{\partial\phi_{\tau\boldsymbol{x}}} + \xi^2(\phi_{\tau+1,\boldsymbol{x}} + \phi_{\tau-1,\boldsymbol{x}} - 2\phi_{\tau\boldsymbol{x}})$$

- **EOM for replica index average of φ ≃ Classical field EOM**

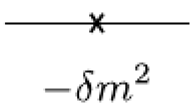
$$\widetilde{\phi}_{\boldsymbol{x}} \equiv \frac{1}{N}\sum_\tau \phi_{\tau\boldsymbol{x}} = \frac{1}{\sqrt{N}}\bar{\phi}_0 \ \rightarrow\ \underline{(\partial^\mu\partial_\mu + m^2)\widetilde{\phi}_{\boldsymbol{x}} + \frac{\lambda}{6}(\widetilde{\phi}_{\boldsymbol{x}})^3} = \mathcal{O}((\delta\phi_{\boldsymbol{x}})^2)$$

=zero in classical field eq.

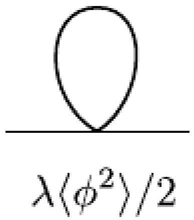
Mass Counterterm

- **Leading order thermal mass**

$$M^2 = m^2 - \delta m^2 + \frac{\lambda}{2}\langle\phi^2\rangle_T = m^2 + \frac{\lambda}{2}\left\langle\phi^2\right\rangle_{\rm ren}$$



$-\delta m^2$



$\lambda\langle\phi^2\rangle/2$

$$\frac{\lambda}{2}\langle\phi^2\rangle_T \underset{\uparrow}{=} \frac{\lambda}{2}\frac{1}{L^3}\sum_{\boldsymbol{k}}\frac{1}{\omega_{\boldsymbol{k}}\sqrt{1+(\omega_{\boldsymbol{k}}/2\xi)^2}}\left[\underbrace{\frac{1}{2}}_{\delta m^2}+\underbrace{\frac{1}{e^{\Omega_{\boldsymbol{k}}/T}-1}}_{\frac{\lambda}{2}\left\langle\phi^2\right\rangle_{\rm ren}}\right]$$

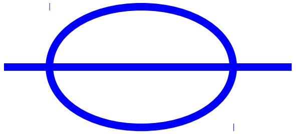
Matsubara sum.

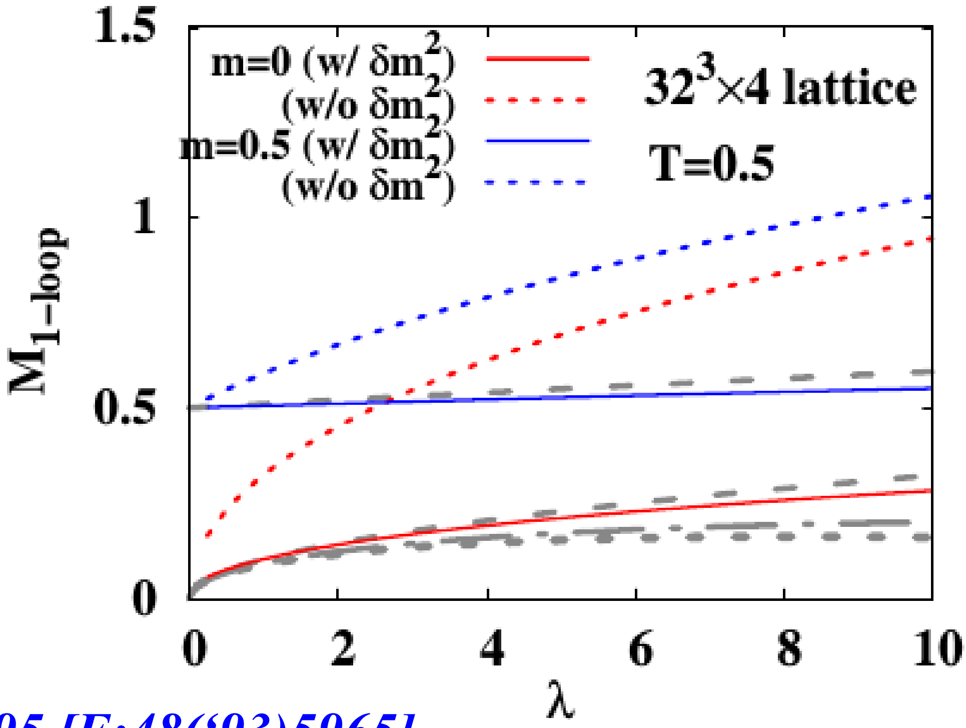
- **Perturbative Calc.** *Kapusta, Gale*

$$M^2_{\rm LO} = m^2 + \lambda T^2/24.$$

$$M^2_{\rm resum} = \frac{\lambda T^2}{24}\left[1 - \frac{3}{\pi}\sqrt{\frac{\lambda}{24}}\right]$$

$$M^2_{2-\rm loop} = \frac{\lambda T^2}{24}\left\{1 - \frac{3}{\pi}\sqrt{\frac{\lambda}{24}} + \frac{\lambda}{(4\pi)^2}\left[\frac{3}{2}\log\left(\frac{T^2}{4\pi\mu^2}\right) + 2\log\left(\frac{\lambda}{24}\right) + \alpha\right]\right\}$$





R. R. Parwani, PRD 45 ('92)4695 [E:48('93)5965].

Numerical Calculation Setup

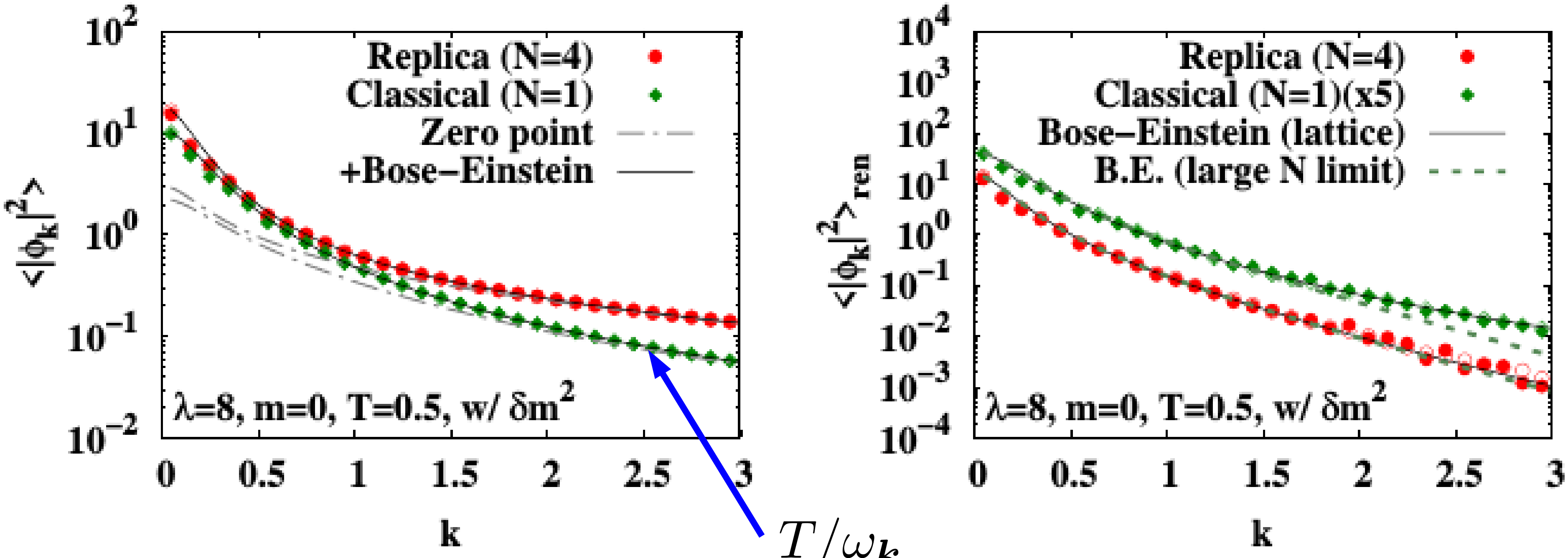
- Lattice size = 32^3 x 4 (L=32, N=4)
- T=0.5 (ξ=NT=2); m=0, 0.5; λ=0.5, 1, 2, 4, 6, 8, 10.
- One loop renormalization of mass, no counterterm for λ.
- Initial conditions are obtained by solving the Langevin equation.
- Solve replica EOM until t=500 with the time step of Δt=0.025.
- Number of replica configurations = 1000
  → 3-6 hours on one core of core i7 PC for a given (m, λ)

Rayleigh-Jeans Divergence

- **Replica evolution calculation with mass counterterm should give correct quantum field calc. results in the large N lim., but momentum dist. does not necessarily damps exponentially at finite N.**

$$\langle|\phi_{\boldsymbol{k}}|^2\rangle_{\rm ren} \simeq \frac{2NT}{k^2}\underline{\exp(-\Omega_{\boldsymbol{k}}/T)} \xrightarrow[k\gg NT]{} 2(NT)^{2N+1}k^{-2(N+1)}$$

$$\exp(-\Omega_{\boldsymbol{k}}/T) \underset{k\gg NT}{\simeq} (\omega_{\boldsymbol{k}}/NT)^{-2N}$$



$T/\omega_{\boldsymbol{k}}$

Rayleigh-Jeans Divergence

- **With N >= 2, free field energy converges in the replica method.**

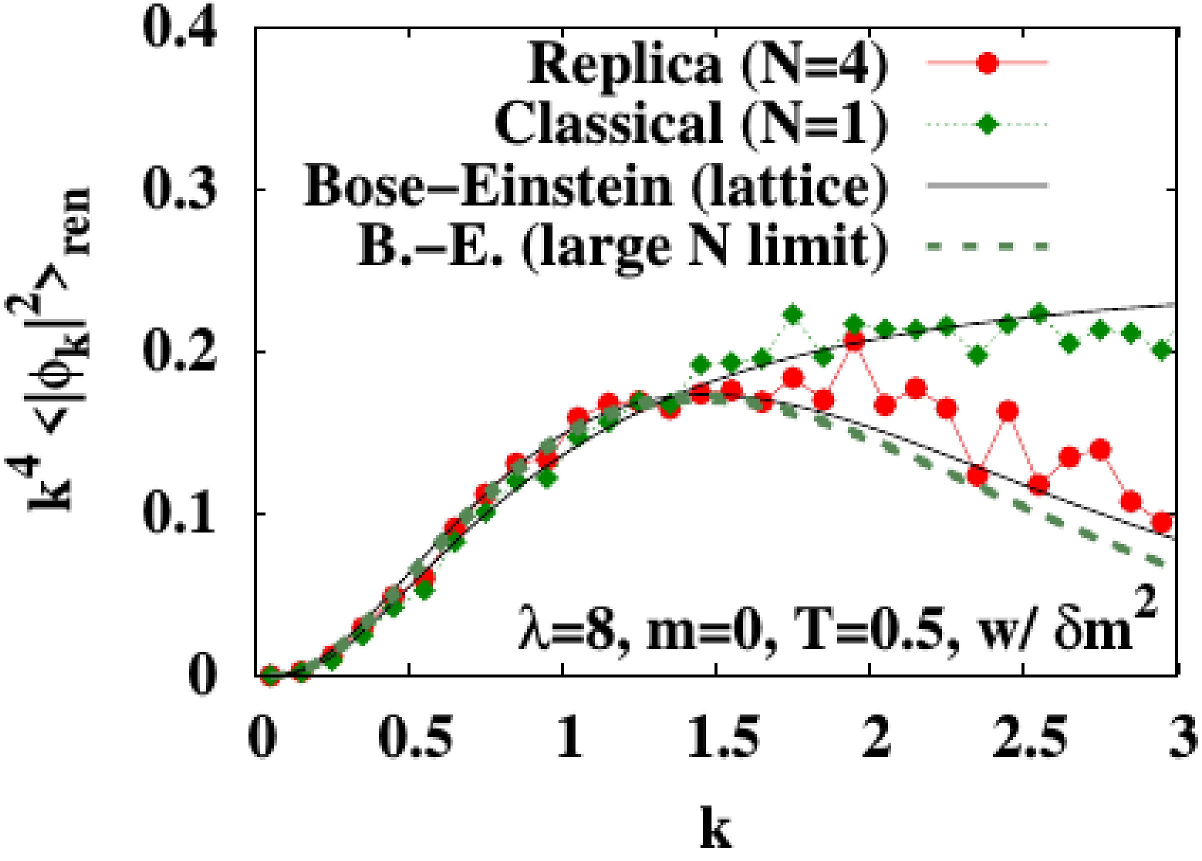
$$\Omega = 2NT\,\mathrm{arcsinh}\,(\omega/2NT) \underset{\omega\gg NT}{\longrightarrow} 2NT\log(\omega/NT)$$

$$\left\langle|\phi_{\boldsymbol{k}}|^2\right\rangle_{\mathrm{ren}} \simeq \frac{2NT}{k^2}\exp(-\Omega_{\boldsymbol{k}}/T) \to 2(NT)^{2N+1}k^{-2(N+1)}$$

$$k^4\left\langle|\phi_{\boldsymbol{k}}|^2\right\rangle_{\mathrm{ren}} \to 2(NT)^{2N+1}k^{-2(N-1)}$$

- **Convergence cond.**
  2(N-1) > 1 → N > 1.5

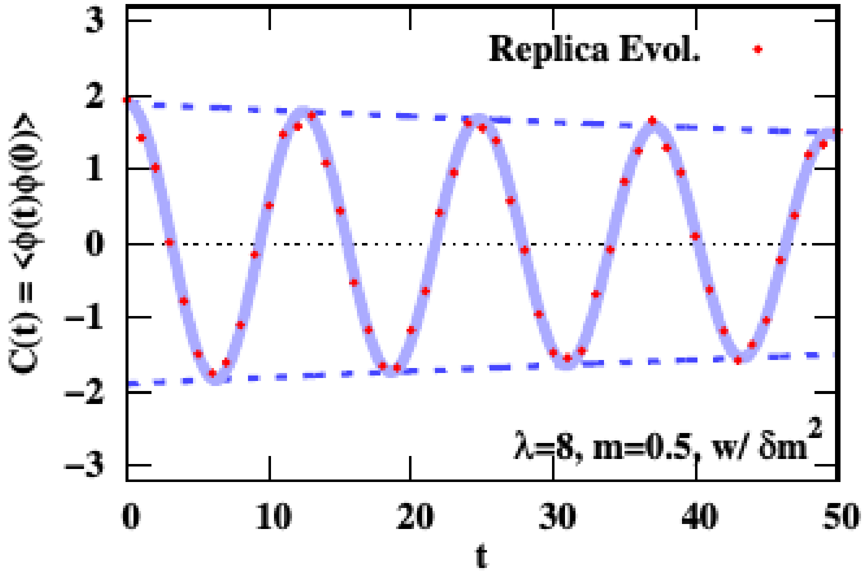
We can remove divergence of energy in the replica method (N>=2) with mass counterterm(s).



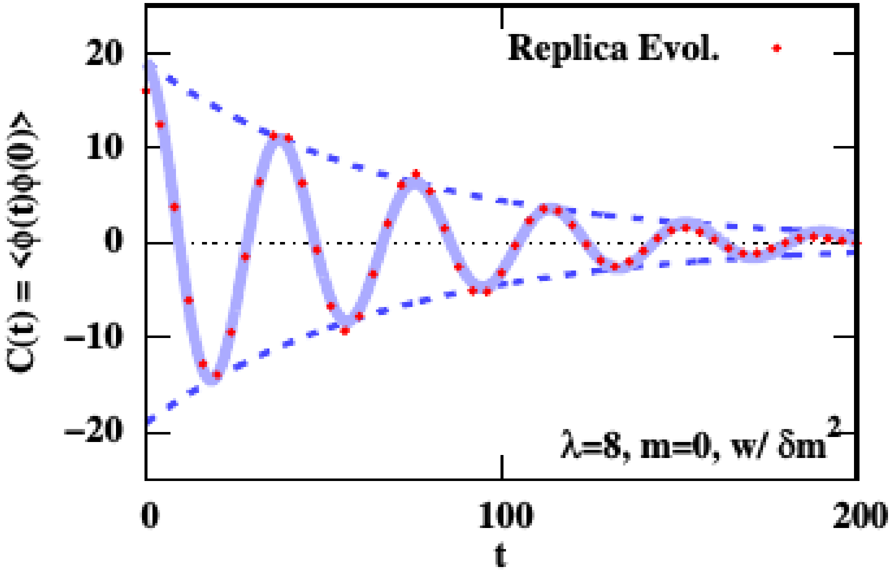
Time-correlation function

- Time-correlation function of free field (zero momentum)

$$C(t) = \frac{1}{L^3}\sum_{\boldsymbol{x},\boldsymbol{y}} \langle \phi_{\boldsymbol{x}}(t)\phi_{\boldsymbol{y}}(0)\rangle$$
$$= \frac{1}{NL^3}\sum_{\tau,\boldsymbol{x},\boldsymbol{y}} \langle \phi_{\tau\boldsymbol{x}}(t)\phi_{\tau\boldsymbol{y}}(0)\rangle$$
$$= \sum_n \frac{T}{M_n^2}\cos M_n t$$
$$(M_n^2 = \omega^2 + 4\xi^2 \sin^2(n\pi/N))$$



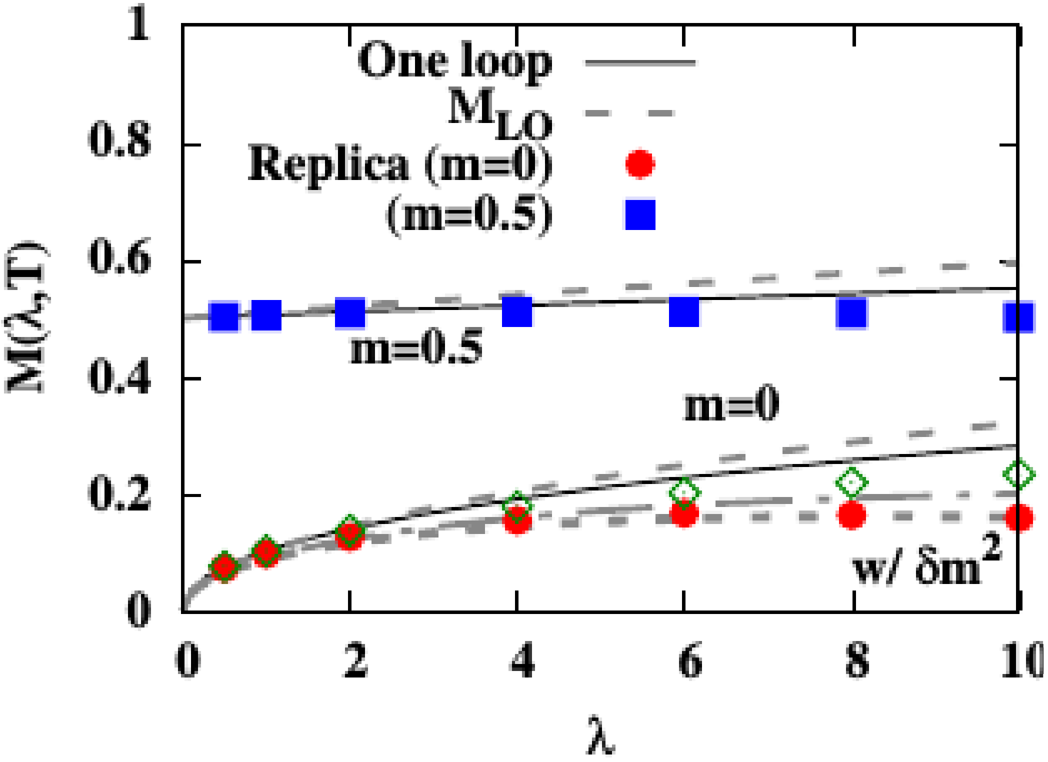
- TCF of interacting field
  - Interaction induces thermal mass
  - Coupling of different momentum modes induces damping.

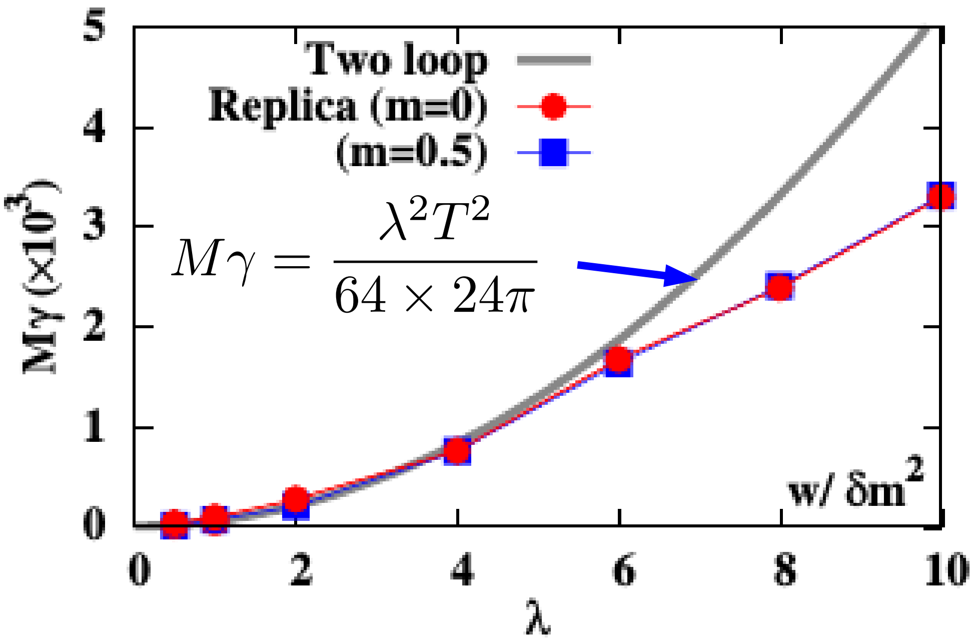


Thermal mass and width

- **Thermal mass in replica method ~ 2-loop calc. results**
- **Thermal width in replica method ~ 2-loop calc. results at λ < 4**

Replica method takes account of higher-order interaction effects over one-loop.





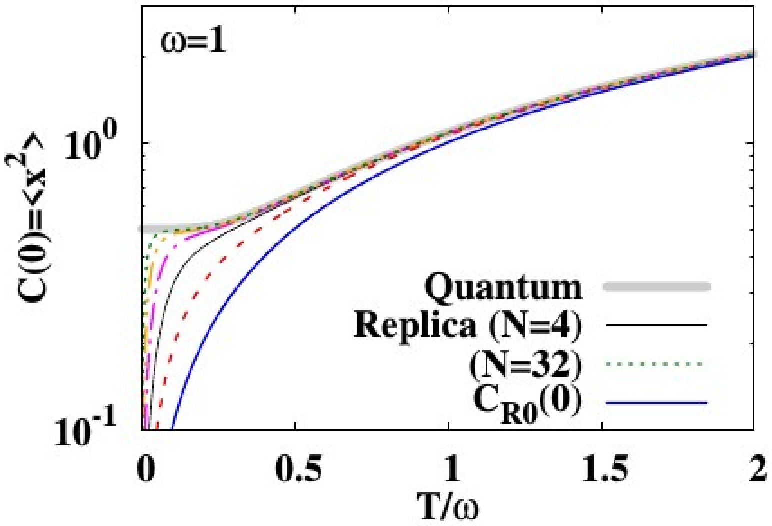
$$M\gamma = \frac{\lambda^2 T^2}{64 \times 24\pi}$$

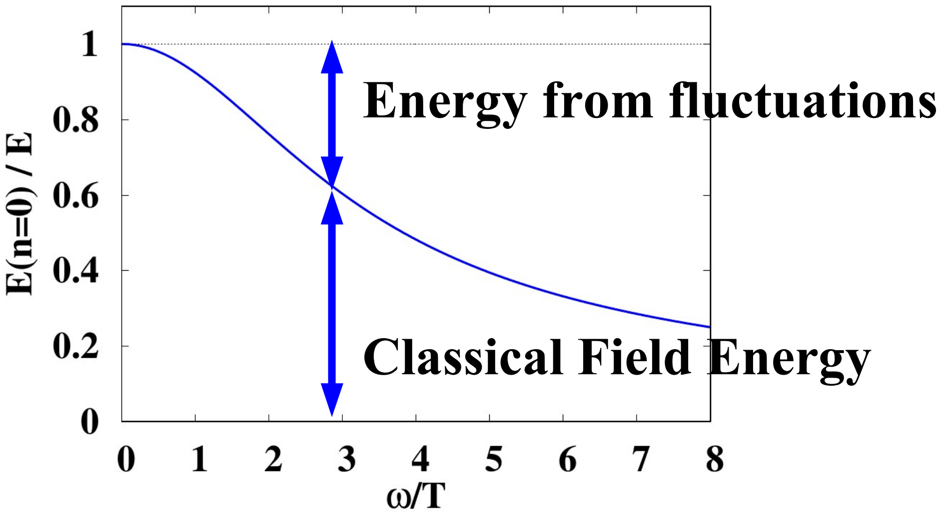
Summary

- **We have proposed a framework, the replica evolution method, in which the N(>1) classical fields interact with each other via τ-derivative term.**
  - **Quantum distribution of the field variable (φ) is described correctly after long-time evolution at large N.**
  - **Replica index average (ave. over τ) of φ approximately follows the classical field equation.**
  - **Time-correlation function is also well described in the temperature region of T/ω>0.5 in the case of harmonic oscillator.**
- **Replica evolution method may be regarded as a version of classical field dynamics with quantum statistical improvement. (minimum claim)**
  - **Real-time evolution of quantum field around equilibrium may be described reasonably well in the temperature region of T/M > 0.5.**
  - **It would be possible to apply it to non-equilibrium evolution. (premature ?)**

To do

- **Complete the manuscript.**
- **Understand the meaning of fluctuations among replicas. (Replica index average = Classical Field)**





- **Comparison with 2PI results, Application to non-equilibrium processes such as entropy production or pressure isotropization, Formal "derivation" of replica method (or its improved one), Coleman / Mermin-Wagner theorem, ...**

Thank you for your attention !

AO

Hidefumi Matsuda

Teiji Kunihiro

Toru T. Takahashi